The set $\mathcal{A}$ is *universal* if there is an universal algorithm for $\mathcal{A}$ contained in $\mathcal{A}$.

Corollary 10. *If $\mathcal{A}$ is universal and $\mathcal{A} \approx \mathcal{B}$ then $\mathcal{B}$ is universal.*

Proof. Suppose U_1, U_2 be basic universal for $\mathcal{A}$, $\mathcal{B}$, and let V_1 be universal for $\mathcal{A}$ and $V_1 \in \mathcal{A}$. We have $V_1 \geqslant U_1$ and since $\mathcal{A} \geqslant \mathcal{B}$ by theorem 8 it follows $U_1 \geqslant U_2$. Therefore $V_1 \geqslant U_2$. On the other hand, since $\mathcal{B} \geqslant \mathcal{A}$ and $V_1 \in \mathcal{A}$ it must exist $V_2 \in \mathcal{B}$ so that $V_2 \geqslant V_1$. So, $V_2 \geqslant U_2$ and hence V_2 is universal for $\mathcal{B}$. Q.E.D.

Using some known constructions one can show that $\mathcal{T}$ is universal. Hence, $\mathcal{O}$, $\mathcal{R}$, $\mathcal{M}$ and all sets similar with $\mathcal{T}$ are universal too.

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Faculty of Mathematics
University of Iași
Iași, R. S. Romania

Analele științifice ale Universității „Al. I. Cuza" Iași
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ON SET FUNCTIONS OF BOUNDED p-VARIATION

BY

VASILE POSTOLICĂ

The concept of p-variation of a real function defined on a real compact interval was introduced by F. Riesz and it has been used to characterize the primitives of the p-absolutely integrable functions ([6], p. 75). Thus, there is a strong connection between the space of functions of bounded p-variation and the space of almost everywhere derivable functions with p-absolutely integrable derivative, space of functions with applications in the theory of distributions [1].

On the other hand, the p-variation leads naturally to the p-absolute continuity; but as shown by the investigations of C. Upton [8] and N. Gheorghiu, T. Precupanu [2], for $p > 1$ these functions coincide with the functions of bounded p-variation.

In this paper we extend the notion of p-variation to set functions, studying in the same time the connection between the set functions of bounded p-variation and the p-absolutely continuous set functions. To this end, we establish an integral representation of the set functions of bounded p-variation [Theorem 1.6].

1. Set function of bounded p-variation. Let $\mathcal{F}$ be a σ-algebra over a space X, μ a finite measure defined on $\mathcal{F}$ and $F : \mathcal{F} \to \overline{R}$ a set function with $F(\varnothing) = 0$. We consider a non-void set $A \in \mathcal{F}$ and we denote the σ-algebra induced by $\mathcal{F}$ on A by $\mathcal{F}_A$. Throughout the paper we make the following calculation conventions:

$$0\,\infty = \infty\,0 = 0, \quad 0^0 = 1, \quad \frac{a}{0} = \infty \text{ for every } a > 0. \tag{1.1}$$

Definition 1.1. *We shall call $\mathcal{F}$-partition of the set A any finite family $(B_i)_{i=\overline{1,m}}$ of disjoint sets $B_i \in \mathcal{F}$ which covers A i. e., $\bigcup_{i=1}^{m} B_i = A$.*

For $p \geqslant 1$ we define the *p-variation of F on A with respect to a $\mathcal{F}$-partition* $\Delta = (A_i)_{i=\overline{1,n}}$ by

$$\sigma_p(F, \Delta) = \left\{ \sum_{i=1}^{n} \frac{|F(A_i)|^p}{[\mu(A_i)]^{p-1}} \right\}^{1/p} \tag{1.2}$$

and we denote

$$\sup_{\Delta} \sigma_p(F, \Delta) = V_p(F, A). \tag{1.3}$$

Definition 1.2. *We say that the function F is of bounded p-variation on the set A with respect to μ, if there exists $K \geqslant 0$, such that*

$$\sigma_p(F, \Delta) \leqslant K, \tag{1.4}$$

for every $\mathcal{F}$-partition Δ of A.

We denote by $BV_p(A)$ the class of functions of bounded p-variation on the set A with respect to μ.

Theorem 1.1. *If $F \in BV_p(A)$, then there exists $M \geqslant 0$ such that*

$$|F(B)| \leqslant M, \quad \forall B \in \mathcal{F}_A. \tag{1.5}$$

Proof. It is sufficient to observe that for every $B \in \mathcal{F}_A$ the family $\{B, A \setminus B\}$ represents a $\mathcal{F}$-partition of A.

Theorem 1.2. *If $F \in BV_p(A)$, then $F \in BV_p(B)$ for every non-void set $B \in \mathcal{F}_A$.*

Proof. Let $B \in \mathcal{F}_A$ be a non-void set and let $(B_i)_{i=\overline{1,n}}$ be an arbitrary $\mathcal{F}$-partition of B. Then $(B_i)_{i=\overline{1,n+1}}$, where $B_{n+1} = A \setminus B$, is a $\mathcal{F}$-partition of A. The result follows.

Theorem 1.3. *If F is countable additive and if $A = \bigcup_{i=1}^{\infty} A_i$, where $(A_i)_{i \in N} \subset \mathcal{F}_A$ is a sequence of non-void mutually disjoint sets, then $F \in BV_p(A)$ if and only if*

$$\sum_{i=1}^{\infty} V_p^p(F, A_i) < +\infty. \tag{1.6}$$

Proof. Let $(A_i^j)_{j \in I_i}$ be a $\mathcal{F}$-partition of the set A_i, for each $i \in N$. Then $((A_i^j)_{i=\overline{1,n},\, j \in I_i}, A \setminus A_0)$, where $A_0 = \bigcup_{i=1}^{n} \bigcup_{j \in I_i} A_i^j$, is a $\mathcal{F}$-partition of A for every $n \in N$.

Hence, if $F \in BV_p(A)$ we have

$$\sum_{j \in I_1} \frac{|F(A_1^j)|^p}{[\mu(A_1^j)]^{p-1}} + \sum_{j \in I_2} \frac{|F(A_2^j)|^p}{[\mu(A_2^j)]^{p-1}} + \dots + \sum_{j \in I_n} \frac{|F(A_n^j)|^p}{[\mu(A_n^j)]^{p-1}} \leqslant \tag{1.7}$$

$$\leqslant V_p^p(F, A), \ \forall n \in N,$$

which implies

$$\sum_{i=1}^{\infty} V_p^p(F, A_i) \leqslant V_p^p(F, A). \tag{1.8}$$

Before proving the converse, we note the following inequality:

$$\left(\sum_{i=1}^{n} a_i\right)^p \left(\sum_{i=1}^{n} b_i\right)^{1-p} \leqslant \sum_{i=1}^{n} a_i^p b_i^{1-p}, \tag{1.9}$$

whenever $a_i, b_i > 0$ and $p \geqslant 1$ [see [4], p. 61].

Suppose now that F satisfies (1.6) and consider an arbitrary $\mathcal{F}$-partition $(A_i^*)_{\overline{i=1,m}}$ of A. Taking into account (1.9) we obtain

$$\sum_{i=1}^{m} \frac{|F(A_1^*)|^p}{[\mu(A_i^*)]^{p-1}} = \sum_{i=1}^{m} \frac{\left|F\left(\bigcup_{h=1}^{\infty}(A_i^* \cap A_k)\right)\right|^p}{\left|\mu\left(\bigcup_{k=1}^{\infty}(A_i^* \cap A_k)\right)\right|^{p-1}} \leqslant \sum_{i=1}^{m} \frac{\left[\sum_{k=1}^{\infty} |F(A_i^* \cap A_k)|\right]^p}{\left[\sum_{k=1}^{\infty} \mu(A_i^* \cap A_k)\right]^{p-1}} =$$

$$= \sum_{k=1}^{\infty} \sum_{i=1}^{m} \frac{|(A_i^* \cap A_k)|^p}{[\mu(FA_i^* \cap A_k)]^{p-1}} \leqslant \sum_{k=1}^{\infty} V_p^p(F, A_k). \tag{1.10}$$

Remark. From the above considerations it follows that if $A = \bigcup_{i=1}^{\infty} A_i$, where $A_i \in \mathcal{F}_A$ are non-void mutually disjoint sets, then

$$\sum_{i=1}^{\infty} V_p^p(F, A_i) = V_p^p(F, A). \tag{1.11}$$

Theorem 1.4. *A countable additive function $F: \mathcal{F} \to \bar{R}$ belongs to $BV_p(A)$ if and only if it may be represented as the difference between two positive monotonous of the same sense set functions of bounded p-variation.*

Proof. From Jordan's theorem ([5], p. 193) it follows that $F = F^+ - F^-$, where

$$F^+(M) = \sup\{F(E);\ E \in \mathcal{F} \text{ and } E \subset M\} \tag{1.12}$$

and

$$F^-(M) = -\inf\{F(E);\ E \in \mathcal{F} \text{ and } E \subset M\}, \tag{1.13}$$

for every $M \in \mathcal{F}$. We shall now prove that $F \in BV_p(A)$ implies F^+, $F^- \in BV_p(A)$.

Indeed, by hypothesis, there exists $M \geqslant 0$ such that

$$\left\{\sum_{i=1}^{n} \frac{|F(A_i)|^p}{\mu(A_i)]^{p-1}}\right\}^{1/p} \leqslant M, \tag{1.14}$$

for any $\mathcal{F}$-partition $(A_i)_{i=\overline{1,n}}$ of A. We can assume without loss of generality that $F^+(A_i) > 0$, for all $i = \overline{1, n}$.

From (1.12) it follows that there exists a set $E_i \subset A_i$ such that

$$F^+(A_i) < 2F(E_i), \quad \forall i = \overline{1, n}.$$

We now remark that $\mu(A_i) > 0$, because, in the contrary case, $A_i \supset E_i$ implies $\mu(E_i) = 0$ and taking into account (1.14), which remains valid if we replace each A_i by the corresponding set E_i, it follows $F(E_i) = 0$, hence $F^+(A_i) < 0$, which is a contradiction.

Then

$$\frac{|F^+(A_i)|^p}{[\mu(A_i)]^{p-1}} \leqslant \frac{2^p|F(E_i)|^p}{[\mu(A_i)]^{p-1}} \leqslant \frac{2^p|F(E_i)|^p}{[\mu(E_i)]^{p-1}},$$

from which we obtain

$$(1.15)\qquad \left\{\sum_{i=1}^{n}\frac{|F^{+}(A_i)|^{p}}{[\mu(A_i)]^{p-1}}\right\}^{1/p}\leqslant 2M,$$

and this shows that $F^{+}\in BV_p(A)$.

In a similar manner it may be proved that $F^{-}\in BV_p(A)$. Conversely, let us suppose $F=F_1+F_2$, where $F_1, F_2\in BV_p(A)$ and consider $(A_i)_{i=\overline{1,n}}$ an arbitrary $\mathcal{F}$-partition of A. Then

$$\sum_{i=1}^{n}\frac{|F(A_i)|^{p}}{[\mu(A_i)]^{p-1}}=\sum_{i=1}^{n}\frac{|F_1(A_i)+F_2(A_i)|^{p}}{[\mu(A_i)]^{p-1}}\leqslant$$

$$\leqslant 2^{p-1}\left\{\sum_{i=1}^{n}\frac{|F_1(A_i)|^{p}}{[\mu(A_i)]^{p-1}}+\sum_{i=1}^{n}\frac{|F_2(A_i)|^{p}}{[\mu(A_i)]^{p-1}}\right\}\leqslant 2^{p-1}\,[V_p^{p}(F_1,A)+V_p^{p}(F_2,A)].$$

and the theorem is proved.

Theorem 1.5. *If $p>1$ and if $F\in BV_p(A)$, then for every $\varepsilon>0$ there exists $\delta>0$ such that*

$$\sum_{i\in I}|F(A_i)|<\varepsilon$$

for all finite families $\{A_i\}_{i\in I}$ of disjoint sets $A_i\in\mathcal{F}_A$ such that

$$\sum_{i\in I}\mu(A_i)<\delta.$$

Proof. From $F\in BV_p(A)$, it follows that there exists $K\geqslant 0$ such that

$$(1.16)\qquad \left\{\sum_{i\in I}\frac{|F(A_i)|^{p}}{[\mu(A_i)]^{p-1}}\right\}^{1/p}\leqslant K.$$

The inequality (1.9) leads to

$$(1.17)\qquad \sum_{i\in I}\frac{|F(A_i)|^{p}}{[\mu(A_i)]^{p-1}}\geqslant\left(\sum_{i\in I}|F(A_i)|\right)^{p}\left(\sum_{i\in I}\mu(A_i)\right)^{1-p},$$

from which, by (1.16) we obtain

$$(1.18)\qquad \sum_{i\in I}|F(A_i)|\leqslant K\left[\sum_{i\in I}\mu(A_i)\right]^{\frac{p-1}{p}},$$

which proves the theorem.

For our further considerations, we suppose the following hypotheses:

(A_1) X is a metric space;
(A_2) $Y\subset X$ is a separable subspace;
(A_3) $\mathcal{F}=\mathcal{B}(Y)$ is the σ-algebra of Borel subsets of Y;
(A_4) $F:\mathcal{F}\longrightarrow\bar{R}$ is a countable additive set function with $F(\emptyset)=0$;
(A_5) $\mu:\mathcal{F}\longrightarrow R_+$ is a measure.

Theorem 1.6. *The function F belongs to $BV_p(Y)$ $(p>1)$ if and only if there exists a function $f\in L^p(Y)$ such that*

$$(1.19)\qquad F(A)=\int_A f\,d\mu,\quad \forall A\in\mathcal{F}.$$

Proof. Let $f\in L^p(Y)$ be the function for which (1.19) is satisfied and let $(A_i)_{i=\overline{1,n}}$ be an arbitrary $\mathcal{F}$-partition of Y. By virtue of Hölder's inequality, we have

$$\sum_{i=1}^{n}\frac{|F(A_i)|^{p}}{[\mu(A_i)]^{p-1}}=\sum_{i=1}^{n}\frac{\left|\int_{A_i}f\,d\mu\right|^{p}}{[\mu(A_i)]^{p-1}}\leqslant\sum_{i=1}^{n}\frac{\left(\int_{A_i}|f|\,d\mu\right)^{p}}{[\mu(A_i)]^{p-1}}\leqslant$$

$$\leqslant\sum_{i=1}^{n}\frac{\left(\int_{A_i}|f|^{p}\,d\mu\right)[\mu(A_i)]^{p-1}}{[\mu(A_i)]^{p-1}}=\sum_{i=1}^{n}\int_{A_i}|f|^{p}\,d\mu=\int_Y|f|^{p}\,d\mu.$$

Therefore $F\in BV_p(Y)$.

Before proving the converse, we shall make some preliminary considerations. Following [7] we call *network* in Y any finite or countable family of disjoint Borel sets which cover Y. A sequence $(\mathfrak{M}_n)$ of networks is called *regular* if each set of $\mathfrak{M}_{n+1}$ is contained in a set of $\mathfrak{M}_n$ and $\delta(\mathfrak{M}_n)\longrightarrow 0$ as $n\longrightarrow\infty$, where $\delta(\mathfrak{M}_n)=\sup_{A\in\mathfrak{M}_n}\sup_{x,y\in A}\{d(x,y)\}$. It is known that *the separability is necessary and sufficient for the existence of a regular sequence of networks in* Y (see [7], p. 153). Let $\mathfrak{M}=\{\mathfrak{M}_k\}$ be such a sequence. If $F\in BV_p(Y)$ it follows that F is absolutely continuous with respect to μ [Theorem 1.5]. Hence (cf. [5], p. 198) there exists a function $f:Y\longrightarrow\bar{R}$, $\mathcal{F}$-measurable and determined up to a set of null measure, such that

$$(1.20)\qquad F(A)=\int_A f\,d\mu,\quad \forall A\in F.$$

We shall prove that $f\in L^p(Y)$. By [3] we have

$$\frac{dF}{d\mu}(x)=f(x),\ \ \mu\text{. a. e.}$$

Let us consider the sequence $f_n:Y\longrightarrow\bar{R}$ defined by

$$(1.21)\qquad f_n(x)=\begin{cases}\dfrac{F(M)}{\mu(M)} & \text{if } \mu(M)\neq 0,\\ +\infty & \text{if } \mu(M)=0 \text{ and } F(M)\geqslant 0,\\ -\infty & \text{if } \mu(M)=0 \text{ and } F(M)<0,\end{cases}$$

if $x\in M$, where the set M runs through $\mathfrak{M}_n$.

By virtue of the above considerations it follows that

$$\lim_{n\to\infty} f_n(x) = f(x)\mu \quad \text{a.e.} \tag{1.22}$$

By Fatou's theorem, if $\mathfrak{M}_n = \{Y_1^n, Y_2^n, \dots, Y_{p_n}^n\}$, we obtain

$$\int_Y |f|^p d\mu \leq \liminf_n \int_Y |f_n|^p d\mu,$$

which, combined with

$$\int_Y |f_n|^p d\mu = \sum_{k=1}^{p_n} \int_{Y_k^n} |f_n|^p d\mu = \sum_{k=1}^{p_n} \frac{|F(Y_k^n)|^p}{[\mu(Y_k^n)]^{p-1}} \leq K^p,$$

implies

$$\int_Y |f|^p d\mu < +\infty$$

as claimed. It is clear that this conclusion remains valid if the network is infinitely countable.

2. Set functions p-absolutely continuous. We now introduce the concept of p-absolutely continuous set function.

Definition 2.1. *We say that the function F is p-absolutely continuous on the set A with respect to μ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\left\{\sum_{i\in I} \frac{|F(A_i)|^p}{[\mu(A_i)]^{p-1}}\right\}^{1/p} < \varepsilon, \tag{2.1}$$

for all finite families $\{A_i\}_{i\in I}$ of disjoint sets $A_i \in \mathcal{F}_A$ such that

$$\sum_{i\in I} \mu(A_i) < \delta.$$

By $AC_p(A)$ we denote the class of all p-absolutely continuous set functions on A.

Theorem 2.1. *If $F \in AC_p(A)(p > 1)$ then for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sum_{i\in I} |F(A_i)| < \varepsilon,$$

for all finite families $\{A_i\}_{i\in I}$ of disjoint sets $A_i \in \mathcal{F}_A$ such that

$$\sum_{i\in I} \mu(A_i) < \delta.$$

The proof follows exactly the same line as the proof of Theorem 1.5. We denote by

$$AC_p^*(A) = \{F \in AC_p(A)\,;\ F \text{ finitely additive}\}. \tag{2.2}$$

Theorem 2.2. *If the set A may be represented as a finite disjoint union of sets with arbitrary small measure then*

$$AC_p^*(A) \subset BV_p(A).$$

Proof. Indeed. $F \in AC_p^*(A)$ implies $F \in AC_p(A)$ and taking $\varepsilon = 1$ in the definition 2.1. it follows that there exists a $\delta_1 > 0$ such that

$$\left\{\sum_{i\in I} \frac{|F(A_i)|^p}{[\mu(A_i)]^{p-1}}\right\}^{1/p} < 1 \tag{2.3}$$

for all finite families $\{A_i\}_{i\in I}$ of disjoint sets $A_i \in \mathcal{F}_A$ such that

$$\sum_{i\in I} \mu(A_i) < \delta_1.$$

By hypothesis, we have $A = \bigcup_{i=1}^m B_i$ with $\mu(B_i) < \delta_1$ for every $i = \overline{1, m}$. Hence $F \in BV_p(A_i)(i = \overline{1, m})$ and taking into account the Theorem 1.3 we have $F \in BV_p(A)$.

Finally, let us consider a separable metric space X and let $\mathcal{B}(X)$ denote the class of Borel subsets of X. Let $\mu : \mathcal{B}(X) \to R_+$ be a measure. We denote by

$$BV_p'(X) = \{F : \mathcal{B}(X) \to \bar{R}\,;\ F \text{ countable additive and } F \in BV_p(X)\} \tag{2.4}$$

and

$$AC_p'(X) = \{G : \mathcal{B}(X) \to \bar{R}\,;\ G \text{ countable additive and } G \in AC_p(X)\}. \tag{2.5}$$

Theorem 2.3. *If X can be represented as a finite disjoint union of sets with arbitrary small measure and $p > 1$, then*

$$BV_p'(X) = AC_p'(X). \tag{2.6}$$

Proof. From the preceding theorem it follows that

$$AC_p'(X) \subset BV_p'(X).$$

Let be $F \in BV_p'(X)$. By virtue of Theorem 1.6 we have the representation

$$F(A) = \int_A f d\mu, \quad \forall A \in \mathcal{B}(X) \tag{2.7}$$

with $f \in L^p(X)$.

If $\{A_i\}_{i\in I}$ is an arbitrary finite family of disjoint sets $A_i \in \mathcal{B}(X)$ then, using Hölder's inequality we obtain

$$\sum_{i\in I} \frac{|F(A_i)|^p}{[\mu(A_i)]^{p-1}} \leq \sum_{i\in I} \int_{A_i} |f|^p d\mu. \tag{2.8}$$

Since the integral is an absolutely continuous set function from the above inequality it follows that $F \in AC_p'(X)$, therefore (2.6) is satisfied.

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Faculty of Mathematics
University of Iași
Iași, R. S. Romania

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MAXIMAL ELEMENTS IN PARTIALLY ORDERED TOPOLOGICAL SPACES AND APPLICATIONS

BY

MIHAI TURINICI

The aim of this note is to give a class of sufficient conditions for the existence of maximal elements in a partially ordered topological space. The results obtained here represent a partial refinement of those established in [1] for uniform structures.

1. Throughout this note, we shall use the following notations:

i) $\mathfrak{P}(X) = \{Y \subset X : Y \neq \emptyset\}$, $\mathfrak{P}^*(X) = \{Y \in \mathfrak{P}(X) ; X \setminus Y \neq \emptyset\}$.

ii) $$F(X, Y) = \{f ; f : X \to Y\},$$

$$R_+ = [0, +\infty[, R_- =]-\infty, 0].$$

iii) $$\mathcal{O}(X) = \{\leqslant \ \in \mathfrak{P}(X^2) ; \leqslant \text{ is a partial order on } X\}.$$

If $\leqslant \ \in \mathcal{O}(X)$, $\leqslant \neq \Delta(X)$, then the relation $< \in \mathfrak{P}(X^2)$ defined by $x < y$ iff $x \leqslant y$ and $x \neq y$ is called the nonreflexive order induced by $\leqslant$. For every $\leqslant \ \in \mathcal{O}(X)$ and $Y \in \mathfrak{P}(X)$, let us denote

iv) $Y(x, \leqslant) = \{y \in Y ; x \leqslant y\}$, $Y(x, <) = \{y \in Y ; x < y\}$, $\forall x \in X$,

v) $$b(Y, \leqslant) = \{z \in X ; z \text{ is an } \leqslant -\text{upper bound of } Y\},$$

vi) $$m(Y, \leqslant) = \{z \in X ; z \text{ is a } \leqslant -\text{maximal element of } Y\}.$$

vii) $$\mathcal{L}(Y, \leqslant) = \{L \in \mathfrak{P}(Y) ; L \text{ is } \leqslant -\text{totally ordered}\}.$$

Remark 1.1. Let $\leqslant, \leqslant_1 \ \in \mathcal{O}(X)$, $\leqslant$ finer than $\leqslant_1$. Then, for every $Y \in \mathfrak{P}(X)$, a) $\mathcal{L}(Y, \leqslant_1) \subset \mathcal{L}(Y, \leqslant)$, b) $m(Y, \leqslant) \subset m(Y, \leqslant_1)$, c) $b(Y, \leqslant_1) \subset b(Y, \leqslant)$.

Remark 1.2. $L \in \mathcal{L}(X, \leqslant) \Rightarrow m(L, \leqslant) = b(L, \leqslant) \cap L$.

Remark 1.3. Let $L \in \mathcal{L}(X, \leqslant)$ be such that $m(L, \leqslant) = \emptyset$, (respectively, $b(L, \leqslant) = \emptyset$). Then, a) $L(x, <) \neq \emptyset$, $\forall x \in L$, b) $m(L(x, <), \leqslant) = \emptyset$ (respectively $b(L(x, <), \leqslant) = \emptyset$), $\forall x \in L$.

2. In this paragraph, (X, τ) is a given topological space and $\leqslant \ \in \mathcal{O}(X)$ is a partial order on X.