(which is valid not only for three sequences f_{0h}, f_{0t}, f_{0z} but for any finite number of sequences $f_{0h}, f_{0t}, \ldots, f_{0z}$).

From (28) it follows that the finitely many sequences $f_{0h}, f_{0t}, \ldots, f_{0z}$, given in (29), are elements of the subring R_0. Consequently, from (31) and (32) it follows that $e_m \in R_0$ and therefore $R_0 = R$, as claimed. But this, in view of (19) and (23) leads to a contradiction. Thus, our assumption in false and $\alpha \geqslant \bar{R}$. Hence, the Theorem is proved.

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ON PSEUDO-IDEALS OF GROUPS

BY

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A nonempty subset A of a semigroup S is called a left pseudo-ideal if and only if

i) $ab \in A$ whenever $a, b \in A$,

ii) $x^2 A \subseteq A$, for every element $x \in S$.

The right pseudo-ideal is defined analogously. In [3], the first author has shown that : i) a left ideal of a semigroup is always a left pseudo-ideal but the converse is not true ; ii) a bi-ideal [1] may not be a pseudo-ideal and a pseudo-ideal may not be a bi-ideal. Also by means of the pseudo-ideal and bi-ideal, the first author has obtained a criterion for determining the regularity of a semigroup.

Now the object of this paper is to study some more properties of pseudo-ideals, especially, in groups. The example given below shows that a group may contain pseudo-ideals. Pseudo-ideals in a group have some properties which do not hold in a semigroup. With the help of these properties of pseudo-ideals, we have obtained the criteria for determining when a semi-group will be a group. Also by using the same properties of the pseudo-ideals, we have proved the following known result :

In a finite field, every element can be expressed as a sum of two squares.

Example : R denotes the set of all non zero rational numbers. It is known that R is a multiplicative group. Suppose $A = \{r^2 \mid r \in R\}$. Then it can be shown that A is a proper pseudo-ideal of R. Again let B be a subset consisting of all the rational numbers r^2, $1/3\, r^2$, $3r^2$ where $r \in R$. Now B is also a proper pseudo-ideal of R.

The above example shows that a group may contain more than one pseudo-ideal.

We recall the following properties of pseudo-ideals. In a group a left pseudo-ideal is a right pseudo-ideal and vice-versa ([3], p. 111).

Every pseudo-ideal of a group is also a normal subgroup but the converse is not true. Hence in a group every pseudo-ideal contains 1 — the identity element of the group ([3], p. 111).

Now we have the following propositions.

Proposition 1. *A semigroup S will be a group if and only if $\bar{B}A = A = A\bar{B}$ for any two pseudo-ideals A, B of S, where $\bar{B} = \{x^2 \mid x \in B\}$.*

Proof. Suppose S is a group and A, B are two pseudo-ideals of S. Then $A\bar{B} \subseteq AS \subseteq A$. Also if $x \in A$, then $x = x \,.\, 1^2 \in A\bar{B}$. Thus $A \subseteq A\bar{B}$. Hence $A\bar{B} = A$. Similarly $\bar{B}A = A$. So $A\bar{B} = A = \bar{B}A$.

Conversely, let S be a semigroup, in which the given hypothesis holds. Let $a \in S$. Now S and Sa being two ideals of S, are also pseudo-ideals. Hence by the given condition, $S = S\,(\bar{Sa}) \subseteq S\,(Sa) = (SS)\,a \subseteq Sa$. Thus $S = Sa$. Similarly $aS = S$. So $Sa = S = aS$ for every element a of S. Hence S is a group.

Proposition 2. *A semigroup S with the identity element 1 will be a group if and only if every psudo-ideal of S contains 1.*

Proof. The necessary part is obvious. To prove the sufficiency part, let every pseudo-ideal of the semigroup S contain 1. Let A, B be any two pseudo-ideals of S. Now $A\bar{B} \subseteq AS \subseteq A$. Also if $x \in A$ then $x = x \,.\, 1^2 \in A\bar{B}$. So $A \subseteq A\bar{B}$. Hence $A\bar{B} = A$. Similarly $A = \bar{B}A$. Thus $A\bar{B} = A = \bar{B}A$. Hence by proposition 1, S is a group.

Proposition 3. *A normal subgroup H of a group G will be a pseudo-ideal if and only if every element of the factor group G/H is of order 2.*

Proof. Let G be a group and H be a normal subgroup which is also a pseudo-ideal of G. Let $aH \in G/H$ where $a \in G$. Now $(aH)^2 = a^2H \subseteq H$. Thus $(aH)^2 = H$. So every element of G/H is of order 2.

Conversely, let H be a normal subgroup of G and every element of G/H is of order 2. Let $g \in G$. Now $g^2H = (gH)^2$. Thus $g^2H \subseteq H$. So H is a pseudo-ideal of G.

It is known ([2], p. 66) that the subgroup generated by all the commutators in G is called the commutator subgroup of G.

Corollary 1. *Every pseudo-ideal of a group G contains the commutator subgroup.*

Proof. The statement follows from the theorem 4.21 ([2], p. 66).

Corollary 2. *A group G will be abelian if the intersection of all the pseudo-ideals of G consists of 1, the identity element of G, only.*

Corollary 3. *If A is a subgroup of index 2 of a group G, then A is a pseudo-ideal of G.*

Proof. The corollary follows since A is a normal subgroup and G/A is of order 2.

Proposition 4. *Let G be a group. Now in G the set of all subsemigroups of G and the set of all pseudo-ideals of G will coincide if and only if the order of each element of G is 2.*

Proof. Let G be a group in which every subsemigroup is also a pseudo-ideal. Now $\{1\}$ being a subsemigroup, it is a pseudo-ideal. Thus $a^2 \,.\, 1 \in \{1\}$ i.e. $a^2 = 1$ where $a \in G$. So every element of G is of order 2. Conversely let every element of G be of order 2. Let A be a subsemigroup of G. Now for every $x \in G$, $x^2 \,.\, A = 1 \,.\, A = A$. Thus A is a pseudo-ideal of G. Hence the proposition.

Proposition 5. *In a group G a subsemigroup will be a pseudo-ideal if and only if it contains a pseudo-ideal of G.*

Proof. Obviously every pseudo-ideal contains itself as a pseudo-ideal of G. Now let H be a subsemigroup of G containing a pseudo-ideal A of G. Let $g \in G$ and $a \in H$. Now $1 \in A$. So $g^2 = g^2 \,.\, 1 \in A$ since A is a pseudo-ideal of G. Thus $g^2a \in H$. So H is a pseudo-ideal of G.

Proposition 6. *If N is a pseudo-ideal of a group G and A is a pseudo-ideal of G containing N, then A/N is a pseudo-ideal of G/N. Conversely, if N is a pseudo-ideal of G, then each pseudo-ideal of G/N has the form A/N for a suitable pseudo-ideal A of G.*

Proof. Let N be a pseudo-ideal of a group G and be contained in another pseudo-ideal A of G. Then N is also a pseudo-ideal of A and hence $A/N \subseteq G/N$. To prove A/N is a pseudo-ideal of G/N, let $aN \in A/N$ where $a \in A$ and $gN \in G/N$ where $g \in G$. Now $(gN)^2 \,.\, (aN) = (g^2 .\ a)\, N \in A/N$ since $g^2 \,.\, a \in A$. So A/N is a pseudo-ideal of G/N. Conversely let N be a pseudo-ideal of G. Let K be a pseudo-ideal of G/N. Suppose is the natural homomorphism of G onto G/N. Let $A = \{a \in G / a\,\varphi \in K\}$. Then A is a pseudo-ideal of G. Now the restriction of φ to A is, of course, a homomorphism of A into G/N and just onto K and the kernel of φ is, of course, N. Now by the first isomorphism theorem ([2], p. 58) $A/N \cong K$. Hence the proposition.

Corollary. *Let N be a pseudo-ideal of a group G. Then there exists a one-one correspondence between the pseudo-ideals of G containing N and the pseudo-ideals of G/N.*

Proposition 7. *Let H be a subsemigroup with the identity element 1 of a group G. A relation ρ is definied on G as follows : $a\,\rho\,b$ if and only if $ab \in H$. Then ρ is a congruence relation if and only if H is a pseudo-ideal of G.*

Proof. Let ρ be a congruence relation defined on a group G as above. Let $a \in H$. Then $a \,.\, 1 = a \in H$, i.e. $a\,\rho\,1$. So for any $x \in G$, $xa\,\rho\,x$ which implies $x\,\rho\,xa$. *Therefore* $x^2 \,.\, a \in H$. Hence H is a pseudo-ideal of G. Conversely, let H be a pseudo-ideal. Since H is a normal subgroup of G, we have a congruence relation defined on G as follows. $a\,\rho' b \Leftrightarrow a^{-1} \,.\, b \in H$. Again since H is a pseudo-ideal $a^{-1} \,.\, b \in H \Leftrightarrow ab \in H$. Thus $a\ \rho b \Rightarrow ab \in H \Rightarrow a\,\rho'\,b$. So ρ and ρ' are identical. Hence ρ is a congruence.

Proposition 8. *Let φ be a homomorphism of a semigroup G onto a group H. Now the hernel of φ will be a pseudo-ideal if and only if the order of each element of H is 2.*

Proof. Let K be the kernel of φ and the order of each element of H be 2. Let $a \in K$ so $a\,\varphi = 1$. Now for any $g \in G$, $(g^2\,a)\,\varphi = (g\,\varphi)^2\,(a\,\varphi) = 1 \,.\, 1 = 1$. So $g^2a \in K$. So K is a pseudo-ideal of G. Conversely, let K be a pseudo-ideal of G. Let $h \in H$, and $g\,\varphi = h$. Now if $a \in K$ then $g^2a \in K$. So $(g^2a)\,\varphi = 1$ i.e.$(g\varphi)^2\,(a\varphi) = 1$ i.e. $(g\varphi)^2 = 1$. So $h^2 = 1$ for every element $h \in H$. Hence the proposition.

Corollary. *The set of all subsemigroups of a group H will coincide with the set of all pseudo-ideals of H if and only if there exists a homomorphism from a semigroup S onto H whose kernel is a pseudo-ideal of S.*

Proof. The corollary follows from the propositions 5 and 9.

Proposition 9. *Every finite cyclic group of even order contains one and only one proper pseudo-ideal.*

Proof. Let G be a finite cyclic group of even order $2n$. Let it be generated by g. Now the set $A = \{1, g^2, g^4, \ldots, g^{2n-2}\}$ forms a proper pseudo-ideal of G. We claim that A is the only proper pseudo-ideal of G. Let there be another pseudo-ideal B of G. Since $1 \in B$, $A \subseteq B$. Now if $A \neq B$ then B contains at least one element g^r where r is an odd integer. In this case $B = G$. Hence the proposition.

Proposition 10. *Every finite cyclic group of odd order does not contain any proper pseudo-ideal.*

Proof. Let G be a finite cyclic group of odd order and A be a pseudo-ideal of G. Then the factor group G/A is also of odd order. But in that case G/A cannot have elements of order 2. Hence G does not contain any proper pseudoideal.

Now using the proposition 9 we shall prove the following known result.

Proposition 11. *In a finite field, every element can be expressed as a sum of two squares.*

Proof. Let F be a finite field. Then $F \cong GF(p^n)$ for some prime integer p. Let $\Gamma = GF(p^n)$. Now the nonzero elements of Γ forms a multiplicative cyclic group of even order. Hence, by proposition 9, the set of all nonzero elements of Γ contains one and only one proper pseudo-ideal. So Γ, which may be considered as a multiplicative cyclic group with externally adjoined zero, contains one and only one proper nonzero pseudo-ideal given by $\bar{\Gamma}$ ($\{x^2 \mid x \in \Gamma\}$. Let $\Gamma^* = \{x^2 + y^2 \mid x, y \in \Gamma\}$. Then Γ^* is also a pseudo-ideal of Γ containing $\bar{\Gamma}$. Hence either $\Gamma^* = \bar{\Gamma}$ or $\Gamma^* = \Gamma$. If $\Gamma^* = \bar{\Gamma}$, then $\bar{\Gamma} = \Gamma^*$ is a subfield of Γ. Hence $\bar{\Gamma} = \Gamma^* = GF(p^m)$ for some prime integer m which divides n. Let $n = mk$. But Γ contains $(p^n + 1)/2$ elements. Thus $(p^n + 1) = 2p^m$ or $1 = p^m - p^n = p^m \{2 - p^{m(k-1)}\}$ which is a contradiction unless $p = 1$. Hence $\bar{\Gamma} \neq \Gamma^*$. So $\Gamma^* = \Gamma$. Hence the proposition.

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ON AN ANALYTIC CHARACTERIZATION OF QUASICONFORMALITY IN NORMED SPACES

BY

MONICA FRUNZĂ

1. **Introduction.** There are several possibilities to define quasiconformality on infinite dimensional normed spaces. The simplest one consists in using conditions of a metric character such as that of G e h r i n g ([6], [2]). These conditions may be imposed either globally or on each bidimensional section. In [3] we have suggested a simple condition of analytic character which is equivalent with quasiconformality on finite dimensional spaces. In the present paper we deal with the same problem on general normed spaces.

2. **Definitions and notations.** Let X be a real normed space and $f: G \to X$ be a homeomorphic mapping of a domain G of X into X. The *linear dilatation* of f in a point $x \in G$ is defined as:

$$\delta_L(x) = \delta(x) = \lim_{r \to 0} \sup \frac{L(x, r)}{l(x, r)},$$

where for $r > 0$ small enough:

$$L(x, r) = \sup \{\|f(y) - f(x)\|, \|y - x\| = r\},$$
$$l(x, r) = \inf \{\|f(y) - f(x)\|, \|y - x\| = r\}.$$

If Π is a bidimensional linear manifold which meets G and $x \in G \cap \Pi$, then $\delta_L^\Pi(x) = \delta^\Pi(x)$ is defined as above by replacing f by its restriction to $G \cap \Pi$.

Let $K \geqslant 1$ be a fixed real number.

Definition 1. ([4], [6]) *A homeomorphic mapping $f: G \to X$ of a domain $G \subset X$ into X is called K-quasiconformal in the sense of Gehring if*

$$\delta(x) \leqslant K, \quad \forall x \in G.$$

Definition 1'. *A homeomorphic mapping $f: G \to X$ of a domain $G \subset X$ into X is called bidimensionally K-quasiconformal in the sense of Gehring if*

$$\delta^\Pi(x) \leqslant K, \quad \forall x \in G \cap \Pi,$$

for any bidimensional linear manifold Π which meets G.

Our analytic characterization of quasiconformality will be formulated in terms of scalar derivatives.