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MAXIMAL ELEMENTS IN PARTIALLY ORDERED TOPOLOGICAL SPACES AND APPLICATIONS

BY

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The aim of this note is to give a class of sufficient conditions for the existence of maximal elements in a partially ordered topological space. The results obtained here represent a partial refinement of those established in [1] for uniform structures.

1. Throughout this note, we shall use the following notations:

i) $\mathfrak{P}(X) = \{Y \subset X : Y \neq \emptyset\}$, $\mathfrak{P}^*(X) = \{Y \in \mathfrak{P}(X) ; X \setminus Y \neq \emptyset\}$.

ii) $$F(X, Y) = \{f ; f : X \to Y\},$$

$$R_+ = [0, +\infty[, R_- =]-\infty, 0].$$

iii) $$O(X) = \{\leqslant \ \in \mathfrak{P}(X^2) ; \leqslant \text{ is a partial order on } X\}.$$

If $\leqslant \ \in O(X)$, $\leqslant \ \neq \Delta(X)$, then the relation $< \ \in \mathfrak{P}(X^2)$ defined by $x < y$ iff $x \leqslant y$ and $x \neq y$ is called the nonreflexive order induced by $\leqslant$. For every $\leqslant \ \in O(X)$ and $Y \in \mathfrak{P}(X)$, let us denote

iv) $Y(x, \leqslant) = \{y \in Y ; x \leqslant y\}$, $Y(x, <) = \{y \in Y ; x < y\}$, $\forall x \in X$,

v) $$b(Y, \leqslant) = \{z \in X ; z \text{ is an } \leqslant -\text{upper bound of } Y\},$$

vi) $$m(Y, \leqslant) = \{z \in X ; z \text{ is a } \leqslant -\text{maximal element of } Y\}.$$

vii) $$\mathcal{L}(Y, \leqslant) = \{L \in \mathfrak{P}(Y) ; L \text{ is } \leqslant -\text{totally ordered}\}.$$

Remark 1.1. Let $\leqslant, \leqslant_1 \ \in O(X)$, $\leqslant$ finer than $\leqslant_1$. Then, for every $Y \in \mathfrak{P}(X)$, a) $\mathcal{L}(Y, \leqslant_1) \subset \mathcal{L}(Y, \leqslant)$, b) $m(Y, \leqslant) \subset m(Y, \leqslant_1)$, c) $b(Y, \leqslant_1) \subset b(Y, \leqslant)$.

Remark 1.2. $L \in \mathcal{L}(X, \leqslant) \Rightarrow m(L, \leqslant) = b(L, \leqslant) \cap L$.

Remark 1.3. Let $L \in \mathcal{L}(X, \leqslant)$ be such that $m(L, \leqslant) = \emptyset$, (respectively, $b(L, \leqslant) = \emptyset$). Then, a) $L(x, <) \neq \emptyset$, $\forall x \in L$, b) $m(L(x, <), \leqslant) = \emptyset$ (respectively $b(L(x, <), \leqslant) = \emptyset$), $\forall x \in L$.

2. In this paragraph, (X, τ) is a given topological space and $\leqslant \ \in O(X)$ is a partial order on X.

Definition 2.1. *Let $L \in \mathcal{L}(X, \leqslant)$, $m(L, \leqslant) = \emptyset$. We say that L converges to $z \in X$ (and we indicate this by $L \to z$) iff $\forall V \in \mathcal{V}(z), \exists x \in L$ such that $L(x, <) \subset V$.*

Let us denote

i) $\lambda(L) = \{z \in X;\ L \to z\}$, $\forall L \in \mathcal{L}(X, \leqslant)$, $m(L, \leqslant) = \emptyset$.

Lemma 2.1. *Let $L \in \mathcal{L}(X, \leqslant)$ and $z \in X$ be such that $m(L, \leqslant) = \emptyset$ and $L \to z$. Then, $L(x, <) \to z$, for every element $x \in L$.*

Proof. Let $x \in L$ be fixed. If $V \in \mathcal{V}(z)$ is given, then, by hypothesis, there exists a $y \in L$ such that $L(y, <) \subset V$. Now, if $y \leqslant x$, $L(x, <)(u, <) = L(u, <) \subset L(y, <) \subset V$, (for a given $u \in L(x, <)$) and if $y > x$ (i.e., $y \in L(x, <)$), $L(x, <)(y, <) = L(y, <) \subset V$. Q.E.D.

The main result of this note may be stated as follows.

Theorem 2.1. *Let $\leqslant \in \mathcal{O}(X)$ and $Y \in \mathfrak{P}(X)$ satisfying*

(2.1) $\quad Y(x, \leqslant)$ *is τ-closed*, $\forall x \in Y$

(2.2) $\quad L \in \mathcal{L}(Y, \leqslant)$, $m(L, \leqslant) = \emptyset \Rightarrow \lambda(L) \neq \emptyset$

Then, $\forall x \in Y$, $\exists z \in Y$ such that a) $x \leqslant z$, b) $z < y$ is false, $\forall y \in Y$.

Proof. Suppose there exists $L \in \mathcal{L}(Y, \leqslant)$, $Y \cap b(L, \leqslant) = \emptyset$. From (2.2) and remark 1.2, $L \to z$ for some $z \in X$ and from lemma 2.1 $L(x, <) \to z$, $\forall x \in L$. But then, from (2.1), $z \in \overline{L(x, <)} \subset \overline{Y(x, \leqslant)} = Y(x, \leqslant)$, $\forall x \in L$, i.e., $z \in Y \cap b(L, \leqslant)$, a contradiction. Therefore, $Y \cap b(L, \leqslant) \neq \emptyset$, $\forall L \in \mathcal{L}(Y, \leqslant)$, and so, Zorn's theorem [4, p. 19] is applicable, completing the proof. Q.E.D.

Remark 2.1. A sufficient condition for (2.1) is

(2.1)' $\quad X(x, \leqslant)$, $(\forall x \in X)$ and Y are τ-closed.

Remark 2.2. A sufficient condition for (2.2) is

(2.2)' $\quad L \in \mathcal{L}(Y, \leqslant_1)$, $m(L, \leqslant_1) = \emptyset \Rightarrow \lambda(L) \neq \emptyset$.

where $\leqslant_1 \in \mathcal{O}(X)$ is finer than $\leqslant$ (see remark 1.1).

Corollary 2.1. *Let $\leqslant \in \mathcal{O}(X)$ and $Y \in \mathfrak{P}(X)$ satisfying (2.1) + (2.2) and suppose $T \in F(Y, Y)$ satisfies*

(2.3) $\quad x \leqslant Tx$, $\forall x \in Y$

Then, $\forall x \in Y$, $\exists z \in Y$ such that, a) $x \leqslant z$, b) $z = Tz$; c) $z < y$ is false for every element $y \in Y$.

Proof. Clearly, $z \in m(Y, \leqslant)$, $z \leqslant Tz \Rightarrow z = Tz$. Q.E.D.

3. In this paragraph, (X, d) is a given metric space, and $\leqslant \in \mathcal{O}(X)$ is a partial order on X.

Definition 3.1. *Let $L \in \mathcal{L}(X, \leqslant)$, $m(L, \leqslant) = \emptyset$. We say that L is a Cauchy net, iff $\forall \varepsilon > 0$, $\exists x \in L$, such that, $y, z \in L(x, <) \Rightarrow d(y, z) < \varepsilon$.*

In what follows, (X, d) is a complete metric space.

Lemma 3.1. *Let $L \in \mathcal{L}(X, \leqslant)$, $m(L, \leqslant) = \emptyset$. Then, $\lambda(L) \neq \emptyset$ if and only if L is a Cauchy net.*

Proof. The necessity is obvious. To prove the sufficiency, let L be a Cauchy net. From definition 3.1, we can construct a sequence $\{x(n); n \in N\} \subset L$ with the properties a) $x(0) < x(1) < \ldots$; b) $d(x(n), y) < 1/2^n$, $\forall y \in L(x(n), <)$, $\forall n \in N$. It easily follows that $\{x(n); n \in N\}$ is a Cauchy sequence and therefore, $x(n) \to z$ as $n \to \infty$, for some $z \in X$. We claim that $L \to z$. Indeed, let $\varepsilon > 0$ be given. There exists $n(\varepsilon) \in N$ such that both $d(x(n), z) < \varepsilon/2$ and $1/2^n < \varepsilon/2$, for every $n \geqslant n(\varepsilon)$. In this case, $y \in L(x(n(\varepsilon)), <) \Rightarrow d(y, z) < d(x(n(\varepsilon)), y) + d(x(n(\varepsilon)), z) < 1/2^{n(\varepsilon)} + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon$, i.e., $L \to z$, as claimed. Q.E.D.

Now, we present a „metrical" version of theorem 2.1.

Theorem 3.1. *Let $\leqslant \in \mathcal{O}(X)$ and $Y \in \mathfrak{P}(X)$, satisfying:*

(3.1) $\quad Y(x, \leqslant)$ *is d-closed*, $\forall x \in Y$,

(3.2) $\quad L \in \mathcal{L}(Y, \leqslant)$, $m(L, \leqslant) = \emptyset \Rightarrow L$ *is a Cauchy net.*

Then, $\forall x \in Y$, $\exists z \in Y$ such that, a) $x \leqslant z$, b) $z < y$ is false, $\forall y \in Y$.

Proof. See theorem 2.1 and lemma 3.1. Q.E.D.

Remark 3.1. A sufficient condition for (3.1) is

(3.1)' $\quad X(x, \leqslant)$ $(\forall x \in X)$ and Y are d-closed.

Remark 3.2. As in the remark 2.2, a sufficient condition for (3.2) is:

(3.2)' $\quad L \in \mathcal{L}(Y, \leqslant_1)$, $m(L, \leqslant_1) = \emptyset \Rightarrow L$ *is a Cauchy net (where $\leqslant_1 \in \mathcal{O}(X)$ is finer than $\leqslant$).*

Corollary 3.1. *Let $\leqslant \in \mathcal{O}(X)$ and $Y \in \mathfrak{P}(X)$ satisfying (3.1) + (3.2) and suppose $T \in F(Y, Y)$ satisfies (2.3). Then, $\forall x \in Y$, $\exists z \in Y$ such that, a) $x \leqslant z$, b) $z = Tz$, c) $z < y$ is false, $\forall y \in Y$.*

4. In this paragraph, by a specification of the partial order on the above metric spaces, we shall obtain some useful particular results.

Definition 4.1. *Let (X, τ) be a topological space and S a nonempty set. A function $f \in F(X^2, S)$ is said to be P-closed ($P \in \mathfrak{P}^*(S)$) if $\{y \in X;\ f(x, y) \in P\}$ is τ-closed, for every $x \in X$.*

Definition 4.2. *Let S be a nonempty set. A function $g \in F(R_+ \times R, S)$ is said to be P-admissible ($P \in \mathfrak{P}^*(S)$) if it satisfies*

(4.1) $\quad g(u, 0) \in P \Leftrightarrow u = 0$; $g(u, v) \in P \Rightarrow v \geqslant 0$,

(4.2) $\quad g(u, a) \in P$, $g(v, b) \in P$, $|u - v| \leqslant w \leqslant u + v \Rightarrow g(w, a + b) \in P$,

(4.3) $\quad \forall \varepsilon > 0$, $\exists \delta > 0$ *such that*, $0 \leqslant v < \delta$, $g(u, v) \in P \Rightarrow u < \varepsilon$.

Theorem 4.1. *Let (X, d) be a complete metric space, $Y \in \mathfrak{P}(X)$, S a nonempty set, $g \in F(R \times R, S)$, $\varphi \in F(X, R_+)$, $P \in \mathfrak{P}^*(S)$, be such that*

(4.4) *the map $(x, y) \mapsto g(d(x, y), \varphi(x) - \varphi(y))$ is P-closed,*

(4.5) *Y is d-closed,*

(4.6) *g is P-admissible.*

Then, $\forall x \in Y$, $\exists z \in Y$ such that, a) $g(d(x, z), \varphi(x) - \varphi(z)) \in P$, b) $g(d(z, y), \varphi(z) - \varphi(y)) \in S \setminus P$, $\forall y \in Y$, $y \neq z$.

Proof. Let us define a relation $\leqslant \in \mathfrak{P}(X^2)$ by

(4.7) $\quad x \leqslant y$ *iff* $g(d(x, y), \varphi(x) - \varphi(y)) \in P$.

From (4.1) + (4.2) it easily follows that $\leqslant \in \mathcal{O}(X)$. Furthermore, clearly, (4.4) + (4.5) $\Rightarrow$ (3.1). Now, let $L \in \mathcal{L}(Y, \leqslant)$ be such that $m(L, \leqslant) = \emptyset$. Suppose $\varepsilon > 0$ is an arbitrary fixed number and let $\delta > 0$ be defined by (4.3). Denote $\alpha = \inf\{\varphi(x);\ x \in L\}$, and let $x \in L$ satisfy

$\alpha \leqslant \varphi(x) < \alpha + \delta$. Let $y, z \in L(x, <)$; without loss of generality we may suppose $y < z$, i.e., $g(d(y,z), \varphi(y) - \varphi(z)) \in P$. From (4.1), $0 \leqslant \varphi(y) - \varphi(z) \leqslant \varphi(x) - \varphi(z) < \alpha + \delta - \alpha = \delta$, and this implies (from (4.3)) $d(y, z) < \varepsilon$, i.e., (3.2) holds, too. Therefore, theorem 3.1 is applicable, and this completes the proof.

Corollary 4.1. *Let (X, d), Y, S, g, φ, P, satisfy only (4.5) + (4.6) and let $\leqslant_1 \in \mathcal{O}(X)$ be such that:*

$$X(x, \leqslant_1) \text{ is } d\text{-closed}, \ \forall x \in X, \tag{4.8}$$

$$x, y \in X,\ x \leqslant_1 y \Rightarrow g(d(x,y), \varphi(x) - \varphi(y)) \in P, \tag{4.9}$$

Then, $\forall x \in Y$, $\exists z \in Y$ such that, a) $x \leqslant_1 z$; b) $z <_1 y$ *is false, for every $y \in Y$.*

Proof. See remarks 3.1 and 3.2.

Corollary 4.2. *Suppose that again (4.4) — (4.6) hold and let $T \in F(Y, Y)$ be such that*

$$g(d(x, Tx), \varphi(x) - \varphi(Tx)) \in P, \ \forall x \in Y \tag{4.10}$$

Then, $\forall x \in Y$, $\exists z \in Y$ such that, a) $g(d(x,z), \varphi(x) - \varphi(z)) \in P$, b) $z = Tz$, c) $g(d(z, y), \varphi(z) - \varphi(y)) \in S \setminus P$, $\forall y \in Y$, $y \neq z$.

5. By a specification of the elements involved in the statement of the above results, we may obtain some known results, as well as some interesting applications.

Theorem 5.1. [1]. *Let (X, d) be a complete metric space and let $Y \in \mathfrak{P}(X)$, $\varphi \in F(X, R_+)$, be such that*

(5.1) *φ is d-lower semicontinuous on X,*

(5.2) *Y is d-closed.*

Then, $\forall x \in Y$, $\exists z \in Y$ such that, a) $d(x,z) \leqslant \varphi(x) - \varphi(z)$, b) $d(z, y) > \varphi(z) - \varphi(y)$, *for every $y \in Y$, $y \neq z$.*

Proof. Take in theorem 4.1, $S = R$, $g(u, v) = u - v$, $\forall u \in R_+$, $\forall v \in R$, $P = R_-$, and observe that (5.1) $\Rightarrow$ (4.4). Q.E.D.

Corollary 5.1. *Let (X, d), Y, φ, satisfy only (5.2), and let $\leqslant_1 \in \mathcal{O}(X)$ be such that*

$$X(x, \leqslant_1) \text{ is } d\text{-closed}, \ \forall x \in X, \tag{5.3}$$

$$x, y \in X,\ x \leqslant_1 y \Rightarrow d(x, y) \leqslant \varphi(x) - \varphi(y), \tag{5.4}$$

Then, $\forall x \in Y$, $\exists z \in Y$ such that, a) $x \leqslant_1 z$, b) $z <_1 y$ *is false, $\forall y \in Y$.*

Corollary 5.2. [2], [5]. *Under the hypotheses (5.1) and (5.2), let $T \in F(Y, Y)$ satisfy*

$$d(x, Tx) \leqslant \varphi(x) - \varphi(Tx), \ \forall x \in Y. \tag{5.5}$$

Then, $\forall x \in Y$, $\exists z \in Y$ such that, a) $d(x,z) \leqslant \varphi(x) - \varphi(z)$; b) $z = Tz$; c) $d(z, y) > \varphi(z) - \varphi(y)$, $\forall y \in Y$, $y \neq z$.

Let (X, d) and (X', d') be a complete metric spaces. Let $f \in F(X, X')$ be a closed mapping [3], [6], and let $c > 0$. It is easy to prove that, a) the map $e \in F(X^2, R_+)$, given by $e(x, y) = \max\{d(x, y); cd'(f(x), f(y))\}$, $\forall x, y \in X$, is a metric on X; b) (X, e) is a complete metric space; c) $Y \in \mathfrak{P}(X)$, Y is d-closed $\Rightarrow$ Y is e-closed; d) if $\psi \in F(X', R_+)$ is d'-lower semicontinuous, then $\varphi = \psi \circ f \in F(X, R_+)$ is e-lower semicontinuous.

Theorem 5.2. *Let (X, d) and (X', d') be complete metric spaces, $f \in F(X, X')$ a closed mapping, $c > 0$, and let $Y \in \mathfrak{P}(X)$, $\varphi \in F(X, R_+)$ satisfy (5.1) + (5.2). Then, $\forall x \in Y$, $\exists z \in Y$ such that,* a) $d(x,z) \leqslant \varphi(x) - \varphi(z)$, $cd'(f(x), f(z)) \leqslant \varphi(x) - \varphi(z)$; b) *for every $y \in Y$, $y \neq z$, either $d(z, y) > \varphi(z) - \varphi(y)$ or $cd'(f(z), f(y)) > \varphi(z) - \varphi(y)$.*

Proof. Clearly, (5.1), (5.2) and the above conclusions imply φ is e-lower semicontinous and Y is e-closed (where e is defined as above). So theorem 5.1 is applicable (with d replaced by e) Q.E.D.

Corollary 5.3. *Under the same hypotheses as in theorem 5.2, let $T \in F(Y, Y)$ satisfy*

$$d(x, Tx) \leqslant \varphi(x) - \varphi(Tx), \ \forall x \in Y, \tag{5.6}$$

$$cd'(f(x), f(Tx)) \leqslant \varphi(x) - \varphi(Tx), \ \forall x \in Y. \tag{5.7}$$

Then, $\forall x \in Y$, $\exists z \in Y$ such that, a) $d(x, z) \leqslant \varphi(x) - \varphi(z)$, $cd'(f(x), f(z)) \leqslant \varphi(x) - \varphi(z)$; b) $z = Tz$; c) *for every $y \in Y$, $y \neq z$, either $d(z, y) > \varphi(z) - \varphi(y)$ or $cd'(f(z), f(y)) > \varphi(z) - \varphi(y)$.*

Remark 5.1. Suppose that, in the statement of theorem 5.2 (respectively, corollary 5.3) φ is replaced by $\psi \circ f$, where $\psi \in F(X', R_+)$ is d'-lower semicontinuous. Then, clearly, conclusions (a) and (b) of theorem 5.2 (respectively, conclusions (a) + (b) + (c) of corollary 5.3) hold, with φ replaced by $\psi \circ f$. Note that, in this way, we have obtained (from corollary 5.3) the main result of [3].

Let $(X, \|\cdot\|)$ be a normed space. For $A, B \in \mathfrak{P}(X)$, $r > 0$, denote diam $(A) = \sup\{d(x, y); x, y \in A\}$, bd (A) = the boundary of A, dist $(A, B) = \inf\{d(x, y); x \in A, y \in B\}$, $S(A, r) = \{x \in X;\ \text{dist}(x, A) \leqslant r\}$. Let $C \in \mathfrak{P}(X)$ be a bounded, closed, convex subset of X and $a \in X \setminus C$. By a *drop* $[a, C]$ we mean the convex hull of $C \cup \{a\}$. It is easy to prove that $[a, C]$ is also a bounded, closed, convex subset of X.

Lemma 5.1. *Let $(X, \|.\|)$ be a normed space, C a bounded closed convex subset of X, and $x \in X \setminus C$, dist $(x, C) > \delta =$ diam (C). Then*

(5.8) $\|x - y\| \leqslant ((\text{dist}(x, C) + \delta)/(\text{dist}(x, C) - \delta))\,(\text{dist}(x, C) - \text{dist}(y, C))$ *for every element $y \in [x, C]$.*

Proof. Let $y \in [x, C]$. Then, $y = (1 - \lambda)x + \lambda c$, for some $\lambda \in [0,1]$ and $c \in C$. As $\|x - c\| \leqslant \|x - c'\| + \|c' - c\|$, $\forall c' \in C$, we get $\|x - c\| \leqslant \text{dist}(x, C) + \delta$. Now, $\|y - c'\| = (1 - \lambda)(x - c') + \lambda(c - c')$ and thus, $\|y - c'\| \leqslant (1 - \lambda)\|x - c'\| + \lambda\|c - c'\|$, $\forall c' \in C$, which gives dist $(y, C) \leqslant (1 - \lambda)$ dist $(x, C) + \lambda\delta$, and this implies $\lambda \leqslant (\text{dist}(x, C) - \text{dist}(y, C))/(\text{dist}(x, C) - \delta)$. Now, $\|x - y\| = \lambda\|x - c\| \leqslant ((\text{dist}(x, C) - \text{dist}(y, C))/(\text{dist}(x, C) - \delta))\,(\text{dist}(x, C) + \delta)$ which is just (5.8), completing the proof. Q.E.D.

Theorem 5.3. *Let $(X, \|.\|)$ be a Banach space, C a bounded closed convex subset of X, and D a closed subset of $X \setminus C$, with $\rho =$ dist $(C, D) > \delta =$ diam (C). Then, $\forall a \in D$, $\exists b \in D$ such that* a) $b \in [a, C]$; b) $b \in$ bd (D); c) $[b, C] \cap D = \{b\}$

Proof. Let $a \in D$ be fixed. Denote $\sigma =$ dist (a, C), $Y = S(C, \sigma) \cap D$ (clearly, Y is closed). Let $\varphi \in F(X, R_+)$ be defined by $\varphi(x) =$

$= ((\sigma + \delta)/(\rho - \delta))$ dist (x, C), $\forall x \in X$. It is a simple matter to verify that φ is continuous on X. Define the relations $\leqslant$ and $\leqslant_1$ on Y by

$$x \leqslant y \text{ iff } y \in [x, C] \tag{5.9}$$

$$x \leqslant_1 y \text{ iff } \|x - y\| \leqslant \varphi(x) - \varphi(y) \tag{5.10}$$

Firstly, $\leqslant$ and $\leqslant_1$ are order relations on Y. Moreover, $Y(x, \leqslant)$ is closed, $\forall x \in Y$, and (by lemma 5.1) $\leqslant_1$ is finer than $\leqslant$. So, corollary 5.1 applies (for other details see [1], [7]). Q.E.D.

Again, let $(X, \|.\|)$ be a normed space. For every $x, y \in X$, $x \neq y$ denote $[x, y] = \{tx + (1-t)y ; t \in [0,1]\}$, $[x, y] = [x, y] \setminus \{x\}$, $[x, y] = [x, y] \setminus \{y\}$, $]x, y[= [x, y] \setminus \{x, y\}$.

Theorem 5.4. [6]. *Let $(X, \|.\|)$ be a Banach space, $x, y \in X$, $x \neq y$ and $T \in F([x, y], X)$ a closed mapping. Suppose there exists $k > 0$, such that, for every $u, v \in [x, y]$, $u \neq v$,*

$$\inf \{\|Tu - Tw\|/\|u - w\| ; w \in]u, v]\} \leqslant k. \tag{5.11}$$

Then T is k-lipschitzian on $[x, y]$, i.e., $\|Tu - Tv\| \leqslant k\|u - v\|$ *for every $u, v \in [x, y]$.*

Proof. Let $u, v \in [x, y]$. Without loss of generality we may suppose $u = x$, $v = y$. Define $\varphi \in F(X, R_+)$ by $\varphi(u) = \|u - y\|$, $\forall u \in X$ and observe that theorem 5.2 applies (with (X, d) and $Y \in \mathfrak{P}(X)$ replaced by $([x, y], \|.\|)$, (X', d') by $(X, \|.\|)$, and c replaced by $1/p$, where $p > k$ is arbitrary fixed). So, for $x \in [x, y]$ there exists $z \in [x, y]$ satisfying (a) + (b) of the quoted theorem. Suppose that $z \neq y$. For every $u \in]z, y]$, the relation $\|z - u\| > \|z - y\| - \|u - y\| = \|z - u\|$ is impossible. Therefore (from conclusion (b)) we must have $(1/p)\|Tz - Tu\| > \|z - y\| - \|u - y\|$, i.e., $\|Tz - Tu\| > p\|z - u\|$, $\forall u \in]z, y]$, which contradicts (5.11). Thus, $z = y$; from conclusion (a), we get $(1/p)\|Tx - Ty\| \leqslant \|x - y\|$, i.e., $\|Tx - Ty\| \leqslant p\|x - y\|$. As $p > k$ was arbitrary, this implies $\|Tx - Ty\| \leqslant k\|x - y\|$, proving our assertion. Q.E.D.

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REGULARLY NONEXPANSIVE MAPPINGS

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The class of nonexpansive mappings in Banach spaces has been largely investigated recently, especially in context of fixed point theory [see e. g. [1], [2], [5], [7], [9]. The main problem in this field is to characterize the class of sets having fixed point property with respect to theese mappings. Our aim here is to define a family of transformations which we shall call regularly nonexpansive ones. We will also investigate some of their properties.

Notations and definitions. Let $(X, \|\,\|)$ be a Banach space and let C be an arbitrary nonempty subset of X. A mapping $T : C \to X$ is said to be nonexpansive iff $\|Tx - Ty\| \leqslant \|x - y\|$ for any $x, y \in C$.

Definition 1. *A mapping $T : C \to X$ is said to be regularly nonexpansive iff for arbitrary $x, y \in C$ the function*

$$\Phi(t) = \Phi_{x,y}(t) = \|(1-t)(x-y) + t(Tx - Ty)\|$$

is nonincreasing with respect to t on the interval $<0, 1>$.

Obviously any regularly nonexpansive mapping is nonexpansive but not vice-versa. Also it is easy to see that for any x, y the function $\Phi(t)$ is convex on the real line, $\lim_{t \to \pm\infty} \Phi(t) = +\infty$ if $x - y \neq Tx - Ty$ and $\Phi(t)$ is constant otherwise.

In view of that, the regularly nonexpansive mappings can be equivalently defined as those satisfying

$$\|Tx - Ty\| \leqslant \|(1-t)(x-y) + t(Tx - Ty)\|$$

for all $x, y \in C$, $t \in <0,1>$.

Let now $J : X \to 2^{X^*}$ be the duality mapping. So

$$J(w) = \{w^* \in X^* : (w, w^*) = \|w\|^2 = \|w^*\|^2\}$$

for all $w \in X$. The following technical lemma is closely patterned on results of Kato [4] and Martin [8] (see also [10]), so we skip the proof.

Lemma 1. *Suppose $z, w \in X$. The following statements are equivalent*

(i) $\|w\| \leqslant \|(1-t)z + tw\|$ for all $t \in <0,1>$,

(ii) $\lim_{t \to 1} \dfrac{\|(1-t)z + tw\| - t\|w\|}{1-t} \geqslant \|w\|$,