6. Kobayashi S. — *Foundations of Differential Geometry* I, Interscience Publisher, New-York-London, 1963.

7. Palais R. S. — *Seminar on the Atiah-Singer index theorem*. Princeton Univ. Press 1965.

8. Penot J-P. — *Connexion linéaire déduite d'une famille de connexions linéaires par un foncteur multilinéaire*. C. R. Acad. Sci. Paris, T. 268, S. A., p. 100—103.

9. Plymen R. J. — *Spinors in Hilbert Spaces*. Math. Proc. Cambridge Phil. Soc. 80 (1976), p. 337—347.

10. Popovici I. — *Représentation irréductibles des fibrés de Clifford*. Ann. Inst. Henri Poincaré, S. A. T. XXV, 1976, p. 35—58.

Reçu le 8.II.1978

Faculté de Mathématiques
Université de Jassy
R. S. România

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXV, s. I a, 1979, f. 2

ON THE HOLOMORPHICALLY SUBPLANAR CURVES

BY

LIVIU NICOLESCU

We define the subcharacteristic vector fields of the deformation algebra of two affine connections. The geometrical interpretation of the subcharacteristic directions fields is given with the help of the holomorphically subplanar curves.

Introduction. The manifolds, mappings, tensor fields and connections about which we discuss are always assumed to be C^∞.

Let M be a n-dimensional manifold ($n > 4$). We denote by $\mathcal{T}^r_s(M)$ the set of all tensor fields of type (r, s) in M and by $\mathcal{F}(M)$ the ring of all real functions on M. The set $\mathcal{T}^r_s(M)$ is a module over $\mathcal{F}(M)$. Particularly for $\mathcal{T}^1_0(M)$ and $\mathcal{T}^0_1(M)$ we shall use the notation $\mathcal{X}(M)$ and $\Lambda(M)$, respectively.

Holomorphically subplanar curves. We consider a tensor field $F \in \mathcal{T}^1_1(M)$ on a differentiable manifold M ($\dim M = n > 4$) such that $F^2 = -\delta$, where $\delta \in \mathcal{T}^1_1(M)$ is the Kronecker tensor. Let $K \in \mathcal{X}(M)$ and ∇ be a symmetric F-connection [1].

We consider a curve $C : J \to M$, where $J \subseteq R$ and let $\dot{C}(t)$ be the tangent vector field along the curve. We consider a regular point $p = C(t_0)$. We suppose that the vectors $\dot{C}(t_0)$, $F_p(\dot{C}(t_0))$, K_p and $F_p(K_p)$ are independent.

Definition 1. *The point $p = C(t_0)$ is called a (F, K, ∇)-holomorphically subplanar point if the vector $(\nabla_{\dot{C}}\dot{C})(t_0)$ belongs to the subspace generated by vectors $\dot{C}(t_0)$, $F_p(\dot{C}(t_0))$, K_p, $F_p(K_p)$, i.e. if there exist four numbers $f_p, g_p, h_p, e_p \in R$ such that*

$$(1) \qquad (\nabla_{\dot{C}}\dot{C})(t_0) = f_p\,\dot{C}(t_0) + g_p F_p(\dot{C}(t_0)) + h_p K_p + e_p F_p(K_p).$$

Definition 2. *Let $C : J \to M$ ($J \subseteq R$) be a regular curve such that for every $t \in J$, the vectors $\dot{C}(t)$, $F_{C(t)}(\dot{C}(t))$, $K_{C(t)}$ and $F_{C(t)}(K_{C(t)})$ are independent. The curve C is called a (F, K, ∇)-holomorphically subplanar curve, if all the points of C are (F, K, ∇)-holomorphically subplanar points.*

Let Γ^i_{jk}, F^i_j and K^i be the components of ∇, F and K, respectively, considered in a system of local coordinates. A curve $x^i = x^i(t)$ is a $(F,$

K, ∇)-holomorphically subplanar curve, if it satisfies the differential equations :

$$d^2 x^i/dt^2 \quad (2) \qquad \frac{d^2 x^i}{d t^2} + \Gamma^i_{jk} \frac{d x^j}{d t}\frac{d x^k}{d t} = f \frac{d x^i}{d t} + g F^i_j \frac{d x^j}{d t} + h K^i + e F^i_j K^j,$$

where f, g, h and e are certain functions of the parameter t.

Theorem of Yamagucki and Sato [5]. Let $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ be two symmetric F-connections in an almost complex manifold M. Let $K \in \mathcal{X}(M)$. The following properties are equivalent :

(i) The $(F, K, \overset{1}{\nabla})$-holomorphically subplanar curves and the $(F, K, \overset{2}{\nabla})$-holomorphically subplanar curves are the same;

(ii) There exist $\rho \in \Lambda(M)$ and $p \in \mathcal{T}^0_2(M)$, $p(X, Y) = p(Y, X)$, $p(X, F(Y)) = p(Y, F(X))$, $(\forall) X, Y \in \mathcal{X}(M)$ such that:

$$(3) \qquad \overset{2}{\nabla}_X Y = \overset{1}{\nabla}_X Y + 2\rho(X) Y + 2\rho(Y) X + 2\tilde{\rho}(X) F(Y) + 2\tilde{\rho}(Y) F(X) + p(X, Y) K - p(X, F(Y)) \tilde{K},$$

where we have put $\tilde{K} = F(K)$ and $\tilde{\rho}(X) = -\rho(F(X))$, $(\forall) X \in \mathcal{X}(M)$.

Subcharacteristic curve. Let $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ be two symmetric F-connections in an almost complex manifold M. Let $A = \overset{2}{\nabla} - \overset{1}{\nabla} \in \mathcal{T}^1_2(M)$. If we define the product of two vector fields X and Y as the vector field $X * Y = A(X, Y)$, we can easily verify that the module $\mathcal{X}(M)$ becomes an algebra over the ring $\mathcal{F}(M)$. This algebra is called the deformation algebra of the paire of connections $(\overset{1}{\nabla}, \overset{2}{\nabla})$, and is denoted by $\mathcal{U}(M, \overset{1}{\nabla}, \overset{2}{\nabla})$ [2], [4]. Let be $K \in \mathcal{X}(M)$.

Definition 3. *An element $X \in \mathcal{U}(M, \overset{1}{\nabla}, \overset{2}{\nabla})$ is called a (F, K)-subcharacteristic vector field if there exist four functions $\lambda, \mu, \nu, \eta \in \mathcal{F}(M)$ such that:*

$$(4) \qquad A(X, X) = \lambda X + \mu F(X) + \nu K + \eta F(K).$$

Theorem 4. *Let $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ be two symmetric F-connections in an almost complex manifold M. Let be $K \in \mathcal{X}(M)$. The following properties are equivalent:*

i) *All the elements of $\mathcal{U}(M, \overset{1}{\nabla}, \overset{2}{\nabla})$ are (F, K)-subcharacteristic vector fields.*

(ii) *There exist $\rho \in \Lambda(M)$ and $p \in \mathcal{T}^0_2(M)$, $p(X, Y) = p(Y, X)$, $p(X, F(Y)) = p(Y, F(X))$, $(\forall) X, Y \in \mathcal{X}(M)$, such that:*

$$\overset{2}{\nabla}_X Y = \overset{1}{\nabla}_X Y + 2\rho(X) Y + 2\rho(Y) X + 2\tilde{\rho}(X) F(Y) + 2\tilde{\rho}(Y) F(X) + p(X, Y) K - p(X, F(Y)) \tilde{K},$$

where we have put $\tilde{K} = F(K)$ and $\tilde{\rho}(X) = -\rho(F(X))$, $(\forall) X \in \mathcal{X}(M)$.

Proof. The proof is symilar to the proof of the Yamaguchi and Sato theorem ([5] p. 234).

Definition 5. *An element $X \in \mathcal{U}(M, \overset{1}{\nabla}, \overset{2}{\nabla})$ is called (F, K)-subcharacteristic directions vector field if:*

(ds_1) *X is (F, K)-subcharacteristic vector field,*

(de_2) *$(\forall)$ $p \in M$, the vectors $X_p, F_p(X_p), K_p, F_p(K_p)$ are independent.*

Theorem 6. *Let $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ be two symmetric F-connections in an almost complex manifold M ($\dim M = n > 4$). Let $K \in \mathcal{X}(M)$. Let $X \in \mathcal{U}(M, \overset{1}{\nabla}, \overset{2}{\nabla})$ such that the vectors X_p, $F_p(X_p)$, K_p and $F_p(X_p)$ are independent, $(\forall) p \in M$. The following properties are equivalent:*

(i) *X is a (F, K)-subcharacteristic directions vector field,*

(ii) *there exist four numbers $\lambda_p, \mu_p, \nu_p, \eta_p \in R$ for every point $p \in M$, such that:*

$$A_p(X_p, X_p) = \lambda_p X_p + \mu_p F_p(X_p) + \nu_p K_p + \eta_p F_p(K_p),$$

where $A = \overset{2}{\nabla} - \overset{1}{\nabla}$.

(iii) *for every point $p \in M$ if $C : J \to M$ is a curve satisfying the conditions:*

(iii_1) *$C(t_0) = p$, $\dot{C}(t_0) = a X_p$, $a \in R - \{0\}$,*

(iii_2) *p is a $(F, K, \overset{r}{\nabla})$-holomorphically subplanar point, then the point p of the curve C is a $(F, K, \overset{s}{\nabla})$-holomorphically subplanar point where $r, s \in \{1, 2\}$ and $r \neq s$.*

Proof. The proof is symilar to the proof of the theorem 4 ([3], p. 311).

Definition 7. *The trajectories of a (F, K)-subcharacteristic directions vector field are called (F, K)-subcharacteristic curves.*

Let A^i_{jk}, F^i_j and K^i be the components of A, F and K, respectivelly, considered in a system of local coordinates. A curve $x^i = x^i(t)$ is a (F, K)-subcharacteristic curve if it satisfies the equations:

$$(5) \qquad A^i_{jk} \frac{d x^j}{d t}\frac{d x^k}{d t} = \lambda \frac{d x^i}{d t} + \mu F^i_j \frac{d x^j}{d t} + \nu K^i + \eta F^i_j K^j,$$

where λ, μ, ν and η are functions of x^i and $d x^i/d t$.

Theorem 8. *The (F, K)-subcharacteristic curves are undeterminated if and only if*

$$A^h_{ij} = 2\rho_{(i}\delta^h_{j)} + 2\tilde{\rho}_{(i}F^h_{j)} + p_{ji} K^h - p_{jr} F^r_i \tilde{K}^h$$

holds for a certain 1-form ρ and a certain symmetric covariant tensor field p such that $p_{jr} F^r_i = p_{ir} F^r_j$, where we have put $\tilde{K}^i = F^i_j K^j$ and $\tilde{\rho}_i = F^j_i \rho_j$

Proof. The proof. is symilar to the proof of the Yamaguchi and Sato theorem ([4], p. 234).

BIBLIOGRAPHY

1. Cruceanu, V. et Miron, R. — *Sur les couples de connexions compatibles avec les structures presque complexes.* An. şt. Univ. „Al. I. Cuza" Iaşi, Sect. I, Tomul XII, Fasc. 1 (1967), p. 78—88.
2. Martin, M. and Nicolescu, L. — *On the* (Δ, F, φ) *subpaths.* Bull. Math. de la Soc. Sci. Math. de la R.S.R. (1979), to appear.
3. Nicolescu, L. — *Des courbes subcaractéristiques sur une variété presque complexe.* An. şt. Univ. Iaşi, XXIII, Fasc. 2 (1977), 309—313.
4. Vaisman, I. — *Sur quelques formules du calcul de Ricci global.* Comm. Math. Helv. 41 (1966—1967), 73—87.
5. Yamaguchi, S. and Sato, S. — *On holomorphically subplanar curves in almost complex manifolds.* Tensor N.S. Vol. 32 (1978), 231—242.

Received 22.XI.1978

Faculty of Mathematics,
University of Bucharest,
Str. Academiei 14,
7000 Bucureşti
R. S. România

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXV, s. I a, 1979, f. 2

DIRECTIONS CARACTÉRISTIQUES ASSOCIÉES À UNE PAIRE DE CONNEXIONS LINÉAIRES SUR UNE VARIÉTÉ DE BANACH

PAR

MIHAI ANASTASIEI et LIVIU NICOLESCU

Les directions caractéristiques et leurs trajectoires, nommées courbes caractéristiques, ont un rôle important dans l'étude des correspondences entre deux espaces affines ou projectives (voir la bibliographie de [3]). Dans le §1 de cette Note nous associons des directions caractéristiques à une paire de connexions linéaires sur une variété de Banach. L'existence des directions caractéristiques en un point de la variété est liée à l'étude des valeurs et vecteurs propres pour les opérateurs quadratiques.

En employant les résultats de [1] nous obtenons, dans le §2, les conditions suffisantes pour l'existence des directions caractéristiques en un point de la variété.

1. Soit M une variété différentiable de classe C^∞ modelée sur un espace de Banach $\mathbf{M}$ et soit $T: TM \to M$ son fibré tangent.

Suppons que M admet des partitions de l'unité différentiables de classe C^∞. Notons par $\mathcal{F}(M)$ l'anneau des fonctions réelles de classe C^∞ sur M et par $\mathcal{X}(M)$ le $\mathcal{F}(M)$-module des champs de vecteurs sur M. Soit (U, φ) une carte locale de M. Pour un champ $X \in \mathcal{X}(M)$ nous notons par $X_\varphi : \varphi(U) \to \mathbf{M}$ sa partie principale et $X_{\varphi(p)} = X_\varphi(p)$ où $p \in U$.

Si nous considérons une autre carte locale (V, ψ) de M, telle que $U \cap V \neq \emptyset$, les applications de transitions de T sont données par

$$(1.1) \qquad D(\psi \circ \varphi^{-1}) : \varphi(U \cap V) \to L(\mathbf{M}),$$

où $L(\mathbf{M})$ est l'espace des opérateurs linéaires sur $\mathbf{M}$ et D est le symbole de différentiation de Fréchet.

Définition 1.1 [2, p. 2]. *On appelle connexion linéaire sur M une application $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$, notée $\nabla(X, Y) = \nabla_X Y)$, $(\forall)$ $X, Y \in \mathcal{X}(M)$, telle que pour toute carte locale (U, φ) de M il existe une application de classe C^∞, $\Gamma_\varphi : \varphi(U) \to L^2(\mathbf{M}; \mathbf{M})$ tel que $\nabla_X Y$ a l'expression locale :*

$$(1.2) \qquad \nabla_X Y\,|_{\varphi(p)} = DY\,|_{\varphi(p)}(X_{\varphi(p)}) + \Gamma_{\varphi(p)}(X_{\varphi(p)}, Y_{\varphi(p)}),$$

où $\Gamma_{\varphi(p)} = \Gamma_\varphi(p)$,

Γ_φ correspond aux symboles de Christoffel.