$= ((\sigma + \delta)/(\rho - \delta))\ \mathrm{dist}\,(x, C)$, $\forall x \in X$. It is a simple matter to verify that φ is continuous on X. Define the relations $\leqslant$ and $\leqslant_1$ on Y by

(5.9) $$x \leqslant y \text{ iff } y \in [x, C]$$

(5.10) $$x \leqslant_1 y \text{ iff } \|x - y\| \leqslant \varphi(x) - \varphi(y)$$

Firstly, $\leqslant$ and $\leqslant_1$ are order relations on Y. Moreover, $Y(x, \leqslant)$ is closed, $\forall x \in Y$, and (by lemma 5.1) $\leqslant_1$ is finer than $\leqslant$. So, corollary 5.1 applies (for other details see [1], [7]). Q.E.D.

Again, let $(X, \|\cdot\|)$ be a normed space. For every $x, y \in X$, $x \neq y$ denote $[x, y] = \{tx + (1 - t)\, y;\ t \in [0,1]\}$, $]x, y] = [x, y] \setminus \{x\}$, $[x, y[= [x, y] \setminus \{y\}$, $]x, y[= [x, y] \setminus \{x, y\}$.

Theorem 5.4. [6]. *Let $(X, \|\cdot\|)$ be a Banach space, $x, y \in X$, $x \neq y$ and $T \in F([x, y], X)$ a closed mapping. Suppose there exists $k > 0$, such that, for every $u, v \in [x, y]$, $u \neq v$,*

(5.11) $$\inf \{\|Tu - Tw\|/\|u - w\|;\ w \in]u, v]\} \leqslant k.$$

Then T is k-lipschitzian on $[x, y]$, i.e., $\|Tu - Tv\| \leqslant k \|u - v\|$ for every $u, v \in [x, y]$.

Proof. Let $u, v \in [x, y]$. Without loss of generality we may suppose $u = x$, $v = y$. Define $\varphi \in F(X, R_+)$ by $\varphi(u) = \|u - y\|$, $\forall u \in X$ and observe that theorem 5.2 applies (with (X, d) and $Y \in \mathfrak{P}(X)$ replaced by $([x, y], \|\cdot\|)$, (X', d') by $(X, \|\cdot\|)$, and c replaced by $1/p$, where $p > k$ is arbitrary fixed). So, for $x \in [x, y]$ there exists $z \in [x, y]$ satisfying (a) + (b) of the quoted theorem. Suppose that $z \neq y$. For every $u \in]z, y]$, the relation $\|z - u\| > \|z - y\| - \|u - y\| = \|z - u\|$ is impossible. Therefore (from conclusion (b)) we must have $(1/p)\|Tz - Tu\| > \|z - y\| - \|u - y\|$, i.e., $\|Tz - Tu\| > p\|z - u\|$, $\forall u \in]z, y]$, which contradicts (5.11). Thus, $z = y$; from conclusion (a), we get $(1/p)\|Tx - Ty\| \leqslant \|x - y\|$, i.e., $\|Tx - Ty\| \leqslant p\|x - y\|$. As $p > k$ was arbitrary, this implies $\|Tx - Ty\| \leqslant k\|x - y\|$, proving our assertion. Q.E.D.

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REGULARLY NONEXPANSIVE MAPPINGS

BY

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The class of nonexpansive mappings in Banach spaces has been largely investigated recently, especially in context of fixed point theory [see e. g. [1], [2], [5], [7], [9]. The main problem in this field is to characterize the class of sets having fixed point property with respect to theese mappings. Our aim here is to define a family of transformations which we shall call regularly nonexpansive ones. We will also investigate some of their properties.

Notations and definitions. Let $(X, \|\ \|)$ be a Banach space and let C be an arbitrary nonempty subset of X. A mapping $T : C \to X$ is said to be nonexpansive iff $\|Tx - Ty\| \leqslant \|x - y\|$ for any $x, y \in C$.

Definition 1. *A mapping $T : C \to X$ is said to be regularly nonexpansive iff for arbitrary $x, y \in C$ the function*

$$\Phi(t) = \Phi_{x,y}(t) = \|(1 - t)(x - y) + t(Tx - Ty)\|$$

is nonincreasing with respect to t on the interval $<0, 1>$.

Obviously any regularly nonexpansive mapping is nonexpansive but not vice-versa. Also it is easy to see that for any x, y the function $\Phi(t)$ is convex on the real line, $\lim\limits_{t \to \pm\infty} \Phi(t) = +\infty$ if $x - y \neq Tx - Ty$ and $\Phi(t)$ is constant otherwise.

In view of that, the regularly nonexpansive mappings can be equivalently defined as those satisfying

$$\|Tx - Ty\| \leqslant \|(1 - t)(x - y) + t(Tx - Ty)\|$$

for all $x, y \in C$, $t \in <0,1>$.

Let now $J : X \to 2^{X^*}$ be the duality mapping. So

$$J(w) = \{w^* \in X^* : (w, w^*) = \|w\|^2 = \|w^*\|^2\}$$

for all $w \in X$. The following technical lemma is closely patterned on results of Kato [4] and Martin [8] (see also [10]), so we skip the proof.

Lemma 1. *Suppose $z, w \in X$. The following statements are equivalent*

(i) $\|w\| \leqslant \|(1 - t)z + tw\|$ for all $t \in <0,1>$,

(ii) $$\lim_{t \to 1} \frac{\|(1 - t)z + tw\| - t\|w\|}{1 - t} \geqslant \|w\|,$$

(iii) *there exists* $j \in J(w)$ *such that* $\| w \|^2 \leqslant (z, j)$.

This allow us to write the following characterization theorem

Theorem 1. *Let* $T : C \to X$. *Then the following statements are equivalent*

(a) T *is regularly nonexpansive,*

(b) $\| Tx - Ty \| \leqslant \| (1-t)(x-y) + t(Tx - Ty) \|$ *for all* $x, y \in C$, $t \in \langle 0,1 \rangle$

(c) $$\lim_{t \to 1} \frac{\| (1-t)(x-y) + t(Tx-Ty) \| - t \| Tx - Ty \|}{1-t} \geqslant \| Tx - Ty \|$$

for all $x, y \in C$ *there exists* $j \in J(Tx - Ty)$ *such that*

(d) *for any* $x, y \in C$ *there exists* $j \in J(Tx - Ty)$ *such that*

$$\| Tx - Ty \|^2 \leqslant (x - y, j).$$

Especially if X is a Hilbert space, then T is regularly nonexpansive iff $\| Tx - Ty \|^2 \leqslant (Tx - Ty, x - y)$ which holds iff $T = 1/2 (I + Q)$, where $Q : C \to X$ is a nonexpansive mapping. The properties of such mappings were described in details in [11].

Finally let us call $T : C \to X$ regular isometry if it is isometrical and regularly nonexpansive, what means that for any x, y the function $\Phi(t)$ is constant on $\langle 0,1 \rangle$.

Some properties of regularly nonexpansive mappings.

Theorem 2. *Suppose* X *is a strictly convex Banach space and* $T : C \to X$ *is regularly nonexpansive. If for some* $x, y \in C$, $\| Tx - Ty \| = \| x - y \|$, *then* $x - y = Tx - Ty$.

Proof: We have

$\| Tx - Ty \| = \| (1-t)(x-y) + t(Tx - Ty) \| = \| x - y \|$ for $t \in \langle 0,1 \rangle$, meaning that the segment joining $Tx - Ty$ and $x - y$ is laying on a sphere. It implies our conclusion.

This theorem implies that in the case of strictly convex space X any regular isometry $T : C \to X$ must be of the form $Tx = x + u$ with fixed u. That is not true in arbitrary Banach space, even if $T : C \to C$, as the following example shows.

Example: Let $X = R^2$ and $\| (x, y) \| = \max (|x|, |y|)$. Put $C =$

$$= \{ (x, y) : y = \operatorname{tg} x,\ x \in \langle 0, \pi/2) \} \text{ and let}$$

$$T(x, y) = (\operatorname{arc\,tg}(y+1), (y+1)).$$

Theorem 3. *Let* $T : C \to C$ *be a regular isometry* [$C \subset X$, X *is an arbitrary Banach space*]. *Then for any* $x \in C$

$$\| x - T^n x \| = n \| x - Tx \| \qquad \text{for } n = 1, 2, \ldots$$

Proof: Let $\| T^i x - T^{i+1} x \| = d$, $i = 0, 1, 2, \ldots$. Obviously $\| x - T^n x \| \leqslant nd$. For $n = 1$ our conclusion is true. Suppose for any $y \in C$ we have $\| y - T^k y \| = kd$ for $k = 1, 2, \ldots n$. Then

$$nd = \| Tx - T^{n+1} x \| = \left\| \frac{x + Tx}{2} - \frac{T^{n+1} x + T^n x}{2} \right\| \leqslant 1/2 \| x - T^{n+1} x \| +$$

$$+ 1/2 \| Tx - T^n x \| = 1/2 \| x - T^{n+1} x \| + 1/2 (n-1) d.$$

So
$$\| x - T^{n+1} x \| \geqslant 2nd - (n-1) d = (n+1) d.$$

which ends the proof. It implies that if C is a bounded set, then identity is the unique regular self isometry on C.

Some fixed points theorems. Theorems 2 and 3 allow us to state the following

Theorem 4. *Suppose* C *is a compact set and* $T : C \to C$ *is regularly nonexpansive. Then* T *has a fixed point.*

Proof: It is known that the set $D = \bigcap_{n=1}^{\infty} T^n(C)$ satisfies $T(D) = D$ and because it is compact then T is isometry on D (see [3]). Hence $Tx = x$ for $x \in D$.

This shows that the class of sets having fixed point property for regularly nonexpansive self-mappings is wider than the class of sets having this property for nonexpansive mappings. Examples of noncompact sets of this type can be given too.

Suppose X is a Hilbert space. Then each regularly nonexpansive mapping $T : C \to C$ can be represented as $T = (I + Q)/2$, where $Q : C \to X$ is a nonexpansive mapping. According to [6], [9] Q admits a nonexpansive extension $\tilde{Q}$: Conv $C \to X$ such that $\tilde{Q}$ (Conv C) $\subset$ Conv Q (C). So $\tilde{T} = (I + \tilde{Q}) / 2$: Conv $C \to$ Conv C.

Theorem 5. *Suppose* C *is an arbitrary nonempty set in Hilbert space such that for any* $z \in$ Conv C *there exists* $y \in C$ *(not necessarily unique) such that* $\| z - y \| = \operatorname{dist}(z, C)$. *Then each regularly nonexpansive mapping* $T : C \to C$ *has a fixed point.*

Proof. Let $\tilde{T}$ be the described above extension of T. Then there exists $z \in$ Conv C such that $z = \tilde{T} z$ ([1], [2]). Take y according to the assumptions of our theorem. Then

$$\operatorname{dist}(z, C) \leqslant \| z - Ty \| = \| \tilde{T} z - \tilde{T} y \| \leqslant \| z - y \| = \operatorname{dist}(z, C).$$

Hence $\| \tilde{T} z - \tilde{T} y \| = \| z - y \|$ what implies $z - y = \tilde{T} z - \tilde{T} y = z - Ty$ and $y = Ty$.

Notice that any weakly compact set C satisfies assumptions of the Theorem 5 ; however not every such set has fixed point property for nonexpansive mappings. Also the unit sphere is an example of not weakly compact set satisfying assumptions of Theorem 5.

Finally let us prove the following for arbitrary space X.

Theorem 6. *Suppose C is nonempty, closed and convex. Then for every nonexpansive mapping $T : C \to C$ there exists a regularly nonexpansive mapping $F : C \to C$ with the same set of fixed points (may be empty).*

Proof. To prove it we shall construct the whole family F_α, $\alpha \in (0,1)$ of such mappings. Take $\alpha \in (0,1)$ and for any fixed $x \in C$ find the unique y satisfying $y = (1-\alpha)x + \alpha Ty$.
It is possible by using Banach contraction principle. Denote such y by $F_\alpha x$. So $F_\alpha : C \to C$ and satisfies $F_\alpha x = (1-\alpha)x + \alpha TF_\alpha x$.
Obviously mappings F_α are nonexpansive for each $\alpha \in (0,1)$. Moreover if we take $p = (1-t)x + tF_\alpha x$ with $t \in (0,1)$
then in view of

$$F_\alpha x = (1-\alpha)x + \alpha TF_\alpha x = (1-\alpha)\left(\frac{-t}{1-t}F_\alpha x + \frac{1}{1-t}p\right) + \alpha TF_\alpha x$$

we see that

$$F_\alpha x = \left(1 - \frac{\alpha - t\alpha}{1 - t\alpha}\right)p + \frac{\alpha - t\alpha}{1 - t\alpha}\, TF_\alpha x$$

showing that $F_\alpha x = F_\beta p$ for $\beta = \dfrac{\alpha - t\alpha}{1 - t\alpha}$.

Analogously for $q = (1-t)y + tF_\alpha y$ we have $F_\alpha y = F_\beta q$.

Hence

$$\| F_\alpha x - F_\alpha y \| = \| F_\beta p - F_\beta q \| \leqslant \| p - q \| = \| (1-t)(x-y) + t(F_\alpha x - F_\alpha y) \| .$$

So all F_α are regularly nonexpansive and obviously their fixed points are exactly those of T.

This theorem does not bring much to the problem of existence of fixed points for nonexpansive mappings on convex domains, but it show that the closed and convex set has fixed point property for nonexpansive mappings if and only if it has this property for a narrower class of regularly nonexpansive ones.

Added in proof. We learned that regularly nonexpansive mappings has been already introduced by R. E. Bruck („Nonexpansive projections on subsets of Banach spaces" Pacific J. Math. 48 (1973), p. 341—357) and called „firmly nonexpansive". Especialy Theorem 6. is known.

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