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ON SOME ORBIT SPACES

BY

I. POP

Our aim is to study in this paper some topological spaces X, the covering space of which is a product $Y \times Y$.

We suppose all considered spaces connected, locally path connected and semi-locally 1-connected. Then, the problem can be reduced to the case of the universal covering spaces, according to:

Lemma. *Let X be a connected, locally path connected and semi-locally 1-connected space, such that there exists a covering projection $p : Y \times Y \to X$, with Y a connected, locally path connected and semilocally 1-connected space. Then, the universal covering space of the space X is in the shape $\tilde{X} \times \tilde{X}$, with $\tilde{X}$ a simply connected space.*

Moreover, the group $\pi_1(X, x_0)$ is isomorphic with the group of the covering transformations of the covering space $\tilde{X} \times \tilde{X}$ and $\pi_1(X, x_0)$ is a properly discontinuous group of homeomorphisms of the space $\tilde{X} \times \tilde{X}$. Since $\tilde{X} \times \tilde{X}$ is a regular covering space, X is homeomorphic with the orbit space $\tilde{X} \times \tilde{X}/\pi_1(X, x_0)$.

According to the above considerations, we consider a properly discontinuous group of homeomorphisms of a product $X \times X$, where X is a simply connected and locally compact space. Denote by $E = X \times X/G$ the orbit space.

We shall prove that under certain conditions, the space E is a locally trivial bundle, such that X is the universal covering space of the fiber and of the base space of this bundle.

Let X be as above, and let $x_0 \in X$. Denote:

$$G_{X \times x_0} = \{g \in G / g(X \times x_0) \subset X \times x_0\},\ G_{x_0 \times X} = \{g \in G / g(x_0 \times X) \subset x_0 \times X\}.$$

Remark 1. $G_{X \times x_0} \cap G_{x_0 \times X} = \{1\}$, where 1 is the unity of the group G.

Theorem. *Let G be a properly discontinuous group of homeomorphisms of the space $X \times X$, where X is a simply connected and locally compact space. Suppose that, for a point $x_0 \in X$, the following conditions are satisfied:*

a) *There exists a homomorphism $\alpha : G \to G$, such that* $\operatorname{Im} \alpha = G_{x_0 \times X}$ *and the equality $g \cdot (x, y) = (x', y')$ implies $\alpha(g) \cdot (x_0, y) = (x_0, y')$; $g \in G$, $(x, y), (x', y') \in X \times X$;*

b) *there exists a mapping* $\beta : G \to G_{X\times x_0}$, *such that the equality* $g\,.\,(x, y) = (x', y)$ *implies* $\beta(g)\,.\,(x, x_0) = (x', x_0)$, $g \in G$, (x, y), $(x', y) \in X \times X$. *Then:*

(i) Ker $\alpha = G_{X\times x_0}$,

(ii) β *is onto,*

(iii) *every element* $g \in G$ *is of the form:* $g = g_1\,.\,g_2$, *with* $g_1 \in G_{X\times x_0}$, $g_2 \in G_{x_0\times X}$, *and* g_1, g_2 *are uniquely determined,*

(iv) $G_{x_0\times X} = \{g \in G / g\,(x, y) = (x', y') \Rightarrow g\,.\,(x_0, y) = (x_0, y')\}$,

(v) *there exists a locally trivial bundle:*

$$F \xrightarrow{\ f\ } E = X \times X/G \xrightarrow{\ p\ } B,$$

such that X *is the universal covering space both for* F *and* B.

Proof. (i) Let $g \in G_{X\times x_0}$. That is, $g\,.\,(x_0, x_0) = (x', x_0)$. From the condition a), it follows $\alpha\,(g)\,.\,(x_0, x_0) = (x_0, x_0)$, which implies $\alpha\,(g) = 1$, that is $G_{X\times x_0} \subset$ Ker α. Then, if $g \in G$, such that $\alpha\,(g) = 1$, let $(x, x_0) \in X \times x_0$ and let $g.\,(x, x_0) = (x', y)$. From the condition a), it follows $\alpha\,(g)\,.\,(x_0, x_0) = (x_0, y) \Rightarrow (x_0, x_0) = (x_0, y)$, therefore $g\,.\,(x, x_0) = (x', x_0)$, that is $g \in G_{X\times x_0}$. So we have $G_{X\times x_0} =$ Ker α.

(ii) Let $g \in G_{X\times x_0}$ and let $g\,.\,(x_0, x_0) = (x', x_0)$. From the condition b), we have $\beta\,(g)\,.\,(x_0, x_0) = (x', x_0)$, and then $\beta\,(g) = g$.

(iii) Let $g \in G$ and let $\alpha\,(g)\,.\,(x_0, x_0) = (x_0, x')$. Then, $\alpha\,(\alpha\,(g))\,(x_0, x_0) = (x_0, x')$, that is $\alpha(g) = \alpha(\alpha(g))$. It follows that $g_1 = \alpha\,(g)\,.\,g^{-1} \in$ Ker $\alpha = G_{X\times x_0}$, which implies $g = g_1^{-1}.\,\alpha(g)$, with $g_1^{-1} \in G_{X\times x_0}$ (since this is a subgroup of the group G by (i)) and $\alpha\,(g) \in G_{x_0\times X}$.

The uniqueness of the decomposition follows from the Remark 1.

iv) Obviously, if $g\,.\,(x, y) = (x', y')$ implies $g\,.\,(x_0, y) = (x_0, y')$, then $g \in G_{x_0\times X}$. Then, let $g \in G_{x_0\times X}$ and let $g.\,(x, y) = (x', y')$ and $g\,.\,(x_0, y) = (x_0, y'')$. We have $\alpha\,(g)\,.\,(x_0, y) = (x_0, y')$ and $\alpha\,(g)\,(x_0, y) = (x_0, y'')$, therefore $y' = y''$, that is $g\,.\,(x, y) = (x', y')$ implies $g\,.\,(x_0, y) = (x_0, y')$.

(v) By condition a) and conclusion (i), $G_{x_0\times X}$ and $G_{X\times x_0}$ are subgroups of the group G. Then, $G_{X\times x_0}$ is a properly discontinuous group of homeomorphisms of the space $X \times x_0$ and $G_{x_0\times X}$ is a properly discontinuous group of homeomorphisms of the space $x_0 \times X$. Denote by $F = (X \times x_0)/G_{X\times x_0}$ and $B = (x_0 \times X)/G_{x_0\times X}$ the orbit spaces of these actions. Define the map:

$$f : F = (X \times x_0)/G_{X\times x_0} \to E = (X \times X)/G,$$

by $f\,((\widetilde{x, x_0})) = (\overline{x, x_0})$, where $(\widetilde{x, x_0})$ is the orbit of the point (x, x_0) by the action of the group $G_{X\times x_0}$ and $(\overline{x, x_0})$ is the orbit of the same point by the action of the group G. It follows easily that f is well defined and continuous. This map is also injective. In fact, if $f\,((\widetilde{x, x_0})) = f\,((\widetilde{x', x_0}))$, it follows that there exists $g \in G$, such that $g.\,(x, x_0) = (x', x_0)$. From the condition b), we hawe $\beta\,(g)\,(x, x_0) = (x', x_0)$, that is $(\widetilde{x, x_0}) = (\widetilde{x', x_0})$. Now, we define the map: $p : E \to B$, by $p\,(\overline{x, y})) = \widehat{(x_0, y)}$ where $\widehat{(x_0, y)}$ is the orbit of the point (x_0, y) by the action of the group $G_{x_0\times X}$. The map p is well defined: if $(\overline{x', y'}) = (\overline{x, y})$, then, there exists $g \in G$ such that $g\,.\,(x, y) = (x', y')$.

From a), it follows that $\alpha\,(g)\,.\,(x_0, y) = (x_0, y')$, and therefore $\widehat{(x_0, y)} = \widehat{(x_0, y')}$. The map p is also continuous. In this way we obtain the sequence of continuous maps:

$$F = (X \times x_0)/G_{X\times x_0} \xrightarrow{\ f\ } E = (X \times X)/G \xrightarrow{\ p\ } B = (x_0 \times X)/G_{x_0\times X}.$$

For $(\widetilde{x, x_0}) \in F$, we have $pf\,(\widetilde{x, x_0}) = p\,(\overline{x, x_0}) = \widehat{(x_0, x_0)}$, that is $f(F) \subset p^{-1}\,(\widehat{(x_0, x_0)})$. Then, let $(\overline{x, y}) \in p^{-1}(\widehat{(x_0, x_0)})$, that is $\widehat{(x_0, y)} = \widehat{(x_0, x_0)}$, which implies $g\,.\,(x_0, y) = (x_0, x_0)$, for an element $g \in G_{x_0\times X}$. It follows $\alpha\,(g) = g$. Let $g\,.\,(x, y) = (x', y')$. Then, $\alpha(g)\,.\,(x_0, y) = (x_0, y')$, that is $g\,.\,(x_0, y) = (x_0, y') \Rightarrow y' = x_0$. Thus, $g\,.\,(x, y) = (x', x_0)$, which implies $(\overline{x, y}) = (\overline{x', x_0}) = f\,(\widetilde{x', x_0})$. In this way we have $p^{-1}\,(\widehat{(x_0, x_0)}) = f\,(F)$, which is homeomorphic to F.

For a point $\widehat{(x_0, y)} \in B$, we have: $p^{-1}\,(\widehat{(x_0, y)}) = \{(\overline{x, y}) \in E\}$. Now we construct a homeomorphism: $h : F \to p^{-1}(\widehat{(x_0, y)})$, defined by $h\,((\widetilde{x, x_0})) = (\overline{x, y})$ (y is fixed). This map is well defined and continuous and injective. In fact, if $h\,((\widetilde{x, x_0})) = h\,((\widetilde{x', x_0}))$, it follows that there exists $g \in G$, such that $g.\,(x, y) = (x', y)$. Then, $\alpha\,(g)\,.\,(x_0, y) = (x_0, y)$, that is $\alpha(g) = 1 \Rightarrow g \in G_{X\times x_0}$, and therefore $g\,.\,(x, x_0) = (x'', x_0)$. From b), we have $\beta\,(g)\,.\,(x, x_0) = (x', x_0)$ and $\beta\,(g)\,.\,(x, x_0) = (x'', x_0)$. Then $x'' = x'$. Thus, we obtain $g\,.\,(x, x_0) = (x', x_0)$, that is $(\widetilde{x, x_0}) = (\widetilde{x', x_0})$. The map h is bijective and his inverse $h^{-1}\,((\overline{x, y_0})) = (\widetilde{x, x_0})$ is continuous.

Now, we consider the diagram:

$$(*)\qquad \begin{tikzcd} (X \times x_0)/G_{X\times x_0} \arrow[r, "f"] & E = (X \times X)/G \arrow[r, "p"] & (x_0 \times X)/G_{x_0\times X} \\ X \arrow[u, "\pi_1"] \arrow[r, "f_0"] & X \times X \arrow[u, "\pi"] \arrow[r, "p_0"] & X \arrow[u, "\pi_2"] \end{tikzcd}$$

where $\pi_1\,(x) = (\widetilde{x, x_0})$, $\pi\,(x, y) = (\overline{x, y})$, $\pi_2(x) = \widehat{(x_0, x)}$, $f_0(x) = (x, x_0)$, $p_0(x, y) = y$. Obviously π_1, π, π_2 are universal covering projections. It follows easily that the diagram (*) is commutative. We can prove that the upper line of this diagram is a locally trivial bundle. For a point $x \in X$, let U be an open neighbourhood, such that $gU \cap U = \emptyset$, if $g \in G_{x_0\times X}$ and $g \neq 1$. Then, $\pi_2(U)$ is an open set in the space B. Define a homeomorphism:

$$\varphi_U : p^{-1}(\pi_2(U)) \to \pi_2(U) \times F, \text{ by}$$

$\varphi_U((\overline{x, y})) = (\widehat{(x_0, y')}, (\widetilde{x', x_0}))$, where $(x', y') \in X \times U$, such that $(\overline{x, y}) = (\overline{x', y'})$. In fact, if $(\overline{x, y}) \in p^{-1}(\pi_2(U))$, then, $p((\overline{x, y})) = \widehat{(x_0, y)} \in$

$\in \pi_2(U)$. It follows $\widehat{(x_0, y)} = \widehat{(x_0, y')}$, for an element $y' \in U$. We have $g \,.\, (x_0, y) = (x_0, y')$, for a $g \in G_{x_0 \times X}$ and $\alpha(g) = g$. If $g \,.\, (x, y) = (x', y'') \Rightarrow$ $\Rightarrow \alpha(g)(x_0, y) = (x_0, y'') \Rightarrow g \,.\, (x_0, y) = (x_0, y'') \Rightarrow y'' = y'$, that is $\overline{(x, y)} =$ $= \overline{(x', y')}$ and $(x', y') \in X \times U$.

Let $(x'', y'') \in X \times U$ be another pair, such that $\overline{(x, y)} = \overline{(x'', y'')}$. Then, $g \,.\, (x', y') = (x'', y'')$, which implies $\alpha(g) \,.\, (x_0, y') = (x_0, y'')$ and by assumption on the open set U, we have $\alpha(g) = 1$, that is $g \in G_{X \times x_0}$, and this implies $g \,.\, (x', x_0) = (x'', x_0)$ that is $(x', x_0) = (x'', x_0)$. Therefore, the map φ_U is well defined and it follows easily that is is also continuous. We prove that φ_U is an injective map. Let $(x, y), (x', y') \in X \times U$ and let $\varphi_U(\overline{(x, y)}) =$ $= \varphi_U(\overline{(x', y')})$, that is $(\widehat{(x_0, y)}, \overline{(x, x_0)}) = (\widehat{(x_0, y')}, \overline{(x', x_0)})$. We have $(x_0, y) =$ $= (x_0, y')$ and, there exists $g_2 \in G_{X \times x_0}$, such that $g_2 \,.\, (x, x_0) = (x', x_0)$. Let $g_2 \,.\, (x, y) = (x'', y'')$. We have $\alpha(g_2)(x_0, y) = (x_0, y'')$. But $\alpha(g_2) = 1$, and therefore $y'' = y$. We obtain $g_2 \,.\, (x, y) = (x'', y)$. From condition b), we have $\beta(g_2) \,.\, (x, x_0) = (x'', x_0)$. But $\beta(g_2)(x, x_0) = (x', x_0)$. Thus, $x'' = x'$ and $g_2 \,.\, (x, y) = (x', y)$, which implies $\overline{(x, y)} = \overline{(x', y')}$.

The inverse map of φ_U is defined by $\varphi_U^{-1}(\widehat{(x_0, y)}, \overline{(x, x_0)}) = \overline{(x, y)}$, and is continuous. Thus φ_U is a homeomorphism.

Finally, $p\varphi_U^{-1}(\widehat{(x_0, y)}, \overline{(x, x_0)}) = p(\overline{(x, y)}) = \widehat{(x_0, y)}$. This finishes the proof of the theorem.

Remark 2. If we identify the group G with the group $\pi_1(E, e_0)$ and the group $G_{x_0 \times X}$ with the group $\pi_1(B, b_0)$, $b_0 = p(e_0)$, then, the homomorphism $\alpha : G \to G_{x_0 \times X}$ can be identified with the homomorphism $p_{\#} : \pi_1(E, e_0) \to$ $\to \pi_1(B, b_0)$. Also, if $i : G_{X \times x_0} \to G$ is the inclusion, then, i can be identified with the monomorphism $f_{\#} : \pi_1(F, e_0) \to \pi_1(E, e_0)$.

In fact, we have the commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & G_{x_0 \times X} \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \psi'} \\ \pi_1(X, e_0) & \xrightarrow{p_{\#}} & \pi_1(B, b_0), \end{array}$$

where ψ and ψ' are well known isomorphisms ([1], pag. 117). Recall that if $g \in G$, then $\psi(g) = [\pi\tilde{\omega}]$, where $\tilde{\omega} : I \to X \times X$ is a path, with $\tilde{\omega}(0) =$ $= (x_0, x_0)$ and $\tilde{\omega}(1) = g \,.\, (x_0, x_0) = (x_1, y_1)$. Similarly, $\psi'(\alpha(g)) = [\pi_2\tilde{\omega}_1]$, with $\tilde{\omega}_1(0) = (x_0, x_0)$ and $\tilde{\omega}_1(1) = \alpha(g) \,.\, (x_0, x_0)$. But $g \,.\, (x_0, x_0) = (x_1, y_1)$ implies $\alpha(g) \,.\, (x_0, x_0) = (x_0, y_1)$, and therefore we can set $\tilde{\omega}_1 = p_0\tilde{\omega}$. In fact, $\tilde{\omega}_1(0) = p_0(x_0, x_0) = (x_0, x_0)$ and $\tilde{\omega}_1(1) = p_0\tilde{\omega}(1) = p_0(x_1, y_1) = (x_0, y_1) =$ $= \alpha(g) \,.\, (x_0, x_0)$. Thus, we obtain $\psi'\alpha(g) = [\pi_2 p_0 \tilde{\omega}] = [p\pi\tilde{\omega}] = p_{\#}[\pi\tilde{\omega}] = p_{\#}\psi(g)$.

In a similar way it is proved the commutativity of the diagram:

$$\begin{array}{ccc} G_{X \times x_0} & \xrightarrow{i} & G \\ {\scriptstyle \psi'}\downarrow & & \downarrow{\scriptstyle \psi} \\ \pi_1(F, e_0) & \xrightarrow{f_{\#}} & \pi_1(E, e_0). \end{array}$$

Corollary 1. *Let X be a simply connected and locally compact space and let G be a properly discontinuous group of homeomorphisms of the product space $X \times X$. If there exists a point $x_0 \in X$ such that for every $g \in G$, the equality $g(x, y) = (x', y')$ implies $g \,.\, (x_0, y) = (x_0, y')$, then, the orbit space $E = (X \times X)/G$ is a locally trivial bundle with the fiber $X \times x_0$ and the base space $(x_0 \times X)/G$.*

Proof. Define $\alpha : G \to G_{x_0 \times X} = G$ by $\alpha = 1_G$. Then, the condition a) of the Theorem is trivially verified. Now, if $g \in G_{X \times x_0}$, we have $g \,.\, (x_0, x_0) =$ $= (x, x_0) \Rightarrow \alpha(g) \,.\, (x_0, x_0) = (x_0, x_0) \Rightarrow g \,.\, (x_0, x_0) = (x_0, x_0) \Rightarrow g = 1$, that is $G_{X \times x_0} = \{1\}$ and the trivial map $\beta : G \to G_{X \times x_0} = \{1\}$ satisfies the condition b) of the Theorem.

Corollary 2. *Let X be a simply connected and locally compact space and let G be a properly discontinuous group of homeomorphisms of the product space $X \times X$. If there exists a point $x_0 \in X$, such that $\forall\, g \in G$, the equality $g(x, y) = (x', y')$ implies $g \,.\, (x, x_0) = (x', x_0)$, then the orbit space $E =$ $= (X \times X)/G$ is a locally trivial bundle with the fiber $x_0 \times X$ and the base space $(x_0 \times X)/G$.*

Proof. Define a new action of the group G on the space $X \times X$ by $(g, (x, y)) \to (p_2 g(y, x), p_1 g(y, x))$, where $p_i : X \times X \to X$, $i = 1, 2$, are the first and the second projection. Then we can apply Corollary 1.

Proposition. *Let X be a simply connected and locally compact space and let G be a properly discontinuous group of homeomorphisms of the product space $X \times X$. Let $x_0, x_0' \in X$ be two points satisfying the conditions a) and b) of the Theorem. Then, the locally trivial bundles*

$$(X \times x_0)/G_{X \times x_0} \xrightarrow{f} (X \times X)/G \xrightarrow{p} (x_0 \times X)/G_{x_0 \times X} \quad \textit{and}$$

$$(X \times x_0')/G_{X \times x_0'} \xrightarrow{f'} (X \times X)/G \xrightarrow{p'} (x_0' \times X)/G_{x_0' \times X},$$

given by the conclusion (v) *of the Theorem, are isomorphic.*

The *proof* of this proposition is obvious.

Corollary 3. *If in the Theorem, the condition b) is replaced by the condition:*

b') *β is a homomorphism and $g \,.\, (x, y) = (x', y')$ implies $\beta(g)(x, x_0) =$ $= (x', x_0)$, then, the bundle considered in the theorem is trivial.*

Proof. Define the map

$h : (X \times X)/G \to (x_0 \times X)/G_{x_0 \times X} \times (X \times x_0)/G_{X \times x_0}$ by $h(\overline{(x, y)}) = (\widehat{(x_0, y)}, \overline{(x, x_0)})$. This map is well defined. In fact, if $g \,.\, (x, y) = (x', y')$, for a $g \in G$, then, $\alpha(g)(x_0, y) = (x_0, y')$, and therefore $\widehat{(x_0, y)} = \widehat{(x_0, y')}$. Then $\beta(g) \,.\, (x, y_0) = (x', x_0)$, that is $\overline{(x, x_0)} = \overline{(x', x_0)}$. The map h is continuous.

Now, let $k : (x_0 \times X)/G_{x_0 \times X} \times (X \times x_0)/G_{X \times x_0} \to (X \times X)/G$, with $k(\widehat{(x_0, y)}, \overline{(x, x_0)}) = \overline{(x, y)}$. Let $\widehat{(x_0, y)} = \widehat{(x_0, y')}$ and $\overline{(x, x_0)} = \overline{(x', x_0)}$. There exist $g_1 \in G_{x_0 \times X}$ and $g_2 \in G_{X \times x_0}$, such that $g_1 . (x_0, y) = (x_0, y')$ and $g_2 . (x, x_0) = (x', x_0)$. Let $g_1 g_2 . (x, y) = (x'', y'')$. Then, $\alpha(g_1)\alpha(g_2)(x_0, y) = (x_0, y'')$. But $\alpha(g_2) = 1$, thus $\alpha(g_1) . (x_0, y) = (x_0, y'')$. We have also $\alpha(g_1) . (x_0, y) = (x_0, y')$ and it follows $y'' = y'$, that is $g_1 g_2 . (x, y) = (x'', y')$. Then, $\beta(g_1)\beta(g_2) . (x, x_0) = (x'', x_0)$. But $\beta(g_1) = 1$, and it follows $\beta(g_2) . (x, x_0) = (x', x_0)$. Also using the relation $\beta(g_2)(x, x_0) = (x', x_0)$, we obtain $x'' = x'$. In this way we have $g_1 g_2 . (x, y) = (x', y')$, that is $\overline{(x', y')} = \overline{(x, y)}$, which proves that the map k is well defined. It is easily verified that this is continuous. We have $hk = 1$ and $kh = 1$, that is h is a homeomorphism.

Finally, we have the commutative diagrams:

$$\begin{tikzcd}
(X \times X)/G \arrow[r, "p"] \arrow[d, shift right, "h"'] & (x_0 \times X)/G_{x_0 \times X} \\
(x_0 \times X)/G_{x_0 \times X} \times (X \times x_0)/G_{X \times x_0} \arrow[u, shift right, "k"'] \arrow[ur, "pr_1"'] &
\end{tikzcd}$$

where pr_1 is the first projection.

Examples. 1. Let be $X = R$, $G = Z \oplus Z$, with the following action on $R^2 : ((m, n), (x, y)) \to (x + m, y + n)$.

Then, $G_{R \times x_0} = \{(m, n) \in Z \oplus Z / x_0 = x_0 + n\} = \{(m, 0)/m \in Z\} \approx Z$, $G_{x_0 \times R} = \{(m, n) \in Z \oplus Z / x_0 = x_0 + m\} = \{(0, n)/n \in Z\} \approx Z$. Define $\alpha : G \to G_{x_0 \times R}$, $\beta : G \to G_{R \times x_0}$, by $\alpha(m, n) = (0, n)$ and $\beta(m, n) = (m, 0)$. These maps satisfy the conditions a) and b'). By the Corollary 3 it follows that $R^2/G = T^2$ is a trivial bundle over $(x_0 \times R)/G_{x_0 \times R} = S^1$ with the fiber $(R \times x_0)/G_{R \times x_0} = S^1$, and R is the universal covering space for S^1.

2. Let $X = R$ and let G be the group with two generators a, b such that $bab = a$. The elements of the group G have the form $a^m b^n$, $m, n \in Z$ and G acts on R^2 by: $(a^m b^n, (x, y)) \to ((-1)^n (x + m), y + n)$. We have $G_{R \times x_0} = \{a^m b^n / x_0 = x_0 + n\} = \{a^m / m \in Z\} \approx Z$, $\forall x_0 \in R$ and

$$G_{x_0 \times R} = \{a^m b^n / x_0 = (-1)^n (x_0 + m)\} = \begin{cases} \{(b^2)^p / p \in Z\}, x_0 \neq 0 \\ \{b^n / n \in Z\}, \quad x_0 = 0. \end{cases}$$

Define $\alpha : G \to G_{0 \times R}$, $\beta : G \to G_{R \times 0}$, by $\alpha(a^m b^n) = a^m$, $\beta(a^n b^m) = b^m$. These maps satisfy the conditions a) and b) of the Theorem. It follows that $R^2/G = K$ — the Klein bottle, is a locally trivial bundle over the space $(0 \times R)/\{a^m\} = S^1$ with the fiber $(R \times 0)/\{b^n\} = S^1$.

3. Let $X = R$ and let $G = Z$. Consider the action of G on R^2 given by $(n, (x, y)) \to (ne^y + x, y)$. Then, $G_{R \times x_0} = \{n \in Z / n(x, x_0) = (x', x_0)\} = Z$, and $G_{x_0 \times R} = \{0\}$. The trivial homomorphism $\alpha : G \to G_{x_0 \times R} = \{0\}$ satisfies the condition a), but it does not exist a map $\beta : G \to G_{R \times 0}$, which satisfies the condition b). However, we can consider the homeomorphism $\varphi : R^2 \to R^2$, $\varphi(x, y) = (xe^{-y}, y)$, and define a new action of the group $G = Z$ on R^2: $(n, (x, y)) \to \varphi(n(\varphi^{-1}(x, y)) = (x + n, y)$. For this action the conditions of the Corollary 2 are satisfied. Moreover, R^2/G is homeomorphic to the orbit space of R^2 under this new action. This orbit space is homeomorphic to a cylinder.

Remark 3. We justify the conditions of the above theorem as follows:

Let us consider the following commutative diagram:

$$\begin{tikzcd}
F \arrow[r, "i"] & E \arrow[r, "p"] & B \\
X \times x_0 \arrow[r, "i_0"] \arrow[u, "p_1"] & X \times X \arrow[r, "p_0"] \arrow[u, "p_2"] & x_0 \times X \arrow[u, "p_3"]
\end{tikzcd}$$

with the first line a locally trivial bundle and p_1, p_2, p_3 universal covering maps. Then, from [2], [3], $i_{\#} : \pi_1(F, e_0) \to \pi_1(E, e_0)$ is injective and one can prove that there exists $\beta : \pi_1(E, e_0) \to \pi_1(F, e_0)$, such that $\beta i_{\#} = 1_{\pi_1(F, e_0)}$, and β satisfies the condition b).

Moreover, if we define $\alpha : G(X \times X/E) \to G(x_0 \times X/B)$ such that the following diagram be commutative:

$$\begin{tikzcd}
G(X \times X/E) \arrow[r, "\alpha"] \arrow[d, "\psi"] & G(x_0 \times X/B) \arrow[d, "\psi'"] \\
\pi_1(E, e_0) \arrow[r, "p_{\#}"] & \pi_1(B, p(e_0)).
\end{tikzcd}$$

then α satisfies the condition a) of the theorem.

In fact, if $g . (x_0, x_0) = (x_1, y_1)$, for a $g \in G(X \times X/E)$, then $p_{\#}(\psi(g)) = [pp_2\tilde{\omega}]$ for $\tilde{\omega}(0) = (x_0, x_0)$ and $\tilde{\omega}(1) = (x_1, y_1)$, and $\psi'(\alpha(g)) = [p_3\tilde{\omega}_1]$, with $\tilde{\omega}_1(0) = (x_0, x_0)$ and $\tilde{\omega}_1(1) = \alpha(g)(x_0, x_0)$. But the equality $[pp_2\tilde{\omega}] = [p_3 p_0 \tilde{\omega}] = [p_3\tilde{\omega}_1]$ and $p_0\tilde{\omega}(0) = (x_0, x_0)$, $\tilde{\omega}_1(0) = (x_0, x_0)$ implies $p_0\tilde{\omega} \simeq \tilde{\omega}_1$ rel $\dot{I} \Rightarrow \tilde{\omega}_1(1) = p_0\tilde{\omega}(1) = (x_0, y_1)$. That is $\alpha(g)(x_0, x_0) = (x_0, y_1)$. Now, let $g . (x, y) = (x', y')$. Then, $p_2 g(x, y) = p_2(x', y') \Rightarrow pp_2(x, y) = pp_2(x', y') \Rightarrow p_3 p_0(x, y) = p_3 p_0(x', y') \Rightarrow p_3(x_0, y) = p_3(x_0, y')$. Thus, there exists $g_1 \in G(x_0 \times X/B)$, such that $g_1 . (x_0, y) = (x_0, y')$. It is easily proved that $\psi'(g_1) = \psi'(\alpha(g))$, which implies $g_1 = \alpha(g)$.

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