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ON THE GEOMETRY OF LEAVES ON A CR-SUBMANIFOLD

BY

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1. Introduction. The study of differential geometry of CR submanifolds has been initiated in [1]. This class of submanifolds of a Kaehler manifold is an essential generalization of complex submanifolds [6] and totally real submanifolds [5]. Fundamental results on the geometry of CR submanifolds have been obtained in ([1] — [4]).

On a CR submanifold there are defined two distributions : the holomorphic distribution and the totally real distribution [1]. B. Y. Chen has proven the totally real distribution is always involutive [4]. In this way, each CR submanifold of a Kaehler manifold is foliated by totally real submanifolds.

The purpose of the present paper is to study the geometry of leaves of both distributions on a CR submanifold. First, we give in §2 basic formulas for the immersion of a CR submanifold in a Kaehler manifold. In §3 we obtain necessary and sufficient conditions in order to characterize totally geodesic leaves of the totally real foliation. A pinching theorem for leaves of the holomorphic distribution is obtained in §4.

2. CR submanifolds of a Kaehler manifold. Let M be a Kaehler manifold of complex dimension m and N be an n — dimensionl Riemannian submanifold immersed in M. The complex structure on M will be denoted by J and both metrics on M and N will be denoted by the same symbol g. For each $x \in N$ denote by T_xN and ν_x,respectively the tangent space and the normal space to N at x.

Definition. *The submanifold N is said to be CR-submanifold of M if there exists a differentiable distribution $D : x \to D_x \subset T_xN$ on N such that the following conditions are fulfilled :*

1. *D is holomorphic, i.e. $J(D_x) = D_x$ for each $x \in N$,*

2. *the complementary orthogonal distribution $D^\perp : x \to D_x^\perp \subset T_xN$ is totally real, i.e. $J(D_x^\perp) \subset \nu_x$, for each $x \in N$.*

If in particular, dim $D^\perp = 0$ we get complex submanifolds and if we have dim $D = 0$, then we get totally real submanifolds. Moreover, each real hypersurface of a Kaehler manifold is a CR-submanifold.

Now, let $\bar{\nabla}$ be the Kaehlerian connection on M and ∇ be the Levi — Civita connection on N. Denote by $\nabla^\perp$ the linear connection induced by

$\widetilde{\nabla}$ on the normal bundle and by h the second fundamental form of the immersion of N in M.

Then we have the well known equations of Gauss and Weingarten

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\widetilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{2.2}$$

for all vector fields X, Y tangent to N and ξ normal to N. For the linear operator A_ξ on the tangent bundle TN we have

$$g(h(X, Y), \xi) = g(A_\xi X, Y). \tag{2.3}$$

Each vector field $J\xi \in TM$ has a unique decomposition

$$J\xi = B\xi + C\xi, \tag{2.4}$$

where $B\xi \in D^\perp$ and $C\xi \in \nu$. Hence, we have defined two morphisms of vector bundles $B: \nu \to D^\perp$ and $C: \nu \to \nu$. It is not difficult to see that these morphisms verify $JB + C^2 = -I$, where I is the identity morphism on the normal bundle. This leads us to $C^3 + C = 0$, which means that on each normal bundle of a CR submanifold immersed in a Kaehler manifold we have a natural Yano's structure [7].

The distributions D and $D^\perp$ are given by the projectors P and Q respectively. Then by using the fact that $\widetilde{\nabla}$ is Kaehlerian, and taking the holomorphic parts, the totally real and the normal parts, from (2.1) and (2.2) we get

$$P(\nabla_X JPY) - P(A_{JQY}X) = JP(\nabla_X Y), \tag{2.5}$$

$$Q(\nabla_X JPY) - Q(A_{JQY}X) = Bh(X, Y), \tag{2.6}$$

$$h(X, JPY) + \nabla^\perp_X JQY = JQ(\nabla_X Y) + Ch(X, Y), \tag{2.7}$$

for all vector fields X, Y tangent to N.

3. Totally geodesic leaves of totally real foliation. As we have seen in the previous paragraph on each CR submanifold of a Kaehler manifold there are defined two real distributions. The integrability of these distributions is characterized by the following theorem.

Theorem 3.1. *Let N be a CR submanifold of a Kaehler manifold M. Then, we have:*

1. *the totally real distribution is involutive* [4],
2. *the holomorphic distribution is involutive if and only if the second fundamental form verifies* [1]

$$h(X, JY) = h(JX, Y) \quad \forall X, Y \in D. \tag{3.1}$$

On totally geodesic leaves of the holomorphic distribution have been obtained results by B. Y. Chen [4]. We are going to get necessary and sufficient conditions for both immersions of a totally real leaf in N and M in order to be totally geodesic.

Let $N^\perp$ be a leaf of the totally real distribution on N. Denote by $h^\perp$ and $\widetilde{h}^\perp$ the second fundamental forms of the immersion of $N^\perp$ in N and M respectivelly. It is not difficult to see that we have

$$\widetilde{h}^\perp(X, Y) = h(X, Y) + h^\perp(X, Y), \quad \forall X, Y \in TN^\perp. \tag{3.2}$$

The CR submanifold N of the Kaehler manifold M is said to be

1. mixed totally geodesic, if we have

$$h(X, Y) = 0, \quad \forall X \in D, \ Y \in D^\perp. \tag{3.3}$$

2. $D^\perp$-totally geodesic, if we have

$$h(X, Y) = 0, \quad \forall X, Y \in D^\perp. \tag{3.4}$$

Now, we can state

Theorem 3.2. *Let $N^\perp$ be a totally real leaf of the CR submanifold N immersed into the Kaehler manifold M. Then $N^\perp$ is totally geodesic immersed in N if and only if we have*

$$h(X, Y) \in C\nu; \ \forall X \in D^\perp, \ Y \in D. \tag{3.5}$$

Proof. The equation of Weingarten for the immersion of $N^\perp$ in N is the following

$$\nabla_X Y = -A^\perp_Y X + \widetilde{D}_X Y, \quad \forall X \in TN^\perp, \ Y \in D, \tag{3.6}$$

where $\widetilde{D}$ is the linear connection induced by Levi-Civita connection ∇ on the holomorphic distribution and $A^\perp_Y : TN^\perp \to TN^\perp$ satisfies

$$g(h^\perp(X, Z), Y) = g(A^\perp_Y X, Z) \quad \forall X, Z \in TN^\perp, \ Y \in D. \tag{3.7}$$

From (3.6) we obtain

$$g(A^\perp_Y X, Z) = -g(Q(\nabla_X Y), Z). \tag{3.8}$$

Next, by a direct computation, from (2.7) we get

$$Q(\nabla_X Y) = C^2 h(X, Y) - Jh(X, JY), \ \forall X \in TN^\perp, Y \in D. \tag{3.9}$$

Taking into account that $C^2 h(X, Y) \in \nu$, from (3.8), (3.9) and (2.4) we obtain

$$g(A^\perp_Y X, Z) = g(Bh(X, JY), Z) \quad \forall X, Z \in TN^\perp, \ Y \in D. \tag{3.10}$$

Thus, from (3.10), it follows that $N^\perp$ is totally geodesic immersed in N if and only if we have

$$Bh(X, JY) = 0, \quad \forall X \in D^\perp, \ Y \in D. \tag{3.11}$$

Since J is an authomorphism of the holomorphic distribution, (3.11) is equivalent to (3.5). The proof is complete.

This theorem implies fundamental results on mixed totally geodesic CR submanifolds.

Theorem 3.3. *Let N be a mixed totally geodesic CR submanifold of a Kaehler manifold M. Then,*

1. *Each leaf of the totally real distribution is totally geodesic immersed in N.*

2. *Each leaf of the totally real distribution is totally geodesic immersed in the Kaehler manifold M if and only if N is $D^{\perp}$-totally geodesic.*

Indeed, the assertion 1 of the theorem follows as a consequence of the Theorem 3.2. and the assertion 2 follows from (3.2) and the assertion 1 We should make the remark that there exists an important class of CR submanifolds (often studied by complex analysts) on which the assertion 1. of the Theorem 3.3. implies N is mixed totally geodesic. We shall define further that class of submanifolds. Suppose the dimensions of the totally real distribution and of the normal space to the CR-submanifold are equal. A such CR-submanifold is called a generic CR-submanifold. It is easily seen that in this case, we have $J = B$ on the normal bundle. This, together with the assertion 1 of the Theorem 3.3. implies.

Theorem 3.4. *A generic CR-submanifold N of a Kaehler manifold M is mixed totally geodesic if and only if each leaf of the totally real distribution is totally geodesic immersed in N.*

Now, we can establish a result on the immersion of each leaf of $D^{\perp}$ in M.

Theorem 3.5. *Let N be a CR submanifold of a Kaehler manifold M Then, each leaf of the totally real distribution is totally geodezic immersed in M if and only if the following conditions are fulfilled*

$$1.\ \nabla^{\perp}_X JY \in JD^{\perp},\quad \forall X, Y \in D^{\perp},$$

$$2.\ h(X, Z) \in C\nu,\quad \forall X \in D^{\perp},\ Z \in TN.$$

Proof. For all $X, Y \in D^{\perp}$, adding (2.6) and (2.7) we obtain

$$h(X, Y) = JQ(A_{JY}X) - J(\nabla^{\perp}_X JY) - Q(\nabla_X Y). \tag{3.12}$$

Now, using (2.4), from (3.12) we have

$$h(X, Y) = JQ(A_{JY}X) - C(\nabla^{\perp}_X JY). \tag{3.13}$$

If each leaf of $D^{\perp}$ is totally geodesic immersed in M, then from (3.2) and Theorem 3.2. follows $h(X, Z) \in C\nu$ for all $X \in D^{\perp}$ and $Z \in D$. In this case, N is $D^{\perp}$-totally geodesic. This together with relation obtained above implies condition 2 of the theorem. Also, from (3.13) follows $C(\nabla^{\perp}_X JY) = 0$, for all $X, Y \in D^{\perp}$, which in fact is equivalent to the condition 1 of the theorem.

Conversely, taking account of condition 2 of Theorem 3.3. it is sufficient to prove that N is $D^{\perp}$-totally geodesic. This follows from (3.13) by using $h(X, Y) \in C\nu$ for all $X, Y \in D^{\perp}$ and condition 1 of the theorem.

Now, we have

Theorem 3.6. *Let N be a generic CR-submanifold of a Kaehler manifold M. Then, each leaf of the totally real distribution is totally geodesic immersed in M if and only if N is mixed totally geodesic and $D^{\perp}$-totally geodesic.*

The proof follows from Theorem 3.5.

4. The geometry of leaves in a complex space form Suppose now M be a complex space form of constant holomorphic sectional curvature c and N be a CR submanifold of M. We say that N is a *proper CR-submanifold* of M, if both distributions have non-null dimensions.

First we recall ([2], Corollary 4.1.).

Proposition 4.1. *There does not exist totally geodezic proper CR-submanifolds of an elliptic or hyperbolic complex space.*

Now, we can state

Theorem 4.1. *Let N be a generic proper CR submanifold of an elliptic or hyperbolic complex space $M(c)$. Suppose the holomorphic distribution is involutive and a leaf $N^{\perp}$ of the totally real distribution is totally geodesic immersed in M.*

Then, the leaf N^* of D passing through the same point as $N^{\perp}$ is never totally geodesic immersed in M.

Proof. Since $N^{\perp}$ is totally geodesic immersed in M, from Theorem 3.6. follows that N is mixed totally geodesic and $D^{\perp}$-totally geodesic. For N^* denote by h^* and $\tilde{h}^*$ the second fundamental forms of immersions of N^* in N and M respectively. Then we have

$$\tilde{h}^*(X, Y) = h(X, Y) + h^*(X, Y),\quad \forall X, Y \in D. \tag{4.1}$$

Now, if N^* is totally geodesic immersed in M, then N is D-totally geodesic, that is, $h(X, Y) = 0$ for all $X, Y \in D$. Therefore, N is totally geodesic immersed in M. By Proposition 4.1. this is a contradiction.

We conclude this paper by a pinching theorem for leaves of the holomorphic distribution. In order to do this, we recall some formulas.

The sectional curvature determined by $\{X, JX\}$ where X is a unit vector from the holomorphic distribution, is called the holomorphic sectional curvature of N and it is denoted by $H_N(X)$.

Then, we have [1]

$$H_N(X) = c + g(h(X, X), h(JX, JX)) - g(h(X, JX), h(X, JX)). \tag{4.2}$$

Now, let CP^3 be the 3-dimensional complex projective space of constant holomorphic sectional curvature 1. Suppose N is a proper CR submanifold of CP^3. Then, either $\dim. D = \dim. D^{\perp} = 2$ or $\dim. D = 4$ and $\dim. D^{\perp} = 1$. But the second case arises when N is a real hypersurface of CP^3 and its geometry was investigated by many people. For this reason we shall study the first case.

The CR submanifold N is said to be *foliate* if the holomorphic distribution is involutive.

Then, we have

Theorem 4.3. *Let N be a foliate CR submanifold of CP^3. If the following conditions are fulfilled:*

$$1.\ H_N = 1, \qquad 2.\ \|h^*\|^2 < \frac{3}{5}$$

then, N^* is totally geodesic immersed in CP^3.

Proof. Since the holomorphic distribution is involutive, from (3.1) and (4.2) we have

$$H_N(X) = 1 - g(h(X, X), h(X, X)) - g(h(X, JX), h(X, JX)), \tag{4.3}$$

for all $X \in D$. The condition 1 of the theorem and (4.3) imply $h(X, Y) = 0$ for all $X, Y \in D$. Then from (4.1) we have

$$\| \widetilde{h}^* \| = \| h^* \| . \tag{4.4}$$

The holomorphic sectional curvature of N^* is given by

$$H_{N^*} = 1 - \frac{1}{2} \| \widetilde{h}^* \|^2. \tag{4.5}$$

Next, using (4.4), (4.5) and the condition 2 of the theorem, we obtain

$$H_{N^*} > \frac{7}{10}. \tag{4.6}$$

Finally, the theorem follows from (4.6) and the Theorem 6.9 from [6].

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COMPLEXES DE QUADRIQUES DANS P_3

PAR

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0. Introduction. Dans plusieurs Notes antérieures [4] — [7], nous avons étudié, dans l'espace projectif P_3, la géométrie différentielle des variétés à un ou deux paramètres, engendrées par une quadrique réelle, qui n'est pas un couple de plans.

Dans le même ordre d'idées, nous allons étudier, dans cette Note, toujours dans P_3, la géométrie différentielle des variétés à trois paramètres, de quadriques ponctuées, dont les équations ont tous les coefficients réels. Soulignons, donc, que nous considérons, comme quadrique génératrice, aussi une quadrique imaginaire ou un couple de plans.

Les variétés de quadriques qui dépendent de trois paramètres seront nommées *complexes de quadriques*. En notant par σ et n, l'espèce, respectivement la signature (l'indice négatif d'inertie) d'une quadrique [2], un complexe engendré par une quadrique (σ, n) sera noté par $V\,(3\,;\,(\sigma, n)\,;\,P_3)$, le premier nombre 3 signifiant le nombre des paramètres dont dépend la quadrique génératrice. Il en résulte que nous allons distinguer huit types de complexes de quadriques dans P_3, selon la classification projective des quadriques.

Pour des raisons de commodité, nous étudions simmultanément les complexes dont les quadriques génératrices ont la même espèce.

En utilisant la méthode du repère mobile de Cartan, nous allons déterminer, pour chacun de ces complexes, en des conditions qui seront précisées, le système complet d'invariants, le repère Frenet, les formules Frenet, le théorème fondamental et les interprétations géométriques des éléments du repère et de certains invariants.

Comme dans l'étude des courbes ou des surfaces en P_3, il y a, — pour chaque type de quadrique génératrice — des complexes différents, ayant des formules Frenet distinctes. Dans cette Note, pour chaque type de quadrique génératrice (σ, n) fixée, nous allons étudier, un seul type de complexe, qui sera noté $V_1\,(3\,;\,(\sigma, n)\,;\,P_3)$. L'étude d'autres complexes, ayant la même quadrique génératrice, sera fait dans un autre travail.

Dans ce qui suit, nous utilisons, plus fréquemment, les indices suivants :