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ALMOST PRODUCT STRUCTURES AND SUBCHARACTERISTIC ALGEBRAS

BY

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1. Introduction. The almost product structures have been investigated by many authors, as for example [1] — [4]. The main purpose of the present paper is to study the space of the vector fields on an almost product manifold, in connection with a tensor field of type (1, 2). For every tensor field of type (1, 2), this space becomes an algebra over the ring of differentiable functions on the manifold (see [5] or [6]). We shall define in section 2 the notion of subcharacteristic algebra (definition 3), which is related to an almost product structure and in section 3 we shall prove the existence of the subcharacteristic algebras.

We shall treat also the case in which a subcharacteristic algebra is a deformation algebra of a pair of linear connections (see [5]). In a natural manner this case leads to a generalisation of the paths with respect to a linear connection (definition 4).

2. Definitions and statement of the main theorems. All geometrical objects in this paper are assumed to be of class C^∞. Let M be a m-dimensional differentiable manifold. We denote by $\mathcal{F}(M)$ the ring of all real functions on M and by $\mathcal{X}(M)$ the set of all vector fields.

Definition 1. *An almost product structure on the manifold M is a system of distributions T_α, $\alpha = 1, \ldots, r$,* i.e. *r differentiable plane fields such that* [1]

(i) *for $\alpha \neq \beta$, the planes of T_α, T_β at any point are disjoint, and*

(ii) *$\sum_\alpha T_\alpha = T$ where T is the distribution of tangent planes on M.*

If the distribution T_α is m_α-dimensional, one can say, using the terminology of [3] that M is an almost product manifold of type $(m_1, \ldots, m_r)$. It follows from (i) and (ii) that $m = \sum_\alpha m_\alpha$.

Definition 2. *Let M be an almost product manifold and let A be a tensor field of type* (1, 2). *We define the product of two vector fields X and Y as the vector field*

$$X * Y = A(X, Y). \tag{2.1}$$

Then o[illegible] easily see that the module $\mathcal{X}(M)$ becomes an algebra over $\mathcal{F}(M)$ [illegible] algebra [illegible] denoted by $\mathcal{U}(M, A)$ (see [6]).

If ∇^1 and ∇^2 are two linear connections on the manifold M, then the algebra $\mathcal{U}(M, \nabla^2 - \nabla^1)$ is called the deformation algebra of the pair (∇^1, ∇^2) (see [5]). From now on let the indice α run over the range $1, \ldots, r$. Let us denote by π_α the projection tensor field coresponding to the distributions T_α and for every tensor field S of type $(1, s)$, let us define the tensor field $\pi_\alpha S$ of type $(1, s)$ by the relation:

$$\pi_\alpha S(X_1, \ldots, X_s) = \pi_\alpha[S(X_1, \ldots, X_s)]. \tag{2.2}$$

Definition 3. *An element X of the algebra $\mathcal{U}(M, A)$ which has the property*

$$\pi_\alpha A(X, X) = f_\alpha \pi_\alpha X, \quad \alpha = 1, \ldots, r, \tag{2.3}$$

where $f_\alpha \in \mathcal{F}(M)$, is called a subcharacteristic vector field of $\mathcal{U}(M, A)$. If all elements of $\mathcal{U}(M, A)$ are subcharacteristic vector fields, then the algebra will be referred to as subcharacteristic algebra.

In section 3 we prove the following:

Proposition 1. *If $\mathcal{U}(M, A)$ is a subcharacteristic commutative algebra, then there exists the 1-forms ω_α, $\alpha = 1, \ldots, r$ such that*

$$A = \textstyle\sum_\alpha (\omega_\alpha \otimes \pi_\alpha + \pi_\alpha \otimes \omega_\alpha). \tag{2.4}$$

In connection with these algebras we treat the case of the deformation algebras. We use for this treatment the following definitions:

Definition 4. *Let ∇ be a linear connection on the almost product manifold M. By a subpath in M with respect to the connection we mean a smooth curve $c: (a, b) \to M$ such that*

$$\pi_\alpha(\nabla_{c_*(t)} c_*(t)) = f_\alpha(t)\,(\pi_\alpha c_*(t)) \tag{2.5}$$

holds for every $t \in (a, b)$ and for every α, where f_α are functions on (a, b) and $c_*(t)$ is the tangent vector of c at the point $c(t)$.

Definition 5. *Let ∇^1 and ∇^2 be two linear connections on M. If the sets of the subpaths in M with respect to the connections ∇^1 and ∇^2 are equal, then the two connections are said to be subprojectively related or briefly SP-related.*

In section 3 we prove also.

Proposition 2. *The linear connections ∇^1 and ∇^2 are SP-related if and only if the deformation algebra $\mathcal{U}(M, \nabla^2 - \nabla^1)$ is subcharacteristic.*

Then, from the propositions 1 and 2 we have.

Theorem. *If the linear connections ∇^1 and ∇^2 are symmetric, then they are SP-related if and only if there exists a symmetric linear connection ∇ and the 1-forms ω_α, $\alpha = 1, \ldots, r$ such that*

$$\begin{aligned} \nabla^1 &= \nabla - \textstyle\sum_\alpha(\omega_\alpha \otimes \pi_\alpha + \pi_\alpha \otimes \omega_\alpha)/2, \\ \nabla^2 &= \nabla + \textstyle\sum_\alpha(\omega_\alpha \otimes \pi_\alpha + \pi_\alpha \otimes \omega_\alpha)/2. \end{aligned} \tag{2.6}$$

3. Proofs.

3.1. *Proof of proposition* 1. Let $\mathcal{U}(M, A)$ be a commutative subcharacteristic algebra and let U be an open subset of M. A frame $(e_1, \ldots, e_m)$ on U is called an adapted frame, if each vector field e_i belongs to one of the distributions T_α. Let $(e_1, \ldots, e_m)$ be an adapted frame and let $(\omega_1, \ldots, \omega_m)$ be its associated co-frame. From now on let the indices i, j, k run over the range $1, \ldots, m$. For a fixed indice α let I_α be the set of all indices i for which e_i belong to T_α. Then we obtain succesively:

$$\pi_\alpha = \sum_{i \in I_\alpha} \omega_i \otimes e_i, \tag{3.1}$$

$$\pi_\alpha A = \sum_{i \in I_\alpha} \Omega^i_\alpha \otimes e_i, \tag{3.2}$$

where Ω^i_α are symmetrical tensor fields of type $(0,2)$

$$\Omega^i_\alpha = \sum_{j,k} \lambda^i_{jk}\, \omega_j \otimes \omega_k, \tag{3.3}$$

where $\lambda^i_{jk} \in \mathcal{F}(U)$ and $\lambda^i_{jk} = \lambda^i_{kj}$.

Lemma 1. *For every i, if $j \notin I_\alpha$ and $k \notin I_\alpha$, $\lambda^i_{jk} = 0$.*

Proof. It can be easily proved, from (2.3) that we have $\pi_\alpha A(e_j, e_k) = 0$, or equivalently $\Omega^i_\alpha(e_j, e_k) = 0$. q.e.d.

With the help of the lemma 1 we shall write instead of (3.3)

$$\Omega^i_\alpha = \sum_{i \in I_\alpha} (\omega_j \otimes \omega^i_j + \omega^i_j \otimes \omega_j),$$

where ω^i_j are 1-forms. For every vector fields $X = \sum_i X^i e_i$ we have (2.3). Therefore

$$\Omega^i_\alpha(X, X) = 2 \sum_{j \in I_\alpha} \omega^i_j(X) X^j = f_\alpha X^i, \quad i \in I_\alpha \tag{3.7}$$

and we obtain:

Lemma 2. *There exists a 1-form ω_α so that for $i, j \in I_\alpha$ we have: $\omega^i_j = \omega_\alpha$ if $j = i$ and $\omega^i_j = 0$ if $j \neq i$.*

From lemma 2, it follows $\Omega^i_\alpha = \omega_\alpha \otimes \omega_i + \omega_i \otimes \omega_\alpha$, and from (3.2) and (1.3) we have

$$\pi_\alpha A = \omega_\alpha \otimes \pi_\alpha + \pi_\alpha \otimes \omega_\alpha. \tag{3.8}$$

The proof of proposition 1, i.e. the relation (2.4) can be obtained from its local formulation (3.8), using also the formula $A = \sum_\alpha \pi_\alpha A$.

3.2. *Proof of proposition* 2. Let X be a vector field, p be a point and let $c: (-\varepsilon, \varepsilon) \to M$ be a ∇^1-subpath, so that $c(0) = p$, $c_*(0) = X_p$ and $\pi_\alpha(\nabla^1_{c_*(t)} c_*(t)) = f^1_\alpha \pi_\alpha(c_*(t))$.

If we put $A = \nabla^2 - \nabla^1$, we remark that the conditions:

$$\pi_\alpha(\nabla^2_{c_*(t)} c_*(t)) = f^2_\alpha \pi_\alpha(c_*(t)) \tag{3.9}$$

$$\pi_\alpha A(c_*(t), c_*(t)) = (f^2_\alpha - f^1_\alpha)\, \pi_\alpha(c_*(t)) \tag{3.10}$$

are equivalent and the proof is completed.

4. Some remarks. We can generalize the concept of subcharacteristic algebra, and implicitly the concept of subpath, if we suppose that the conditions (2.3), respectively (2.5), are satisfied only for some indices. Then the formulas (2.4) and (2.6) must be properly modified and as particular cases we obtain the connection transformations studied by N o r d e n [7], M i r o n [8], Y a n o [9] and others. For more details see V r ă n c e a n u ([10], [illegible]) and also the references contained in [10].

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ON THE GEOMETRY OF LEAVES ON A *CR*-SUBMANIFOLD

BY

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1. Introduction. The study of differential geometry of CR submanifolds has been initiated in [1]. This class of submanifolds of a Kaehler manifold is an essential generalization of complex submanifolds [6] and totally real submanifolds [5]. Fundamental results on the geometry of CR submanifolds have been obtained in ([1] — [4]).

On a CR submanifold there are defined two distributions: the holomorphic distribution and the totally real distribution [1]. B. Y. Chen has proven the totally real distribution is always involutive [4]. In this way, each CR submanifold of a Kaehler manifold is foliated by totally real submanifolds.

The purpose of the present paper is to study the geometry of leaves of both distributions on a CR submanifold. First, we give in §2 basic formulas for the immersion of a CR submanifold in a Kaehler manifold. In §3 we obtain necessary and sufficient conditions in order to characterize totally geodesic leaves of the totally real foliation. A pinching theorem for leaves of the holomorphic distribution is obtained in §4.

2. CR submanifolds of a Kaehler manifold. Let M be a Kaehler manifold of complex dimension m and N be an n — dimensionl Riemannian submanifold immersed in M. The complex structure on M will be denoted by J and both metrics on M and N will be denoted by the same symbol g. For each $x \in N$ denote by T_xN and ν_x,respectively the tangent space and the normal space to N at x.

Definition. *The submanifold N is said to be CR-submanifold of M if there exists a differentiable distribution $D : x \to D_x \subset T_xN$ on N such that the following conditions are fulfilled:*

1. *D is holomorphic, i.e. $J(D_x) = D_x$ for each $x \in N$,*
2. *the complementary orthogonal distribution $D^\perp : x \to D_x^\perp \subset T_xN$ is totally real, i.e. $J(D_x^\perp) \subset \nu_x$, for each $x \in N$.*

If in particular, dim $D^\perp = 0$ we get complex submanifolds and if we have dim $D = 0$, then we get totally real submanifolds. Moreover, each real hypersurface of a Kaehler manifold is a CR-submanifold.

Now, let $\tilde{\nabla}$ be the Kaehlerian connection on M and ∇ be the Levi — Civita connection on N. Denote by $\nabla^\perp$ the linear connection induced by