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THEORY OF LOADED COSSERAT ELASTIC CYLINDERS

BY

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1. Introduction. The Saint-Venant's problem for inhomogeneous and anisotropic cylinders in Cosserat elasticity was solved by D. Ieşan in [3]. In this paper, following [4] we extend these results to the case of loaded cylinders. A Cosserat elastic cylinder acted by body forces, body couples and surface loadings is considered.

2. Statement of the Problem. Throughout this paper, Ω denotes the interior of a right cylinder of length l and with generic cross-section Σ. B is the lateral surface of the cylinder and L the boundary of Σ. A rectangular Cartesian coordinate system, Ox_k, $(k = 1, 2, 3)$ is used. We assume that Ox_3 is parallel to the generators of Ω and x_1Ox_2 plane contains one of the terminal sections. Let $\Sigma^{(0)}$ be the cross-section located at $x_3 = 0$ and $\Sigma^{(l)}$ the cross-section located in the plane $x_3 = l$.

We shall employ the usual summation and differentiation conventions: greek subscripts are understood to run over integers (1,2); latin subscripts are understood to run over integers (1, 2, 3) (unless otherwise is specified); summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.

In this paper we consider the linear theory of Cosserat elastic solids. The basic equations of this theory are [1]:

— the equilibrium equations

$$t_{ji,j} + f_i = 0, \quad m_{ji,j} + \varepsilon_{ijk}\, t_{jk} + l_i = 0, \tag{2.1}$$

— the constitutive equations

$$\begin{aligned} t_{ij} &= A_{ijkl}\, e_{kl} + B_{ijkl}\, \varkappa_{kl}, \\ m_{ij} &= B_{klij}\, e_{kl} + C_{ijkl}\, \varkappa_{kl}, \end{aligned} \tag{2.2}$$

— the geometrical equations

$$e_{ij} = u_{j,i} + \varepsilon_{jik}\, \varphi_k, \quad \varkappa_{ij} = \varphi_{j,i}, \tag{2.3}$$

where we have used the notations: u_i — the components of the displacement vector, φ_i-components of the microrotation vector, e_{ij} and $\varkappa_{ij}$-kinematic characteristics of the strain, t_{ij}-components of the stress tensor, m_{ij}-components of the couple-stress tensor, ε_{ijk}-alternating symbol, A_{ijkl}, B_{ijkl}, C_{ijkl}-characteristic coefficients of the material. The elastic coefficients satisfy the symmetry relations

$$A_{ijkl} = A_{klij}, \qquad C_{ijkl} = C_{klij}. \tag{2.4}$$

We consider the following conditions on the lateral surface of the cylinder:

$$t_{ji}n_j = \widetilde{t}_i, \qquad m_{ji}n_j = \widetilde{m}_i, \tag{2.5}$$

where n_i are the direction cosines of the outward normal to B.

Let the loading applied on $\Sigma^{(0)}$ be statically equivalent to a force $R\,(R_i)$ and a moment $M\,(M_i)$. On $\Sigma^{(l)}$ there are applied tractions and couples in a way which satisfies the equilibrium conditions of a rigid body. Thus for $x_3 = 0$, we have the following conditions:

$$\int_\Sigma t_{\alpha 3}\, d\sigma = -R\alpha, \tag{2.6}$$

$$\int_\Sigma t_{33}\, d\sigma = -R_3, \tag{2.7}$$

$$\int_\Sigma (x_\alpha t_{33} - \varepsilon_{3\alpha\beta} m_{3\beta})\, d\sigma = \varepsilon_{3\alpha\beta} M_\beta \tag{2.8}$$

$$\int_\Sigma (\varepsilon_{3\alpha\beta} t_{3\beta} x_\alpha + m_{33})\, d\sigma = -M_3. \tag{2.9}$$

We assume that the body forces, the body couples and surface loadings are polynomials of degree r in the axial coordinate

$$f_i = \sum_{k=0}^{r} F_{ik}(x_1, x_2)\, x_3^k, \quad l_i = \sum_{k=0}^{r} L_{ik}(x_1, x_2)\, x_3^k \tag{2.10}$$

$$\widetilde{t}_i = \sum_{k=0}^{r} P_{ik}(x_1, x_2)\, x_3^k, \quad \widetilde{m}_i = \sum_{k=0}^{r} M_{ik}(x_1, x_2)\, x_3^k, \tag{2.11}$$

where $F_{ik}, L_{ik}, P_{ik}, M_{ik}$ are given.

In that follows we assume that $A_{ijkl}, B_{ijkl}, C_{ijkl}$ are C^∞-functions on x_1 and x_2 in Ω an $d\Sigma$ is C^∞-smooth (Fichera [2]).

The problem consists in the determination of a solution of equations (2.1) — (2.3) which satisfies the conditions (2.5) — (2.9), where A_{ijkl}, C_{ijkl}, f_i, l_i, t_i, m_i are given by (2.4), (2.10), (2.11) and R_i, M_i are constants.

Denote by (A) the problem of determination of a solution of equations (2.1) — (2.3) with the body forces and body couples given by

$$f_i = F_{in}(x_1, x_2)\, x_3^n, \quad l_i = L_{in}(x_1, x_2)\, x_3^n, \tag{2.12}$$

which satisfies the conditions (2.6) — (2.9) on $x_3 = 0$ and the conditions

$$t_{i\alpha}n_\alpha = P_{in}(x_1, x_2)\, x_3^n \; m_{i\alpha}n_\alpha = M_{in}(x_1, x_2)\, x_3^n, \text{ on } B, \tag{2.13}$$

where n is a positive integer or zero.

We denote by $(B^{(0)})$ the problem (A) for $n = 0$ and by $(B^{(s)})$ $(s \geq 1)$ the problem (A) for $n = s$ and $R_i = M_i = 0$. To solve the initial problem

we use the method of induction: first we solve the problem $(B^{(0)})$ and then we shall find the solution of the problem $(B^{(n+1)})$ assuming that the solution of the problem $(B^{(n)})$ $(n \geq 1)$ is known.

3. Auxiliary Plane Strain Problems. Saint-Venant's problem for Cosserat elastic solids was solved by means of some generalized plane strain problems. The basic equations of a generalized plane strain problem are

— the equilibrium equations:

$$t_{\alpha i,\alpha} + f_i = 0, \; m_{\alpha i,\alpha} + \varepsilon_{ijk} t_{jk} + l_i = 0, \text{ in } \Sigma, \tag{3.1}$$

— the constitutive equations:

$$\begin{gathered} t_{\alpha i} = A_{\alpha ijk}e_{jk} + B_{\alpha i\beta j}\varkappa_{\beta j}, \; t_{3\alpha} = A_{3\alpha jk}e_{jk} + B_{3\alpha\beta j}\varkappa_{\beta j}, \\ t_{33} = A_{33jk}e_{jk} + B_{33\beta j}\varkappa_{\beta j}, \\ m_{\alpha i} = B_{jk\alpha i}e_{jk} + C_{\alpha i\beta j}\varkappa_{\beta j}, \; m_{3i} = B_{jk3i}e_{jk} + C_{3i\beta j}\varkappa_{\beta j}, \end{gathered} \tag{3.2}$$

— the geometrical equations

$$\begin{gathered} e_{\alpha i} = u_{i,\alpha} + \varepsilon_{i\alpha k}\varphi_k, \quad e_{3i} = \varepsilon_{i3k}\varphi_k, \\ \varkappa_{\alpha i} = \varphi_{i,\alpha}, \quad \varkappa_{3i} = 0. \end{gathered} \tag{3.3}$$

The involved functions depend only on x_1 and x_2.

We consider the boundary conditions

$$t_{\alpha i}n_\alpha = p_i, \quad m_{\alpha i}n_\alpha = q_i, \quad \text{on } L, \tag{3.4}$$

where p_i, q_i are prescribed functions of x_1 and x_2.

It is known that a necessary and sufficient conditions for the existence of a solution of problem (3.1) — (3.3) with boundary conditions (3.4) are (see [2]):

$$\begin{gathered} \int_\Sigma f_i d\sigma + \int_L p_i ds = 0, \\ \int_\Sigma (\varepsilon_{3\alpha\beta}\, x_\alpha f_\beta + l_3)\, d\sigma + \int_L (\varepsilon_{3\alpha\beta} x_\alpha p_\beta + q_3)\, ds = 0. \end{gathered} \tag{3.5}$$

We denote by $v_i^{(s)}$, $\varphi_i^{(s)}$ the solutions of four plane strain problems $P^{(s)}$ $(s = 1, 2, 3, 4)$ corresponding to the following systems of loadings:

$$\begin{aligned} f_i^{(\beta)} &= (\varepsilon_{3\alpha\beta}A_{\nu i33}x_\alpha - B_{\nu i3\beta})_{,\nu}, \\ f_i^{(3)} &= (\varepsilon_{\alpha\beta3}A_{\nu i3\alpha}x_\beta - B_{\nu i33})_{,\nu}, \; f_i^{(4)} = A_{\alpha i33,\alpha}, \\ l_i^{(\beta)} &= (\varepsilon_{3\alpha\beta}B_{33\nu i}x_\alpha - C_{\nu i3\beta})_{,\nu} + \varepsilon_{ijk}(\varepsilon_{3\alpha\beta}A_{jk33}x_\alpha - B_{jk33}), \\ l_i^{(3)} &= (\varepsilon_{\alpha\beta3}B_{3\alpha\nu i}x_\beta - C_{\nu i33})_{,\nu} + \varepsilon_{ijk}(\varepsilon_{\alpha\beta3}A_{jk3\alpha}x_\beta - B_{jk33}), \\ l_i^{(4)} &= B_{33\alpha i,\alpha} + \varepsilon_{ijk}A_{jk33}, \\ p_i^{(\beta)} &= -(\varepsilon_{3\alpha\beta}A_{\nu i33}x_\alpha - B_{\nu i3\beta})\, n_\nu, \quad q_i^{(\beta)} = -(\varepsilon_{3\nu\beta}B_{33\alpha i}x_\nu - C_{\alpha i3\beta})\, n_\alpha, \\ p_i^{(3)} &= -(\varepsilon_{\alpha\beta3}A_{\nu i3\alpha}x_\beta - B_{\nu i33})\, n_\nu, \quad q_i^{(3)} = -(\varepsilon_{\nu\beta3}B_{3\nu\alpha i}x_\beta - C_{\alpha i33})\, n_\alpha, \\ p_i^{(4)} &= -A_{\alpha i33}n_\alpha, \qquad q_i^{(4)} = -B_{33\alpha i}n_\alpha. \end{aligned} \tag{3.6}$$

It is easy to verify that there are satisfied conditions (3.5) for existence of solutions of the problems $P^{(s)}$ $(s = 1, 2, 3, 4)$. In what follows we assume that the functions $v_i^{(s)}$ and $\varphi_i^{(s)}$ $(s = 1, 2, 3, 4)$ are known.

4. Uniformly Loaded Cylinders. In this section we solve the problem $B^{(0)}$. We assume that the body forces, body couples and the surface loadings have the form

$$f_i = G_i(x_1, x_2), \qquad l_i = L_i(x_1, x_2), \qquad \widetilde{t_i} = g_i(x_1, x_2), \qquad \widetilde{m_i} = h_i(x_1, x_2), \tag{4.1}$$

where G_i, L_i, g_i and h_i are prescribed functions.

We look for a solution in the form:

$$\begin{aligned} u_\alpha &= \varepsilon_{\alpha\beta3}\left(-\frac{1}{2}a_\beta x_3^2 + a_3 x_\beta x_3 - \frac{1}{6}b_\beta x_3^3 + \frac{1}{2}b_3 x_\beta x_3^2 - \frac{1}{24}c_\beta x_3^4 + \right.\\ &\left.+\frac{1}{6}c_3 x_\beta x_3^3\right) + \sum_{s=1}^{4}\left(a_s + b_s x_3 + \frac{1}{2}c_s x_3^2\right)v_\alpha^{(s)} + x_3 v_\alpha(x_1, x_2) + w_\alpha(x_1 x_2),\\ u_3 &= \varepsilon_{3\alpha\beta}\left(a_\beta x_\alpha x_3 + \frac{1}{2}b_\beta x_\alpha x_3^2 + \frac{1}{6}c_\beta x_\alpha x_3^3\right) + \left(a_4 x_3 + \frac{1}{2}b_4 x_3^2 + \right.\\ &\left.+\frac{1}{6}c_4 x_3^3\right) + \sum_{s=1}^{4}\left(a_s + b_s x_3 + \frac{1}{2}c_s x_3^2\right)v_3^{(s)} + x_3 v_3(x_1, x_2) + w_3(x_1, x_2),\\ \varphi_i &= -\left(a_i x_3 + \frac{1}{2}b_i x_3^2 + \frac{1}{6}c_i x_3^3\right) + \sum_{s=1}^{4}\left(a_s + b_s x_3 + \frac{1}{2}c_s x_3^2\,\varphi_i^{(s)}\right) + \\ &+ x_3\psi_i(x_1, x_2) + \varkappa_i(x_1, x_2), \end{aligned} \tag{4.2}$$

where $v_i^{(s)}$, $\varphi_i^{(s)}$ are the displacement and microrotation components from the auxiliary plane strain problems considered in section 3, v_i, w_i, ψ_i, $\varkappa_i$ are unknown functions and a_s, b_s, c_s $(s = 1, 2, 3, 4)$ are unknown constants.

Using (2.2), (2.3), from (4.2) we obtain:

$$\begin{aligned} t_{ij} &= \sum_{s=1}^{4}\left(a_s + b_s x_3 + \frac{1}{2}c_s x_3^2\right)\sigma_{ij}^{(s)} + S_{ij}x_3 + N_{ij} + U_{ij}^{(0)} + U_{ij}^{(1)}x_3\\ m_{ij} &= \sum_{s=1}^{4}\left(a_s + b_s x_3 + \frac{1}{2}c_s x_3^2\right)\mu_{ij}^{(s)} + P_{ij}x_3 + R_{ij} + W_{ij}^{(0)} + W_{ij}^{(1)}x_3 \end{aligned} \tag{4.3}$$

where:

$$\sigma_{ij}^{(\beta)} = t_{ij}^{(\beta)} + \varepsilon_{3\alpha\beta}A_{ij33}x_\alpha - B_{ij3\beta}; \quad \sigma_{ij}^{(3)} = \varepsilon_{\alpha\beta3}A_{ij3\alpha}x_\beta - B_{ij33} + t_{ij}^{(3)};$$
$$\sigma_{ij}^{(4)} = t_{ij}^{(4)} + A_{ij33},$$

$$\begin{aligned} \mu_{ij}^{(\beta)} &= m_{ij}^{(\beta)} + \varepsilon_{3\alpha\beta}B_{33ij}x_\alpha - C_{ij3\beta}; \quad \mu_{ij}^{(3)} = \varepsilon_{\alpha\beta3}B_{3\alpha ij}x_\beta - C_{ij33} + m_{ij}^{(3)};\\ \mu_{ij}^{(4)} &= m_{ij}^{(4)} + B_{33ij}, \end{aligned} \tag{4.4}$$

$t_{ij}^{(s)}$, $m_{ij}^{(s)}$ $(s = 1, 2, 3, 4)$ being respectively the components of the stress tensor and couple-stress tensor of the problem $P^{(s)}$, and

$$\begin{aligned} S_{kl} &= A_{kl\,ji}\eta_{ji} + B_{klji}\rho_{ji}, \quad P_{kl} = B_{jikl}\eta_{ji} + C_{klji}\rho_{ji},\\ \eta_{ji} &= v_{i,j} + \varepsilon_{ijk}\psi_k, \quad \rho_{ji} = \psi_{i,j}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} N_{kl} &= A_{klji}\theta_{ji} + B_{klji}\mu_{ji}, \quad R_{kl} = B_{jikl}\theta_{ji} + C_{klji}\mu_{ji},\\ \theta_{ji} &= w_{i,j} + \varepsilon_{ijk}\varkappa_k, \quad \mu_{ji} = \varkappa_{i,j}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} U_{kl}^{(0)} &= \sum_{s=1}^{4} b_s\left(A_{kl3i}v_i^{(s)} + B_{kl3i}\psi_i^{(s)}\right) + \left(A_{kl3i}v_i + B_{kl3i}\psi_i\right),\\ U_{kl}^{(s)} &= \sum_{s=1}^{4} c_s\left(A_{kl3i}v_i^{(s)} + B_{kl3i}\psi_i^{(s)}\right),\\ W_{kl}^{(0)} &= \sum_{s=1}^{4} b_s\left(B_{jikl}v_i^{(s)} + C_{kl3i}\psi_i^{(s)}\right) + \left(B_{3ikl}v_i + C_{kl3i}\psi_i\right),\\ W_{kl}^{(1)} &= \sum_{s=1}^{4} c_s\left(B_{3ikl}v_i^{(s)} + C_{kl3i}\psi_i^{(s)}\right). \end{aligned} \tag{4.7}$$

Taking into account the equations satisfied by the functions $\sigma_{ij}^{(s)}$, $\mu_{ij}^{(s)}$ $(s = 1, 2, 3, 4)$, the equilibrium equations reduce to

$$\begin{aligned} &U_{\nu j,\nu}^{(0)} + N_{\nu j,\nu} + \sum_{s=1}^{4} b_s\sigma_{3j}^{(s)} + S_{3j} + U_{3j}^{(1)} + G_j = 0,\\ &W_{\nu j,\nu}^{(0)} + \varepsilon_{jkr}U_{kr}^{(0)} + R_{\nu j,\nu} + \varepsilon_{jkr}N_{kr} + \sum_{s=1}^{4} b_s\mu_{3j}^{(s)} + \\ &\qquad + P_{3j} + W_{3j}^{(1)} + L_j = 0, \end{aligned} \tag{4.8}$$

$$\begin{aligned} &U_{\nu j,\nu}^{(1)} + S_{\nu j,\nu} + \sum_{s=1}^{4} c_s\sigma_{3j}^{(s)} = 0,\\ &W_{\nu j,\nu}^{(1)} + \varepsilon_{jkr}U_{kr}^{(1)} + P_{\nu j,\nu}\varepsilon_{jkr} + S_{kr} + \sum_{s=1}^{4} c_s\mu_{3j}^{(s)} = 0. \end{aligned} \tag{4.9}$$

Similarly, the conditions of the lateral surface B become

$$U_{\nu j}^{(0)}n_\nu + N_{\nu j}n_\nu = g_j, \quad W_{\nu j}^{(0)}n_\nu + R_{\nu j}n_\nu = h_j, \quad \text{on } L, \tag{4.10}$$

$$U_{\nu j}^{(1)}n_\nu + S_{\nu j}n_\nu = 0, \quad W_{\nu j}^{(1)}n_\nu + P_{\nu j}n_\nu = 0, \quad \text{on } L. \tag{4.11}$$

Let us consider the generalized plane strain problem (4.9), (4.5) and boundary conditions (4.11). The necessary and sufficient conditions for existence of solution reduce to:

$$\sum_{s=1}^{n} c_s \int_\Sigma \sigma_{3j}^{(s)}\,d\sigma = 0 \tag{4.12}$$

$$\sum_{s=1}^{n} c_s \int_\Sigma \left[\varepsilon_{3\alpha\beta}x_\alpha\,\sigma_{3\beta}^{(s)} + \mu_{33}^{(s)}\right]d\sigma = 0. \tag{4.13}$$

Analogously as in classical theory, [3] we can prove that

$$\int_\Sigma \sigma_{3\alpha}^{(s)}\, d\sigma = 0, \quad (s = 1, 2, 3, 4). \tag{4.14}$$

From (4.12), (4.13) and (4.14) we obtain :

$$\sum_{s=1}^{n} D_{3s} c_s = 0, \tag{4.12'}$$

$$\sum_{s=1}^{n} D_{4s} c_s = 0, \tag{4.13'}$$

where

$$D_{3s} = \int_\Sigma \sigma_{33}^{(s)}\, d\sigma , \tag{4.15}$$

$$D_{4s} = \int_\Sigma [\varepsilon_{3\alpha\beta} x_\alpha \sigma_{3\beta}^{(s)} + \mu_{33}^{(s)}]\, d\sigma . \tag{4.16}$$

For the generalized plane strain problem (4.8), (4.6) and (4.10) the necessary and sufficient conditions for existence of solution become

$$\int_\Sigma \left[\sum_{s=1}^{4} b_s \sigma_{3j}^{(s)} + S_{3j} + U_{3j}^{(1)}\right] d\sigma + \int_\Sigma G_j d\sigma + \int_L g_j ds = 0, \tag{4.17}$$

$$\begin{aligned} &\int_\Sigma \Big[\varepsilon_{3\alpha\beta} x_\alpha \big(U_{\nu\beta,\nu}^{(0)} + \sum_{s=1}^{4} b_s \sigma_{3\beta}^{(s)} + S_{3\beta} + U_{3\beta}^{(1)} + G_\beta\big) + \big(W_{3\nu,\nu}^{(0)} + \\ &+ \varepsilon_{3kr} U_{kr}^{(0)}\big) + \sum_{s=1}^{4} b_s \mu_{33}^{(s)} + P_{33} + W_{33}^{(1)} + L_3\Big] d\sigma + \int_L [\varepsilon_{3\alpha\beta} x_\alpha (g_\beta - \\ &- U_{\nu\beta}^{(0)} n_\nu) + (h_3 - W_{\nu 3}^{(0)} n_\nu)]\, ds = 0. \end{aligned} \tag{4.18}$$

Using the equilibrium equations, the divergence theorem and (4.8) we can write

$$\begin{aligned} &\int_\Sigma \left[\sum_{s=1}^{4} b_s \sigma_{3\alpha}^{(s)} + S_{3\alpha} + U_{3\alpha}^{(1)}\right] d\sigma = \int_\Sigma (t_{3\alpha,3})_{x_3=0}\, d\sigma = \int_\Sigma x_\alpha t_{33,33}\, d\sigma - \\ &- \varepsilon_{3\alpha\beta} \int_\Sigma m_{i\beta,i3}\, d\sigma = \int_\Sigma x_\alpha t_{33,33}\, d\sigma - \varepsilon_{3\alpha\beta} \sum_{s=1}^{4} c_s \int_\Sigma \mu_{3\beta}^{(s)} d\sigma = \\ &= \sum_{s=1}^{4} c_s \int_\Sigma (x_\alpha \sigma_{33}^{(s)} - \varepsilon_{3\alpha\beta} \mu_{3\beta}^{(s)})\, d\sigma. \end{aligned} \tag{4.19}$$

Thus the first two conditions from (4.17) reduce to

$$\sum_{s=1}^{4} D_{\alpha s} c_s = -\int_\Sigma G_\alpha d\sigma - \int_L g_\alpha ds. \tag{4.20}$$

where

$$D_{\alpha s} = \int_\Sigma (x_\alpha \sigma_{33}^{(s)} - \varepsilon_{3\alpha\beta} \mu_{3\beta}^{(s)})\, d\sigma. \tag{4.21}$$

In [4], it was proved that $\det D_{ij} > 0$ so that the system (4.12), (4.13′) and (4.20) uniquely determines the constants c_s $(s = 1, 2, 3, 4)$. Thus the conditions (3.5) are satisfied and in what follows we assume that v_i, ψ_i, η_{ij}, ρ_{ij}, S_{ij}, P_{ij}, $U_{ij}^{(0)}$, $U_{ij}^{(1)}$, $W_{ij}^{(0)}$ and $W_{ij}^{(1)}$ are known.

The remaining condition from (4.17) leads to

$$\sum_{s=1}^{4} D_{3s} b_s = -\int_\Sigma G_3\, d\sigma - \int_L g_3 ds - \int_\Sigma (S_{33} + U_{33}^{(1)})\, d\sigma . \tag{4.22}$$

Taking into account that

$$\begin{aligned} \int_\Sigma \varepsilon_{3\alpha\beta} x_\alpha U_{\nu\beta,\nu}^{(0)}\, d\sigma &= \int_\Sigma \varepsilon_{3\alpha\beta} [(x_\alpha U_{\nu\beta}^{(0)})_{,\nu} - \delta_{\alpha\nu} U_{\nu\beta}^{(0)}]\, d\sigma = \\ &= \int_L \varepsilon_{3\alpha\beta} x_\alpha U_{\nu\beta}^{(0)} n_\nu ds - \int_\Sigma \varepsilon_{3\nu\beta} U_{\nu\beta}^{(0)} d\sigma , \end{aligned} \tag{4.23}$$

(4.18) reduces to

$$\begin{aligned} \sum_{s=1}^{4} D_{4s} b_s = &- \int_\Sigma \varepsilon_{3\alpha\beta} x_\alpha (S_{3\beta} + U_{3\beta}^{(1)} + G_\beta)\, d\sigma - \int_\Sigma (W_{\nu 3,\nu}^{(0)} + \\ &+ P_{33} + W_{33}^{(1)} + L_3)\, d\sigma - \int_L [\varepsilon_{3\alpha\beta} x_\alpha (g_\beta - U_{\nu\beta}^{(0)} n_\nu) + \\ &+ (h_3 - W_{\nu 3}^{(1)} n_\nu)]\, ds. \end{aligned} \tag{4.24}$$

Thus the conditions (3.5) are satisfied and in what follows we assume that w_i, $\varkappa_i$, θ_{ij}, μ_{ij}, N_{ij} and R_{ij} are known functions.

It remains to satisfy conditions (2.6) – (2.9). Using the equilibrium equation and divergence theorem, we obtain :

$$\begin{aligned} \int_\Sigma t_{31}\, d\sigma &= \int_\Sigma (t_{13} - m_{j2,j} + l_2)\, d\sigma = \int_\Sigma [t_{13} + x_1 (t_{i3,i} + G_3) - \\ &- m_{j2,j} + l_2]\, d\sigma = \int_\Sigma [(x_1 t_{\beta 3})_{,\beta} + (x_1 t_{33})_{,3}]\, d\sigma - \int_\Sigma m_{j2,j} d\sigma + \\ &+ \int_\Sigma (x_1 G_3 + l_2)\, d\sigma = \int_L (x_1 g_3 + h_2)\, ds + \int_\Sigma (x_1 t_{33} - m_{32})_{,3}\, d\sigma + \\ &+ \int_\Sigma (x_1 G_3 + l_2)\, d\sigma. \end{aligned} \tag{4.25}$$

Analogously :

$$\begin{aligned} \int_\Sigma t_{32} d\sigma &= \int_\Sigma (t_{23} + m_{i1,i} + l_1)\, d\sigma = \int_\Sigma [t_{23} + x_2 (t_{i3,i} + G_3) + \\ &+ m_{i1,i} + l_1]\, d\sigma = \int_\Sigma [(x_2 t_{\beta 3})_{,\beta} + (x_2 t_{33})_{,3}]\, d\sigma + \end{aligned} \tag{4.26}$$

$$(4.26)\qquad \begin{aligned} &+\int_{\Sigma}(x_2G_3+l_1)\,d\sigma+\int_{\Sigma}m_{41,4}\,d\sigma = (x_2g_3+h_1)\,ds+\\ &+\int_{\Sigma}(x_2t_{33}+m_{31})_{,3}\,d\sigma+\int_{\Sigma}(x_2G_3+l_1)\,d\sigma \end{aligned}$$

From (4.3), (2.6), (4.21), (4.25) and (4.26), we obtain :

$$(4.27)\qquad \begin{aligned} \sum_{s=1}^{4}D_{1s}b_s &= -R_1-\int_{L}(x_1g_3-h_2)\,ds-\int_{\Sigma}(x_1G_3+l_2)\,d\sigma-\\ &-\int_{\Sigma}[x_1(S_{33}+U_{33}^{(1)})-(P_{32}+W_{32}^{(1)})]\,d\sigma, \end{aligned}$$

$$(4.28)\qquad \begin{aligned} \sum_{s=1}^{4}D_{2s}b_s &= -R_2-\int_{L}(x_2g_3+h_1)\,ds-\int_{\Sigma}(x_2G_3+l_1)\,d\sigma-\\ &-\int_{\Sigma}[x_2(S_{33}+U_{33}^{(1)})-(P_{31}+W_{31}^{(1)})]\,d\sigma. \end{aligned}$$

The system (4.22), (4.24), (4.27) and (4.28) uniquely determines the constants $b_s\,(s=1,2,3,4)$.

From (2.7), (2.8), (2.9) and (4.3) we have

$$(4.29)\qquad \sum_{s=1}^{4}D_{3s}a_s = -R_3-\int_{\Sigma}(N_{33}+U_{33}^{(0)})\,d\sigma,$$

$$(4.30)\qquad \sum_{s=1}^{4}D_{\alpha s}a_s = \varepsilon_{3\alpha\beta}M_\beta-\int_{\Sigma}[x_\alpha(N_{33}+U_{33}^{(0)})-\varepsilon_{3\alpha\beta}(R_{3\beta}+W_{3\beta}^{(0)})]\,d\sigma,$$

$$(4.31)\qquad \sum_{s=1}^{4}D_{4s}a_s = -M_3-\int_{\Sigma}[\varepsilon_{3\alpha\beta}x_\alpha(N_{3\beta}+U_{3\beta}^{(0)})+R_{33}+W_{33}^{(0)}]\,d\sigma.$$

Thus $a_s\,(s=1,2,3,4)$ are also determined from the system (4.29), (4.30), (4.31) and $B^{(0)}$ problem is solved.

5. Recurrence Process. Let us establish the solution of the problem $B^{(n+1)}$ assuming that the solution of the problem $B^{(n)}$ (in which $R_i=M_i=0$) is known. As the solution of the problem $B^{(n)}$ is known for any body forces, body couples and any loadings on the lateral surface, we can know the solution u_i^{*}, φ_i^{*} of the following problem :

$$(5.1)\qquad \begin{aligned} &t_{ji,j}+F_{i(n+1)}(x_1,x_2)\,x_3^{n}=0,\\ &m_{ji,j}+\varepsilon_{ijk}t_{jk}+L_{i(n+1)}(x_1,x_2)\,x_3^{n}=0, \end{aligned}$$

$$(5.2)\qquad t_{ij}=A_{ijkl}e_{kl}+B_{ijkl}\varkappa_{kl},\qquad m_{ij}=B_{klij}e_{kl}+C_{ijkl}\varkappa_{kl},$$

$$(5.3)\qquad e_{ij}=u_{j,i}+\varepsilon_{jik}\varphi_k,\qquad \varkappa_{ij}=\varphi_{j,i},$$

with boundary conditions on the lateral surface

$$(5.4)\qquad t_{ji}n_j=P_{i(n+1)}(x_1,x_2)\,x_3^{n},\qquad m_{ji}n_j=M_{i(n+1)}(x_1,x_2)\,x_3^{n}$$

and with the conditions on the $x_3=0$ end of the cylinder

$$(5.5)\qquad \int_{\Sigma}t_{i3}\,d\sigma=0;\quad \int_{\Sigma}(x_\alpha t_{33}-\varepsilon_{3\alpha\beta}m_{3\beta})\,d\sigma=0;\quad \int_{\Sigma}(\varepsilon_{3\alpha\beta}t_{3\beta}x_\alpha+m_{33})\,d\sigma=0.$$

Let u_i, φ_i be the solution of the $B^{(n+1)}$ problem

$$(5.6)\qquad t_{ji,j}+F_{i(n+1)}(x_1,x_2)\,x_3^{n+1}=0,\quad m_{ji,j}+\varepsilon_{ijk}t_{jk}+L_{i(n+1)}(x_1,x_2)\,x_3^{n+1}=0,$$

with the constitutive equations (5.2), geometricaly equations (5.3) and with boundary conditions on the lateral surface of the cylinder

$$(5.7)\qquad t_{ji}n_j=P_{i(n+1)}(x_1,x_2)\,x_3^{n+1},\qquad m_{ji}n_j=M_{i(n+1)}(x_1,x_2)\,x_3^{n+1}.$$

On the $x_3=0$ end of the cylinder we have

$$(5.8)\qquad \begin{aligned} &\int_{\Sigma}t_{\alpha3}\,d\sigma=0,\quad \int_{\Sigma}t_{33}\,d\sigma=0,\quad \int_{\Sigma}(x_\alpha t_{33}-\varepsilon_{3\alpha\beta}m_{3\beta})\,d\sigma=0,\\ &\int_{\Sigma}(\varepsilon_{3\alpha\beta}t_{3\beta}x_\alpha+m_{33})\,d\sigma=0. \end{aligned}$$

We look for a solution for $B^{(n+1)}$ problem in the form

$$(5.9)\qquad u_i=(n+1)\left[\int_0^{x_3}u_i^{*}\,dx_3+v_i\right],\quad \psi_i=(n+1)\left[\int_0^{x_3}\varphi_i^{*}\,dx_3+\psi_i\right].$$

v_i, ψ_i being unknown functions.

The components of the stress and couple-stress tensors are

$$(5.10)\qquad \begin{aligned} t_{ij}&=(n+1)\left[\int_0^{x_3}t_{ij}^{*}\,dx_3+\tau_{ij}+\theta_{ij}\right],\\ m_{ij}&=(n+1)\left[\int_0^{x_3}m_{ij}^{*}\,dx_3+\nu_{ij}+\varkappa_{ij}\right] \end{aligned}$$

where

$$(5.11)\qquad \begin{aligned} &\tau_{ij}=A_{ijkl}\eta_{kl}+B_{ijkl}\mu_{kl};\qquad \nu_{ij}=B_{klij}\eta_{kl}+C_{ijkl}\mu_{kl};\\ &\eta_{ij}=v_{j,i}+\varepsilon_{jik}\psi_k;\qquad \mu_{ij}=\psi_{j,i};\\ &\theta_{ij}=A_{ij3l}u_l^{*}(x_1,x_2,0)+B_{ij3l}\varphi_l^{*}(x_1,x_2,0),\\ &\varkappa_{ij}=B_{3lij}u_l^{*}(x_1,x_2,0)+C_{ij3l}\varphi_l^{*}(x_1,x_2,0). \end{aligned}$$

By using (5.10), the equilibrium equations reduce to :

$$(5.12)\qquad \begin{aligned} &\tau_{ji,j}+(\theta_{ji,j}+t_{3i}^{*}(x_1,x_2,0))=0,\\ &\nu_{ji,j}+\varepsilon_{ijk}\tau_{jk}+(\varkappa_{ji,j}+\varepsilon_{ijk}\theta_{jk}+m_{3i}^{*}(x_1,x_2,0))=0, \end{aligned}\qquad \text{on } \Sigma,$$

and the conditions on B become:

$$(5.13)\qquad \tau_{ji}n_j=-\theta_{ji}n_j,\qquad \nu_{ji}n_j=-\varkappa_{ji}n_j,\quad \text{on } L$$

On the end of the cylinder lieng in the plane $x_3 = 0$ we have the conditions

$$\int_\Sigma \tau_{3\alpha} d\sigma = -\frac{1}{n+1}\int_\Sigma \theta_{3\alpha} d\sigma, \tag{5.14}$$

$$\int_\Sigma \tau_{33} d\sigma = -\frac{1}{n+1}\int_\Sigma \theta_{33}\, d\sigma \tag{5.15}$$

$$\int_\Sigma (x_\alpha \tau_{33} - \varepsilon_{3\alpha\beta}\nu_{3\beta})\, d\sigma = \frac{-1}{n+1}\int_\Sigma (x_\alpha \theta_{33} - \varepsilon_{3\alpha\beta}\varkappa_{3\beta})\, d\sigma \tag{5.16}$$

$$\int_\Sigma (\varepsilon_{3\alpha\beta} x_\alpha \tau_{3\beta} + \nu_{33})\, d\sigma = -\frac{1}{n+1}\int_\Sigma (\varepsilon_{3\alpha\beta} x_\alpha \theta_{3\beta} + \varkappa_{33})\, d\sigma \tag{5.17}$$

The body forces, the body couples, the loadings on the lateral surface of the cylinder and the loadings on the end of the cylinder of the problem (5.11) — (5.17) depend only on x_1 and x_2. As in sect. 4 a such problem was solved, the considered problem is entirely solved.

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THE EQUATIONS OF RADIATION FIELD. INVARIANCE GROUPS

BY

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1. Introduction. In [1] Pomraning shows that under certain rather unrestrictive conditions, the thermal radiation in a medium can be characterized by a function, named *specific intensity*, which depends on time and point, but also on the frecvency and the direction of propagation of the radiation, and which satisfies an integro-differential equation, given in Section 2. In hydrodynamic problems of interest, there are three integral quantities which characterize the radiation field at a given time and point. These correspond to the first three angular moments of the specific intensity. In Section 3, we write the equations these quantities satisfy, equations that are used in radiative magnetohydrodynamics [2], [3]. Here, however, they appear as a separate system, uncoupled with hydrodynamics equations. In the present paper, we show how the coupling appears, finding out the effect of the medium motion on the radiation field. In Section 4, we introduce the parameters which are used to characterize the interaction between the radiation fields and medium and we establish the constitutive equations. Since the angular dependence of the specific intensity is generally very complicated, we give the simplest approximate descriptions of radiative transfer, in Section 5. The quadridimensional form of the radiation field equations is deduced in Section 6, and we analyse the radiation field dynamics for the media in motion, in Section 7. Finally, in Section 8, the invariance group of the radiation field equations for a medium with homogeneous absorbtion is derived and some of its properties are given.

In this paper the indices $i, j, k = 1, 2, 3$ and $\alpha, \beta, \gamma, \delta = 0, 1, 2, 3$. The summation convention will always be used unless the contrary is stated explicitly.

2. The equation of radiative transfer. The *equation of transfer* or so-called *transport equation*, is the mathematical statement of the conservation of photons.

At any time t, six variables are required to specify the position of the photon in the phase space, namely three position variables, which we denote by the vector $\mathbf{r}(x_1, x_2, x_3)$, and three momentum variables. Instead of the latter ones we use frecvency ν and the direction of travel of the photon Ω (it requires two angular variables to specify Ω). The equation of radiative transfer has then the form [1]: