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FREQUENCY-DOMAIN CRITERIA FOR DISSIPATIVITY OF SOME THIRD ORDER DIFFERENTIAL EQUATIONS

BY

ANTHONY UYI AFUWAPE

1. Introduction. In the works of I. Barbalat and A. Halanay [1], I. Barbalat [3] generalisations of the frequency conditions for the dissipativity for general non-linear differential equations are given. Applications of the generalisations were given to certain third order equations with one non-linear function.

We shall consider here the following equations of the third order with two non linear functions:

$$x''' + \alpha x'' + g(x') + h(x) = p(t, x, x', x''), \tag{1.1}$$

$$x''' + \alpha x'' + g_1(x)\, x' + h(x) = p_1(t, x, x', x''), \tag{1.2}$$

where g, g_1 and h are continuous functions and g, h and $\int_0^x g_1(s)\, ds$ are „almost linear" in their arguments. Using the generalised theorem and choosing parameters, we shall that (1.1) and (1.2) are uniformly dissipative.

For completness we shall state the following version of the generalized theorem as given in [1], which will be used.

Theorem (Generalised Yacubovich). *Consider the real system*

$$\frac{dx}{dt} = Ax - B\varphi(\sigma) + p(t, x), \quad \sigma = C^* x. \tag{1.3}$$

A being an nxn stable matrix, B and C are n x m matrices, $\varphi(\sigma) = \mathrm{col}\,(\varphi_j(\sigma_j))$ $(j = 1, 2, \ldots, m)$ *and suppose for* $|\alpha| \geqslant \lambda_0$
$0 \leqslant \alpha\varphi_j(\alpha) \leqslant \mu_0^j \alpha^2$, $-\alpha_1^j \leqslant \varphi_j^1(\alpha) \leqslant \alpha_2^j$, $\mu_0^j \leqslant \alpha_2^j$, $\alpha_1^j \geqslant 0$; $|p(t, x)| < \rho_0$
for all (t, x).

If there exists diagonal matrices $D_1 > 0$, $D_2, D_3 \geqslant 0$, *such that*

$$\{D_1 + \mathrm{Re}\,[[D_1\, \mathrm{diag}\,(\mu_0^j) + i\omega D_2]\, G(i\omega)] + \omega^2 [\mathrm{Re}\,[D_3(I + \mathrm{diag}\,(\alpha_2^j - \alpha_1^j)\, G(i\omega))] - G^*(-i\omega)\, D_3\, \mathrm{diag}\,(\alpha_1^j \alpha_2^j)\, G(i\omega)] > 0 \tag{1.4}$$

for all $\omega \in \mathcal{R}$, *where I is the unit n x n matrix,* $G(i\omega) = C^*(i\omega I - A)^{-1} B$.
Suppose further that

$$\lim_{|\lambda| \to \infty} D_2 \frac{1}{\lambda^2}\, \mathrm{diag}\left[\int_0^\lambda \varphi_j(\alpha)\, d\alpha - \tfrac{1}{2}\lambda\varphi(\lambda)\right] \geqslant 0; \tag{1.5}$$

then the system is uniformly dissipative.

The following definition will also be useful:

Definition: *Under the frequency-domain criteria (1.4) we shall say that the system (1.3) has the dual system*

$$\frac{dx}{dt} = A^*x - C\varphi(\tilde{\sigma}) + \tilde{p}(t, x), \quad \tilde{\sigma} = B^*x \tag{1.6}$$

where A^*, B^* are complex conjugates of matrices A and B and C is given above.

It must be remarked that under the frequency-domain criteria if a system satisfies the frequency-domain criteria, so is also the dual (see [4; pp. 166–167]).

2. Main Results. Theorem 1. *Suppose we have in* (1.1)

$$\alpha > 0, \text{ and} \tag{2.1}$$

$$|p(t, x, x', x'')| \leqslant \rho_0. \tag{2.2}$$

If there exists strictly positive numbers b and c such that $\alpha b > c$ and positive numbers μ_1, μ_2 such that for $|z| \geqslant \lambda_0$

$$b \leqslant \frac{g(z)}{z} \leqslant b + \mu_2;\ b \leqslant g'(z) \leqslant b + \mu_1, \tag{2.3}$$

$$c \leqslant \frac{h(z)}{z} \leqslant c + \mu_2;\ c \leqslant h'(z) \leqslant c + \mu_2, \tag{2.4}$$

then the equation (1.1) is uniformly dissipative.

Proof: Let $g(z) = bz + \hat{g}(z)$ and $h(z) = cz + \hat{h}(z)$.
For convenience we shall use the following equivalent, to (1.1) system

$$\begin{aligned} x_1' &= -bx_2 - cx_3 - \hat{g}(x_2) - \hat{h}(x_3) + \hat{p}(t, x_1, x_2, x_3), \\ x_2' &= x_1 - \alpha x_2, \quad x_3' = x_2 \end{aligned} \tag{2.5}$$

where $\hat{p}$ is defined by $\hat{p}(t, x_1, x_2, x_3) = p(t, x_3'' + \alpha x_3', x_2, x_3)$. In utilising the theorem in section 1, let

$$D_1 = \begin{pmatrix} \frac{\tau_1}{\mu_1} & 0 \\ 0 & \frac{\tau_1}{\mu_2} \end{pmatrix}; \quad D_2 = \begin{pmatrix} \frac{s}{\tau_1} & 0 \\ 0 & \frac{t}{\tau_2} \end{pmatrix},$$
$$D_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \operatorname{diag}(\mu_0^j) = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \tag{2.6}$$

where $\tau_1 > 0$, $\tau_2 > 0$ and s and t are parameters to be determined. From (25) we have

$$A = \begin{pmatrix} 0 & -b & -c \\ 1 & -a & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}; \tag{2.7}$$

$$C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad p(t, x) = \begin{pmatrix} \hat{p}(t, x_1, x_2, x_3) \\ 0 \\ 0 \end{pmatrix}$$

and $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$; thus giving $\sigma = C^*x = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$ and $\varphi(\sigma) = \begin{pmatrix} \hat{g}(x_2) \\ \hat{h}(x_3) \end{pmatrix}$.

Now

$$G(i\omega) = C^*(i\omega I - A)^{-1}B = \begin{pmatrix} \frac{i\omega}{\Delta} & \frac{i\omega}{\Delta} \\ \frac{1}{\Delta} & \frac{1}{\Delta} \end{pmatrix}$$

where Δ is the determinant of $(i\omega I - A)$ given by $\Delta \equiv (c - \alpha\omega^2) + i\omega(b - \omega^2)$.
The condition (1.4) becomes

$$\pi(\omega) > 0 \tag{2.8}$$

where $\pi(\omega) = (\pi_1(\omega), \pi_2(\omega))$

$$\text{with } \pi_1(\omega) = \begin{pmatrix} \frac{\tau_1}{\mu_1} - \frac{\omega^2}{|\Delta|^2}[s_1\tau_1(c - \alpha\omega^2) + \tau_1(b - \omega^2)] \\ \frac{1}{2}\left[\left(\frac{\tau_2}{\Delta} - \frac{\omega^2 s_1\tau_1}{\bar{\Delta}}\right) - i\omega\left(\frac{\tau_1}{\bar{\Delta}} - \frac{t_1\tau_2}{\Delta}\right)\right] \end{pmatrix},$$

$$\text{and } \pi_2(\omega) = \begin{pmatrix} \frac{1}{2}\left[\left(\frac{\tau_2}{\bar{\Delta}} - \frac{\omega^2 s_1\tau_1}{\Delta}\right) + i\omega\left(\frac{\tau_1}{\Delta} - \frac{t_1\tau_2}{\bar{\Delta}}\right)\right] \\ \frac{\tau_2}{\mu_2} + \frac{\tau_2}{|\Delta|^2}[(c - \alpha\omega^2) + \omega^2 t_1(b - \omega^2)] \end{pmatrix},$$

on putting $s_1 = s/\tau_1$ and $t_1 = t/\tau_2$.

In order that (2.8) be satisfied, the first element of $\pi_1(\omega)$ and the determinant of $\pi(\omega)$ must be positive.

Choose $s_1 \in (1/\alpha, b/c)$ and

$$\frac{1}{\mu_1} > \frac{\omega^2[\omega^2(1 - \alpha s_1) + (s_1 c - b)]}{(c - \alpha\omega^2)^2 + (b - \omega^2)^2\omega^2} \tag{2.9}$$

The first element of $\pi_1(\omega)$ gives

$$\frac{\sigma_1}{\mu_1|\Delta|^2}\{\omega^6 + \omega^4[\alpha^2 - 2b - \mu_1(1 - \alpha s_1)] + \omega^2[b^2 - 2\alpha c - \mu_1(s_1 c - b)] + c^2\}$$

which is positive for all $\omega \in \mathcal{R}$.

In addition to (2.9), choose $t_1 > \alpha/b$ and

$$\frac{1}{\mu^2} > \frac{\omega^4 t_1 + \omega^2(\alpha - bt_1) - c}{(c - \alpha\omega^2)^2 + \omega^2(b - \omega^2)^2} \tag{2.10}$$

The determinant of $\pi(\omega)$ will be positive if

$$\frac{1}{|\Delta|^2}\Big\{\omega^4\Big[\mu_1(\alpha s_1 - 1) - \mu_2 t_1 - \frac{\mu_1\mu_2 s_1^2 \lambda}{4}\Big] + \omega^2\Big[\mu_2(bt_1 - \alpha) + \mu_1(b - s_1 c) - \frac{\mu_1\mu_2}{4\lambda}(\lambda - t_1)^2 - \frac{s_1\mu_1\mu_2}{2}\Big] + \Big[\mu_2 c - \frac{\mu_1\mu_2}{4\lambda}\Big]\Big\} + 1 > 0 \tag{2.11}$$

where $\lambda = \tau_1/\tau_2$ (2.11) will be true if

$$\frac{\mu_1}{4c} < \lambda < \frac{4[\mu_1(\alpha s_1 - 1) - \mu_2 t_1}{\mu_1\mu_2 s_1^2}. \tag{2.12}$$

Hence the conclusion follows.

Theorem 2. *Consider the system (1.2).*

Suppose

$$\alpha > 0 \text{ and} \tag{2.13}$$

$$|p_1(t, x, x', x'')| \leqslant \rho_0. \tag{2.14}$$

If there exists strictly positive numbers μ_1, μ_2 such that for $|x| \geqslant \lambda_0$

$$b \leqslant \frac{\int_0^x g_1(s)\,ds}{x} \leqslant b + \mu_1; \tag{2.15}$$

$$c \leqslant \frac{h(x)}{x} \leqslant c + \mu_2;\ c \leqslant h'(x) \leqslant c + \mu_2, \tag{2.16}$$

then the equation (1.2) *is uniformly dissipative.*

Proof. It is enough to show that (1.2) is dual to (1.1).

Let $\int_0^x g_1(s)\,ds = bx + \hat{g}_1(x)$ and $h_1(x) = cx + \hat{h}_1(x)$. Choose an equivalent system to (1.2) as

$$x_1' = x_2,\ x_2' = -bx_1 - \alpha x_2 + x_3 - \hat{g}_1(x_1),\ x_3' = -cx_1 - \hat{h}_1(x_1) + \hat{p}_1(t, x_1, x_2, x_3), \tag{2.17}$$

where $\hat{p}_1(t, x_1, x_2, x_3) = p_1(t, x_1, x_2, x_3 - bx_1 - \alpha x_2 - \hat{g}_1(x_1))$.

From (1.17) we have (1.2) in the form (1.3) with

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ -b & -\alpha & 1 \\ -c & 0 & 0 \end{pmatrix};\quad B_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix};\quad C_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, hence $\sigma_1 = C_1{}^* x = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}$ and $\varphi(\sigma_1) = \begin{pmatrix} \hat{g}_1(x_1) \\ \hat{h}_1(x) \end{pmatrix}$.

Comparing with (2.7), we have $A_1 = A^*$, $B_1 = C$, $C_1 = B$. This shows that (1.2) is the dual of (1.1) and theorem 2 follows from [4].

Remarks. The result of theorem 2 compares with the boundedness results of Ezeilo [5] and Swick [6] wich were shown by specific constructions of unbounded positive definite Lyapunov functions, with negative definite derivatives with respect to t.

In [5], Ezeilo used the function

$$2V = 2\alpha\int_0^x h(s)\,ds + 25\int_0^x G_1(s)\,ds + G_1^2(x_1) - 2\,\delta x_1 G_1(x_1) + \delta x_2^2 + x_3^2 + 2x_2 h(x_1) + 2[G_1(x_1) - \delta x_1]\,x_3 - 2D_6 x_2 \Psi_d(x_3 + G_1(x_1) - \delta x_1) + D_7 x_2 \Psi_{2\Delta}(x_1)$$

where $D_6^* = D_6 + A_1$, D_7 are constants, while in [6], Swick used the function

$$V = \int_0^x [\alpha G_1(s) - h(s)]\,ds + q_0(x_1, x_2, x_3)$$

with q_0 a quadratic form in x_1, x_2, x_3 and $G_1(x) = \int_0^x g_1(s)\,ds$ in each case.

These Lyapunov functions go in the line of the theory of Frequency — domain criteria which though narrow in its scope of applicability, remains very powerful and an alternative approach when the Lyapunov function in mind is a quadratic form plus integrals of the non-linear parts.

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Department of Mathematics
University of Ife
Ile-Ife, Nigeria