alors la paire (∇^1, ∇^2) admet les directions caractéristiques. Si ∇^2 est une connexion constante, c'est-à-dire l'application $v \to \overset{2}{\Gamma}_{\varphi(v)}(v, v)$ est une application constante non nulle, la paire (∇^1, ∇^2) admet un continu de directions caractéristiques.

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CR-SUBMANIFOLDS OF GENERALIZED COMPLEX SPACE FORMS

BY

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§ **1. Introduction.** Let M be a Riemannian submanifold immersed in $\overline{M}$. At each point $p \in M$, let $\mathfrak{D}_p$ be the maximal holomorphic subspace of the tangent space T_pM, i.e. $J\mathfrak{D}_p = \mathfrak{D}_p$. If the dimension of $\mathfrak{D}_p$ is the same for all $p \in M$, we have a holomorphic distribution $\mathfrak{D}$ on M.

Recently in [2], A. B e j a n c u introduced the notion of a CR-submanifold of $\overline{M}$. M is a CR-submanifold of an almost Hermitian manifold $\overline{M}$ if there exists on M a C^∞ holomorphic distribution $\mathfrak{D}$, such that its orthogonal complement $\mathfrak{D}^\perp$ is totally real in $\overline{M}$ (6), i.e. $J\mathfrak{D}^\perp_p \subset T^\perp_p M$, where $T^\perp_p M$ is the normal space at p.

In the present paper, we generalized some results obteined in [2], [3] and [4] for CR-submanifolds of a generalized complex space form. In §3 we give a characterization of CR-submanifolds of generalized complex space forms in terms of the restriction of the curvature. In §§ 4 and 5, we study different kinds of sectional curvature, Ricci tensor and scalar curvature of a CR-submanifold of a generalized complex space form. Finally in §6, totally umbilical CR-submanifolds are studied in detail.

§2. **Preliminaries.** Let M be a Riemannian submanifold in an almost Hermitian manifold $\overline{M}$. We denote by ∇ (respectively, $\overline{\nabla}$) covariant differentiation with respect to the metric on M (respectively, on $\overline{M}$). The curvature tensor R of ∇ is given by

$$R(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X, Y]}$$

and we denote by $\overline{R}$ the curvature tensor with respect to $\overline{\nabla}$. We write

$$R(X, Y, Z, W) = g(R(X, Y)Z, W).$$

Moreover the sectional curvature of M is

$$K_M(X \wedge Y) = -\frac{R(X, Y, X, Y)}{g(X, X)\, g(Y, Y) - g(X, Y)^2}. \tag{2.1}$$

The second fundamental form σ is given by

$$\sigma(X, Y) = \overline{\nabla}_X Y - \nabla_X Y \tag{2.2}$$

for vector fields X, Y tangent to M. For a normal vector field ξ on M, we have

$$\overline{\nabla}_X \xi = - A_\xi X + \nabla^{\perp}_X \xi, \tag{2.3}$$

where $- A_\xi X$ (respectively, $\nabla^{\perp}_X \xi$) is the tangencial (respectively, normal) component of $\overline{\nabla}_X \xi$. Moreover

$$g(\sigma(X, Y), \xi) = g(A_\xi X, Y). \tag{2.4}$$

For the second fundamental form σ, we define the covariant differentiation $\tilde{\nabla}$ with respect to the connection in $(TM) \oplus (T^{\perp}M)$ by

$$(\tilde{\nabla}_X \sigma)(Y, Z) = \nabla^{\perp}_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{2.5}$$

for all X, Y, Z tangent to M. The equations of Gauss and Codazzi are then given respectively by

$$R(X, Y, Z, W) = \overline{R}(X, Y, Z, W) + g(\sigma(X, W), \sigma(Y, Z)) - - g(\sigma(X, Z), \sigma(Y, W)), \tag{2.6}$$

$$(\overline{R}(X, Y)Z)^{\perp} = (\tilde{\nabla}_X \sigma)(Y, Z) - (\tilde{\nabla} Y \sigma)(X, Z) \tag{2.7}$$

where $\perp$ in (2.7) denotes the normal component.

A RK-manifold $\overline{M}$ (9) is an almost Hermitian manifold whose Riemannian curvature tensor verified

$$\overline{R}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \overline{R}(J\overline{X}, J\overline{Y}, J\overline{Z}, J\overline{W}),$$

where $\overline{X}, \overline{Y}, \overline{Z}$ and $\overline{W}$ are vector fields tangent to $\overline{M}$.

It is known (9) that if $\overline{M}$ is a RK-manifold, then $\overline{M}$ has (pointwise) constant type, if and only if there exists a C^∞ function α such that

$$\lambda(\overline{X}, \overline{Y}) = \alpha\{g(\overline{X}, \overline{X}) g(\overline{Y}, \overline{Y}) - g(\overline{X}, \overline{Y})^2 - g(J\overline{X}, \overline{Y})^2\}, \tag{2.8}$$

where $\lambda(\overline{X}, \overline{Y}) = \overline{R}(\overline{X}, \overline{Y}, J\overline{X}, J\overline{Y}) - \overline{R}(\overline{X}, \overline{Y}, \overline{X}, \overline{Y})$, for all $\overline{X}, \overline{Y}$ vector fields tangent to $\overline{M}$. Furthermore, $\overline{M}$ has global constant type if and only if (2.8) holds with a constant function α. This α is called the constant type of $\overline{M}$.

Among the Kähler manifolds, those, of constant holomorphic sectional curvature are of particular interest and are called complex space forms. Such manifolds belong to the class formed by the RK-manifolds of constant holomorphic sectional curvature μ and constant type α. We call these manifolds *generalized complex space forms*. Let $\overline{M}(\mu, \alpha)$ be a generalized complex space form of constant holomorphic sectional curvature μ and constant type α. Then we have

$$\overline{R}(\overline{X}, \overline{Y})\overline{Z} = \frac{\mu + 3\alpha}{4}\{g(\overline{Y}, \overline{Z})\overline{X} - g(\overline{X}, \overline{Z})\overline{Y}\} + \frac{\mu - \alpha}{4}\{g(\overline{X}, J\overline{Z})J\overline{Y} - g(\overline{Y}, J\overline{Z})J\overline{X} + 2g(\overline{X}, J\overline{Y})J\overline{Z}\}, \tag{2.9}$$

where $\overline{X}, \overline{Y}$ and $\overline{Z}$ are vector fields tangent to $\overline{M}$.

Let M be a CR-submanifold of an almost Hermitian manifold $\overline{M}$. The distribution $\mathfrak{D}$ (respectively, $\mathfrak{D}^{\perp}$) can be defined by a projector P (respectively, Q) which satisfy the well-known conditions

$$P^2 = P, Q^2 = Q, \ PQ = QP = 0, g \circ (P \times Q) = 0. \tag{2.10}$$

Let ξ be any normal vector field on M. We put

$$J\xi = b\xi + c\xi \tag{2.11}$$

where $b\xi$ (respectively, $c\xi$), is the $\mathfrak{D}^{\perp}$-component (respectively, the normal, component) of $J\xi$. It is known (2) that $c^3 + c = 0$ on the normal bundle, that is the structure introduced by K. Yano [11].

§3. Characterization of CR-submanifolds. We give the following characterization of CR-submanifolds in a generalized complex space form in terms of curvature tensor of the ambient space.

A proper CR-submanifold M of an almost Hermitian manifold $\overline{M}$ is a CR-submanifold with both distributions $\mathfrak{D}$ and $\mathfrak{D}^{\perp}$ of non-null dimensions.

Theorem 3.1. *Let M be a submanifold immersed in $\overline{M}(\mu, \alpha)$ with $\mu \neq \alpha$. Then M is a proper CR-submanifold if and only if the maximal holomorphic subspaces $\mathfrak{D}_p = T_pM \cap J(T_pM)$, $p \in M$, define a nontrivial differentiable distribution $\mathfrak{D}$ on M such that*

$$\overline{R}(\mathfrak{D}, \mathfrak{D}, \mathfrak{D}^{\perp}, \mathfrak{D}^{\perp}) = 0 \tag{3.1}$$

where $\mathfrak{D}^{\perp}$ denotes the orthogonally complementary distribution of $\mathfrak{D}$ in M.

Proof. If M is a proper CR-submanifold for any $X, Y \in \mathfrak{D}$, and $Z \in \mathfrak{D}^{\perp}$, (2.9) gives $\overline{R}(X, Y)Z = [(\mu - \alpha)/2]\, g(X, JY)JZ$. Because JZ is normal to M for any $Z \in \mathfrak{D}^{\perp}$, we obtain (3.1).

Conversely, if the maximal holomorphic subspaces $\mathfrak{D}_p = T_pM \cap J(T_pM)$, $p \in M$, define a nontrivial differentiable distribution $\mathfrak{D}$ such that (3.1) holds, then (2.9) implies that $0 = \overline{R}(JX, X, Z, W) = [(\mu - \alpha)/2]\, g(X, X)\, g(JZ, W)$ for all $X \in \mathfrak{D}$ and $Z, W \in \mathfrak{D}^{\perp}$. From this we see that $J\mathfrak{D}^{\perp}_p$ is perpendicular to $\mathfrak{D}^{\perp}_p$. Since $\mathfrak{D}$ is holomorphic, $J\mathfrak{D}^{\perp}_p$ is also perpendicular to $\mathfrak{D}_p$. Therefore $J\mathfrak{D}^{\perp}_p \subset T^{\perp}_pM$. This shows that M is CR-submanifold.

A submanifold M of an almost Hermitian manifold $\overline{M}$ is said to be *invariant* if we have

$$(\overline{R}(X, Y))\, T_pM \subset T_pM$$

for all $X, Y \in T_pM$, $p \in M$.

Theorem 3.2. *Let M be a proper CR-submanifold immersed in $\overline{M}(\mu, \alpha)$. Then M is an invariant submanifold if and only if $\mu = \alpha$.*

Proof. For any $X, Y \in \mathfrak{D}$ and $Z \in \mathfrak{D}^{\perp}$, from (2.9) we have $\overline{R}(X, Y)Z = [(\mu - \alpha)/2]\, g(X, JY)\, JZ$. If M is invariant, $\overline{R}(X, Y)Z$ is tangent to M. Because JZ is normal to M, we obtain $\mu = \alpha$.

Conversely, if $\mu = \alpha$, then from (2.9), we have $\overline{R}(X, Y)Z = \mu\{g(Y, Z)X - g(X, Z)Y\}$, for any X, Y and Z tangent to M, giving the result.

§4. Sectional curvature of a CR-submanifold. If $\overline{M}(\mu, \alpha)$ is a generalized complex space form, then (2.6) and (2.9) give

$$R(X, Y, Z, W) = \frac{\mu + 3\alpha}{4}\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} +$$

$$(4.1) \quad + \frac{\mu - \alpha}{4}\{g(X, JPW)g(Y, JPZ) - g(X, JPZ)g(Y, JPW) -$$

$$- 2g(X, JPY)g(Z, JPW)\} - g(\sigma(X, Z), \sigma(Y, W)) + g(\sigma(X, W), \sigma(Y, Z)),$$

where X, Y, Z and W are vector fields tangent to M.

The sectional curvature K_M determined by orthonormal vector X and Y is given by

$$(4.2) \quad K_M(X \wedge Y) = \frac{\mu + 3\alpha}{4} + \frac{3\mu - \alpha}{4} g(PX, JPY)^2 + g(\sigma(X, X), \sigma(Y, Y)) - g(\sigma(X, Y), \sigma(X, Y)).$$

The holomorphic sectional curvature H of M determined by a unit vector X, $H(X) = K_M(X \wedge JX)$ is given by

$$(4.3) \quad H(X) = \mu + g(\sigma(X, X), \sigma(JX, JX)) - g(\sigma(X, JX), \sigma(X, JX)).$$

If $\overline{M}(\mu, \alpha)$ is a nearly Kaehler manifold, i.e. $(\overline{\nabla}_{\overline{X}} J)\overline{X} = 0$ for all $\overline{X}$ tangent to $\overline{M}$, then from (2.2) and (2.3) we have

$$(4.4) \quad \sigma(X, JX) = JQ\nabla_X X + c\sigma(X, X)$$

for any vector vector field on M which lie in $\mathfrak{D}$. As a consequence of (4.4) we obtain

$$\sigma(JX, JX) = JQ\nabla_{JX} X + c^2\sigma(X, X).$$

Therefore, the holomorphic sectional curvature H of M determined by a unit vector $X \in \mathfrak{D}$ is given by

$$H(X) = \mu + \|b\sigma(X, X)\|^2 - \|\sigma(X, X)\|^2 - \|c\sigma(X, X)\|^2 - \|Q\nabla_X X\|^2 +$$

$$+ g(\sigma(X, X), JQ\nabla_{JX} X) - 2g(c\sigma(X, X), JQ\nabla_X X).$$

Remark. If in particular M is a complex submanifold of $\overline{M}$ (i.e. $Q = 0$, $J = c$), then we get

$$H(X) = \mu - 2\|\sigma(X, X)\|^2.$$

Definition. *A $\bar{\sigma}$-submanifold N of an almost Hermitian manifold $\overline{M}$* (10), *is a complex submanifold satisfying*

$$\bar{\sigma}(JX, Y) = \bar{\sigma}(X, JY) = J\bar{\sigma}(X, Y),$$

where X and Y are vector fields tangent to N.

Remark. $\bar{\sigma}$ is the second fundamental form of N in $\overline{M}$.

Proposition 4.1. *Let M be a CR-submanifold of an almost Hermitian manifold $\overline{M}$. If $\mathfrak{D}$ is a integrable distribution and the integral submanifolds are $\bar{\sigma}$-submanifolds, then*

$$\sigma(X, JY) = \sigma(JX, Y)$$

for all $X, Y \in \mathfrak{D}$.

Proof. Let N be an integral submanifold and σ' the second fundamental form of N in M. Since N is $\bar{\sigma}$-submanifold, we have $\bar{\sigma}(JX, Y) = \bar{\sigma}(X, JY)$. Now $\bar{\sigma} = \sigma + \sigma'$ and hence

$$(4.5) \quad \sigma(X, JY) - \sigma(JX, Y) = \sigma'(JX, Y) - \sigma'(X, JY),$$

but the left hand side is normal to M in $\overline{M}$ and the right hand side is tangent to M (normal to N in M). Therefore both sides of equation (4.5) vanish, giving the desired condition.

Corollary 4.2. *Let M be a CR-submanifold of a nearly Kaehler manifold $\overline{M}$. If $\mathfrak{D}$ is an integrable distribution, then*

$$\sigma(X, JY) = \sigma(JX, Y)$$

for all $X, Y \in \mathfrak{D}$.

The proof follows from the Proposition 4.1 and [10], (Theorem 5).

Theorem 4.3. *Let M be a CR-submanifold immersed in $\overline{M}(\mu, \alpha)$. If $\mathfrak{D}$ is an integrable distribution and the integral submanifolds are $\bar{\sigma}$-submanifolds, then the holomorphic sectional curvature verifies $H(X) \leq \mu$ for all $X \in \mathfrak{D}$.*

Proof. From (4.3) and Proposition 4.1 we have

$$H(X) = \mu - (\|\sigma(X, X)\|^2 + \|\sigma(X, JX)\|^2)$$

giving the result.

Theorem 4.4. *Let M be a CR-submanifold immersed in $\overline{M}(\mu, \alpha)$. If $\overline{M}(\mu, \alpha)$ is nearly Kaehlerian manifold and $\mathfrak{D}$ is an integrable distribution, then the holomorphic sectional curvature satisfies $H(X) \leq \mu$, for all $X \in \mathfrak{D}$.*

The proof follows from Theorem 4.3 and [10], (Theorem 5).

Definition. The distribution $\mathfrak{D}$ is called parallel with respect to the Riemannian connection ∇ on M if $\nabla_X Y \in \mathfrak{D}$ for all $X, Y \in \mathfrak{D}$.

Remark. If $\mathfrak{D}$ parallel with respect to ∇, then it is integrable, thus, Theorem 4.3 and 4.4 are also valid.

Definition. *The CR-submanifold M is called $\mathfrak{D}$-totally geodesic (respectively, $\mathfrak{D}^{\perp}$-totally geodesic) if $\sigma(X, Y) = 0$ for each $X, Y \in \mathfrak{D}$ (respectively, $X, Y \in \mathfrak{D}^{\perp}$).*

Theorem 4.5. *Let M be a CR-submanifold immersed in $\overline{M}(\mu, \alpha)$. We suppose that*

a) $\mathfrak{D}$ is integrable and the integral submanifolds are $\bar{\sigma}$-submanifolds.

b) $H(X) = \mu$ *for each* $X \in \mathfrak{D}$.
Then, M is $\mathfrak{D}$-totally geodesic. Moreover, if M is $\mathfrak{D}$-totally geodesic, then $H(X) = \mu$ *for all* $X \in \mathfrak{D}$.
The proof follows from (4.3) and Proposition 4.1.

Theorem 4.6. *Let M be a CR-submanifold immersed in* $\overline{M}(\mu, \alpha)$. *If* $\overline{M}(\mu, \alpha)$ *is a nearly Kaehler manifold and the following conditions are fulfilled*

a) $\mathfrak{D}$ *is integrable*

b) $H(X) = \mu$ *for all* $X \in \mathfrak{D}$

then M is $\mathfrak{D}$-totally geodesic. Moreover, if M is $\mathfrak{D}$-totally geodesic, then $H(X) = \mu$.

Let $\{E_1, \ldots, E_m\}$ be a orthonormal local frame field on M, such that $\{E_1, \ldots, E_p, E_{p+1} = JE_1, \ldots, E_{2p} = JE_p\}$ (respectively, $\{E_{2p+1}, \ldots, E_{2p+q}\}$) is a local frame field in $\mathfrak{D}$ (respectively, in $\mathfrak{D}^\perp$).

Definition. *The CR-submanifold M is called $\mathfrak{D}$-minimal (respectively, $\mathfrak{D}^\perp$-minimal)* $\sum_{i=1}^{2p} \{\sigma(E_i, E_i)\} = 0$ (respectively, $\sum_{i=1}^{q} \{\sigma E_{2p+i}, E_{2p+i})\} = 0$).

Theorem 4.7. *Let M be a $\mathfrak{D}^\perp$-minimal CR-submanifold immersed in* $\overline{M}(\mu, \alpha)$. *Then M is $\mathfrak{D}^\perp$-totally geodesic if and only if*

$$K_M(X \wedge Y) = \frac{\mu + 3\alpha}{4} \tag{4.6}$$

for all $X, Y \in \mathfrak{D}^\perp$.

Proof. If M is $\mathfrak{D}^\perp$-totally geodesic (4.6) follows from (4.2).

If $K_M(X \wedge Y) = \dfrac{\mu + 3\alpha}{4}$ for each $X, Y \in \mathfrak{D}^\perp$, since M is $\mathfrak{D}^\perp$-minimal from (4.2), we have

$$\| \sigma(E_{2p+i}, E_{2p+j}) \|^2 = 0$$

for all $1 \leqslant i, j \leqslant q$. Therefore $\sigma(X, Y) = 0$ for all $X, Y \in \mathfrak{D}^\perp$.

Definition. *A CR-submanifold M is called CR-totally geodesic if* $\sigma(X, Y) = 0$ *for any* $X \in \mathfrak{D}$ *and* $Y \in \mathfrak{D}^\perp$. *The sectional curvature determined by orthonormal vectors* $X \in \mathfrak{D}$ *and* $Y \in \mathfrak{D}^\perp$ *is called CR-sectional curvature.*

Theorem 4.8. *Let M be a CR-submanifold immersed in* $\overline{M}(\mu, \alpha)$. *If the CR-sectional curvature is given by*

$$K_M(X \wedge Y) = \frac{\mu + 3\alpha}{4}$$

for all $X \in \mathfrak{D}$ *and* $Y \in \mathfrak{D}^\perp$, *and one the following conditions is satisfied*

a) M is $\mathfrak{D}$-minimal,

b) M is $\mathfrak{D}^\perp$-minimal,

then, M is CR-totally geodesic.

Proof. If $K_M(X \wedge Y) = \dfrac{\mu + 3\alpha}{4}$ for all $X \in \mathfrak{D}$ and $Y \in \mathfrak{D}^\perp$, from (4.2) we have

$$\sum_{i,j} \| \sigma(E_i, E_{2p+j}) \|^2 = g\left(\sum_{i=1}^{2p} \sigma(E_i, E_i), \sum_{j=1}^{q} \sigma(E_{2p+j}, E_{2p+j}) \right)$$

for all $1 \leqslant i \leqslant 2p$ and $1 \leqslant j \leqslant q$. If M is $\mathfrak{D}$-minimal (respectively, $\mathfrak{D}^\perp$-minimal) we have $\sigma(X, Y) = 0$ for all $X \in \mathfrak{D}$ and $Y \in \mathfrak{D}^\perp$.

Theorem 4.9. *If M is a CR-submanifold immersed in* $\overline{M}(\mu, \alpha)$ *and the following conditions are satisfied*

a) M is $\mathfrak{D}^\perp$-minimal,

b) $\mathfrak{D}$ is integrable and the integral submanifolds are $\bar{\sigma}$*-submanifolds,*

c) $H(X) = \mu$ *for all* $X \in \mathfrak{D}$,

d) $K_M(X \wedge Y) = \dfrac{\mu + 3\alpha}{4}$ *for all* $X \in TM$ *and* $Y \in \mathfrak{D}^\perp$,

then M is totally geodesic. Moreover, if M is totally geodesic a), c) and d) holds.

Theorem 4.10. *Let M be a CR-submanifold immersed in* $\overline{M}(\mu, \alpha)$. *If* $\overline{M}$ *is nearly Kaehlerian and the following conditions are satisfied*

a) M is $\mathfrak{D}^\perp$-minimal,

b) $\mathfrak{D}$ is integrable,

c) $H(X) = \mu$ *for all* $X \in \mathfrak{D}$,

d) $K_M(X \wedge Y) = \dfrac{\mu + 3\alpha}{4}$ *for all* $X \in TM$ *and* $Y \in \mathfrak{D}^\perp$,

then M is totally geodesic. Moreover, if M is totally geodesic a), c) and d) holds.

§5. Ricci tensor and scalar curvature of a CR-submanifold. It is well-known that the Ricci tensor of a Riemannian manifold M is given by

$$S(X, Y) = \sum_{i=1}^{m} R(X, E_i, E_i, Y),$$

where $\{E_1, \ldots, E_m\}$ is an orthonormal local frame field and $X, Y \in TM$.

Let M be a CR-submanifold immersed in $\overline{M}(\mu, \alpha)$. Then from (2.6) we get

$$\begin{aligned} S(X, Y) = {} & \left(\frac{m+2}{4} \mu + \frac{3(m-2)}{4} \alpha \right) g(PX, PY) + \\ & + \frac{m-1}{4} (\mu + 3\alpha) g(QX, QY) + \sum_{i=1}^{m} \{ g(\sigma(X, Y), \sigma(E_i, E_i)) - \\ & - g(\sigma(X, E_i), \sigma(Y, E_i)) \}, \end{aligned} \tag{5.1}$$

for all $X, Y \in TM$.

In this way the scalar curvature of M is given by

$$\begin{aligned} \sigma = {} & \frac{m^2 - m + 6p}{4} \mu + \frac{3(m^2 - m - 2p)}{4} \alpha + \\ & + \sum_{i,j=1}^{m} \{ g((\sigma(E_j, E_j), \sigma(E_i, E_i)) - g(\sigma(E_i, E_j), \sigma(E_i, E_j)) \}. \end{aligned} \tag{5.2}$$

If $\{\xi_1, \ldots, \xi_{2n-m}\}$ is a local frame of normal sections, then we have

$$\sigma(X, Y) = \sum_{a=1}^{2n-m} g(A_{\xi_a} X, Y \xi_a); \tag{5.3}$$

thus, from (5.3) we get

$$S(X,Y) = \left(\frac{m+2}{4}\mu + \frac{3(m-2)}{4}\alpha\right) g(PX,PY) + \frac{m-1}{4}(\mu + \tag{5.4}$$
$$3\alpha)\, g(QX,QY) + \sum_{\alpha=1}^{2n-m}\{(TrA_{\xi_\alpha}) g(A_{\xi_\alpha}X, Y) - g(A_{\xi_\alpha}X, A_{\xi_\alpha}Y),$$

$$\sigma = \frac{m^2 - m + 6p}{4}\mu + \frac{3(m^2 - m - 2p)}{4}\alpha + m^2 \| L \|^2 - \| \sigma \|^2, \tag{5.5}$$

where $L = \mathrm{Tr}\,\sigma/m$ is the mean curvature vector.

Therefore we have

Theorem 5.1. *Let M be a minimal CR-submanifold immersed in $\overline{M}$ (μ, α). Then*

a) $S - \left(\frac{m+2}{4}\mu + \frac{3(m-2)}{4}\alpha\right) g\circ(PxP) - \frac{m-1}{4}(\mu + 3\alpha)\, g\circ(QxQ)$ *is negative semidefinite,*

b) $\rho \leqslant \frac{m^2 - m + 6p}{4}\mu + \frac{3(m^2 - m - 2p)}{4}\alpha.$

Theorem 5.2. *A minimal CR-submanifold M immersed in $\overline{M}(\mu, \alpha)$ is totally geodesic if and only if one of the following conditions is satisfied*

a) $S = \left(\frac{m+2}{4}\mu + \frac{3(m-2)}{4}\alpha\right) g\circ(PxP) - \frac{m-1}{4}(\mu + 3\alpha)\, g\circ(QxQ),$

b) $\rho = \frac{m^2 - m + 6p}{4}\mu + \frac{3(m^2 - m - 2p)}{4}\alpha.$

§6. Totally umbilical CR-submanifolds. Suppose M is totally umbilical, that is

$$\sigma(X,Y) = g(X,Y)\,L,$$

where L is the mean curvature vector. Then we have

Theorem 6.1. *Let M be a totally umbilical CR-submanifold immersed in $\overline{M}$ (μ, α). Then M is a space of constant curvature if and only if $\mu = \alpha$.*

Proof. If M is totally umbilical, for any plane of the tangent space the sectional curvature is either

$$K_M = \frac{\mu + 3\alpha}{4} + \| L \|^2 \quad \text{or} \tag{6.1}$$

$$H = \mu + \| L \|^2. \tag{6.2}$$

Then, the theorem follows from (6.1) and (6.2).

Theorem 6.2. *There exist no totally umbilical proper CR-submanifolds immersed in $\overline{M}$ (μ, α) with $\mu \neq \alpha$.*

Proof. Suppose there exists a totally umbilical proper CR-submanifold M of $\overline{M}$ (μ, α) with $\mu \neq \alpha$. Let X and Y be two non-null vector fields from $\mathfrak{D}$ and $\mathfrak{D}^\perp$ respectively. Then for the normal part of $\overline{R}(X, JX)Y$ we get

$$(\overline{R}(X, JX)Y)^\perp = -\frac{\mu - \alpha}{2} g(X,X)\, JY \neq 0. \tag{6.3}$$

Since M is totally umbilical, from (2.7) we have

$$(R(X, JX)Y)^\perp = g(JX, Y)\, \nabla^\perp_X L - g(X,Y)\, \nabla^\perp_{JX} L = 0,$$

which contradicts (6.3). Thus the proof is complete.

Corollary 6.3. *There exist no totally geodesic proper CR-submanifolds immersed in $\overline{M}$ (μ, α) with $\mu \neq \alpha$.*

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