C) *Complexes de couples de plans distincts* $V_1(3; (2; n); P_3)$, $(n = 0,1)$. Avec (3.6), on obtient

$$(4.7) \qquad dQ = p_{6+i}Q + T_1\theta^1 + \varepsilon_2 T_2\theta^2 + T_3\theta^3, \quad T_1 = x^0x^1 + \\ + \varepsilon_2 x^2x^3, \; T_2 = x^0x^2, \; T_3 = x^1(x^2 + x^3).$$

La quadrique $T_1 = 0$ est un hyperboloïde, $T_2 = 0$ et $T_3 = 0$ sont des couples de plans distincts. Voici l'interprétation du repère. A_0A_3 et A_1A_3 sont les centres des couples de plans, $Q = 0$, respectivement $T_2 = 0$. A_3 est l'intersection de A_0A_3 et A_1A_3. A_0 est l'intersection de A_0A_3 avec le centre du couple de plans $T_3 = 0$. A_0 est situé sur $T_1 = 0$. Le plan $x^1 = 0$ est tangent à $T_1 = 0$, au point A_0. Le plan $x^1 = 0$ coupe la quadrique $T_1 = 0$, suivant A_0A_2 et A_0A_3. Le plan $x^2 = 0$, déterminé par A_0A_3 et A_1A_3 est l'un des plans du couple $T_2 = 0$; on a ainsi l'autre plan de ce couple, $x^0 = 0$. A_0A_2 coupe le plan $(x^0 = 0)$ au point A_2. A_1 est le pôle du plan $(x^0 = 0)$ par rapport à $T_1 = 0$.

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CONVEX FUNCTIONS WITH RESPECT TO A DISTRIBUTION ON A RIEMANNIAN MANIFOLD

BY

FULGA POP

In this Note we define the notion of *convex function with respect to a distribution* on a Riemannian manifold. Some examples and basic properties are given. Particularly, we investigate the *vertical* and *horizontal convexity* of the vertical lift of a function defined on a Riemannian manifold.

In a next paper we shall study the convexity of the complete lift of a function and of Pfaff forms.

§ 1. D-convex functions. Let M be a Riemannian manifold with respect to the metric g and the Levi-Civita connection ∇, having the local expressions g_{ij} and Γ^k_{ij} respectively.

Let $D = D(M)$ be a distribution on the manifold M (a subbundle of the tangent bundle TM).

Definition. *A smooth function* $f : M \to R$ *is called a D-convex function if for every point* $x \in M$ *and for every vector field* $X \in D$, *we have*:

$$(1) \qquad < \nabla_X df, X >_x \geqslant 0.$$

If the function f is a TM-convex function, then it is a convex function, [1] (In [1] the function f is called a convex function if $-f$ fulfills the condition (1) for every vector field X on the manifold M. Moreover, in [1] there are defined convex functions which are not smooth).

According to [2], [3], we denote:
$< \nabla_X df, Y > = \text{Hess}\,(f)\,(X, Y)$, *for* X, Y two vector fields on M.

Examples. 1. Let $f, g : R \to R$ be the functions $f(x) = x^2 + bx + c$ and $g(x) = \sin x$. We consider the function $h : R^2 \to R$ defined by $h(x, y) = f(x) + g(y)$. Then, for a vector field $X = (X^1, X^2) \in T(R^2)$, we have:

$$\text{Hess}\,(h)\,(X, X) = 2\,(X^1)^2 - (X^2)^2 \sin y.$$

It is obvious that the function h is not convex, but if we consider the 1-dimensional distribution D on R^2, generated by a vector field $X_0 = (X_0^1, X_0^2)$, with $X_0^2 = u(x, y)\,X_0^1$, such that $|u(x, y)| \leqslant \sqrt{2}$, then h is a D-convex function. In fact, if $X \in D$, then, for every point $z = (x, y) \in R^2$ we have:

$$\text{Hess}\,(h)_z\,(X, X) = (X^1)^2\,(2 - u^2 \sin y) \geqslant 0.$$

(We can take for example $u(x, y) = (2 + x^2 + y^2)/(1 + x^2 + y^2)$.

2. Let $f: R^2 \to R$ be the function defined by $f(x, y) = (1 + x^2)y$. For a vector field $X = (X^1, X^2) \in T(R^2)$, we have:

$$\text{Hess}\,(f)\,(X, X) = 2X^1\,(X^1 y + 2xX^2).$$

It is obvious that f is a not convex function. But, if we consider the distribution D on R^2, generated by the vector field $X_0 = x\dfrac{\partial}{\partial x} + \dfrac{1-y}{2}\,\dfrac{\partial}{\partial y}$, then, for $X = uX_0$, and every point $z \in R^2$, we have:

$$\text{Hess}\,(f)_z\,(X, X) = u^2\,\text{Hess}\,(f)_z\,(X_0, X_0) = 2u^2X^2 \geqslant 0,$$

that is f is a D-convex function.

3. Let M be a connected complete finite-dimensional Riemannian manifold and let X be a Killing vector field on M. We consider the function $f: M \to R$ defined by $f(x) = g(X_x, X_x) = \| X \|_x^2$. Then, from [2] we have:

$$1/2\ \text{Hess}\,(f)\,(Y, Y) = -R\,(X, Y, X, Y) + \| \nabla_Y X \|^2.$$

It follows that f as above defined is a convex function with respect to the distribution D on M generated by the vector field X, because for $Y = u\,.\,X$, we have $\text{Hess}\,(f)_x(Y, Y) = 2\| \nabla_Y X \|^2 \geqslant 0$.

Proposition 1. *Let M, M' be two Riemannian manifolds with Levi-Civita connections ∇ and ∇' respectively. Let D be a distribution on M and D' a distribution on M'. Let $F: M \to M'$ be a smooth map and $f: M' \to R$ a smooth function.*

Suppose that the following conditions are satisfied:

a) $F_*(D) \subset D'$, b) For every $X \in D$:

$$F_*\,(\nabla_X X) = \nabla'_{F_* X} F_* X, \tag{2}$$

(*for example $F_*/D: D \to D'$ is an affine colineation-we could call F a D-affine colineation). If f is a D'-convex function, then the composition $f \circ F$ is a D-convex function. Moreover, if we have b) and the condition*

a') $$F_*(D) = D',$$

then, the D-convexity of the composition $f \circ F$ implies the D'-convexity of the function f.

Proof. Let $X \in D$ and $x \in M$. Then: $\text{Hess}\ (f \circ F)_x(X, X) = X(d(f \circ F)\,X) - d\,(f \circ F)\,(\nabla_X X) = X\,(d\,(f \circ F)\,X) - df\,(F_*\,(\nabla_X X))$. But: $X\,(d\,(f \circ F)\,X) = X\,(df\,(F_*\,X) \circ F) = F_*X\,(df\,(F_*\,X))$, and by using (2) we obtain:

$$\text{Hess}\ (f \circ F)_x(X, X) = \text{Hess}\ (f)_{F(x)}\,(F_*\,X, F_*X), \tag{3}$$

which proves the first part of the proposition.

Now, if $D' = F_*(D)$, then the map F is onto and for every $X' \in D'$, we have $X' = F_*X$, for a certain $X \in D$. We can apply the above formula and the second part of the proposition follows.

Proposition 2. *Let D be a distribution on a Riemannian manifold and let I be an integral manifold of D.*

If $f: M \to R$ is a D-convex function, then the restriction $f' = f/I: I \to R$ is a convex function on I.

Proof. Let ∇' be the Levi-Civita connection induced on the integral manifold I by the Levi-Civita connection of the Riemannian manifold M. Let $i: I \to M$ be the inclusion map. Then, we have:
$i_*(T\,(I)) \subset D$ and $i_*\,(\nabla'_X X) = \nabla_{i_* X}\, i_* X$, for every vector field $X \in T\,(I)$.

Thus, we can apply the Proposition 1 from which, it follows that $f \circ i = f'$ is a $T\,(I)$-convex function, that is a convex function on I.

If a distribution D is completely integrable, then it defines a foliation of the manifold M.

Proposition 3. *Let (M, D) be a Riemannian foliated manifold and let $f: M \to R$ be a smooth function. Then f is a D-convex function (in this case we call f a foliated-convex function) if and only if the restriction of the function f to every leaf of the foliation D is a convex function.*

Proof. If f is a D-convex function and if F_x is the leaf of the foliation D at the point $x \in M$, then, by the Proposition 2, it follows that the restriction f/F_x is a convex function on F_x.

Conversely, if for any leaf F of the foliation D, the restriction f/F is a convex function, then, if $x \in M$ and $X \in D_x$, let F_x be the leaf of D at the point x. If $i: F_x \to M$ is the inclusion, then as in the proof of Proposition 1, we have:

$$\text{Hess}\ (f)_x\,(X, X) = \text{Hess}\,(f)_x\,(i_*X, i_*X) = \text{Hess}\ (f \circ i)_x\,(X, X) \geqslant 0,$$

which proves that f is a D-convex function.

§ 2. Convex functions on the tangent bundle of a Riemannian manifold. The case of the vertical lift of a function. Let M be a Riemannian manifold and let TM be the tangent bundle with the natural projection $\pi: TM \to M$.

We use the notations and the results from [4].

Consider the tangent bundle TM as a Riemannian manifold with the Riemannian metric G of Sasaki. Let $\tilde{\nabla}$ be the Levi-Civita connection of G. Let V and H be, respectively, the vertical and horizontal distributions on the Riemannian manifold TM with the metric G.

If $f: M \to R$ is a smooth function, then the vertical lift of f is the function:

$$f^v = f \circ \pi: TM \to R. \tag{4}$$

Definition. *A smooth function $\tilde{f}: TM \to R$ is called a vertical (horizontal) convex function if $\tilde{f}$ is a convex function on TM with respect to the vertical (horizontal) distribution.*

Proposition 4. *The vertical lift of a smooth function $f: M \to R$ is a vertical convex function on TM.*

Proof. Using the fact that the vertical distribution V on TM is generated by the vertical lifts of the vector fields on M and by the formula $[X^v, Y^v] = 0$, for any vector fields X, Y on M, it follows that the distribution V is completely integrable on TM. Thus we can apply the Proposition 3. For this, we remark that for a point $z \in TM$, the leaf F_z of the foliation

V is $F_z = \pi^{-1}(\pi(z))$. Therefore $f^v/F_z = (f \circ \pi)/F_z = f(\pi(z)) = \text{const.}$, and therefore it is obvious that f^v/F_z is a convex function. By Proposition 3, we obtain that f^v is a V-convex function on the manifold TM.

Remark. If we consider the null distribution D_0 on the manifold M, then any smooth function $f: M \to R$ is a D_0-convex function. Then, we have $V = \operatorname{Ker} \pi_*$, that is $\pi_*(V) = D_0$. Using the formula $\tilde{\nabla}_{X^v} Y^v = 0$, it follows that $\tilde{\nabla}_{\tilde{X}} \tilde{X} = 0$ for every vertical field $\tilde{X}$ and therefore we can write $\pi_*(\tilde{\nabla}_{\tilde{X}} \tilde{X}) = \nabla_{\pi_* \tilde{X}} \pi_* \tilde{X}$. In this way we can apply the Proposition 1 and we obtain again the Proposition 4.

Proposition 5. *Let $f: M \to R$ be a smooth function on a Riemannian manifold. Then, its vertical lift $f^v: TM \to R$ is a horizontal convex function if and only if the function f is convex.*

Proof. Let $\pi: TM \to M$ be the natural projection. If X is a vector field on the manifold M and if X^h is its horizontal lift, then we have $\pi_* X^h = X$. From this and by the fact that the distribution H is generated by the horizontal lifts of the vector fields on M, it follows:

$$\pi_*(H) = TM. \tag{5}$$

Then, if X, Y are two vector field on M, we have: $\tilde{\nabla}_{X^h} Y^h = (\nabla_X Y)^h - 1/2\, vR(X, Y)$, where $vR(X, Y)$ is a vertical vector field. Because $\pi_*(V) = D_0$ and $\pi_*(\nabla_X Y)^h = \nabla_X Y$, we obtain:

$$\pi_*(\tilde{\nabla}_{X^h} Y^h) = \nabla_X Y. \tag{6}$$

For a horizontal vector field $\tilde{X}$ on TM we have:

$$\pi_*(\tilde{\nabla}_{\tilde{X}} \tilde{X}) = \nabla_{\pi_* \tilde{X}} \pi_* \tilde{X}. \tag{7}$$

The formulas (5) and (7) show that we can apply the Proposition 1. It follows that f is TM-convex function if and only if $f \circ \pi = f^v$ is a H-convex function.

Remark. If we consider the null section $s_0: M \to TM$, then $M_0 = s_0(M)$ is an integral manifold of the horizontal distribution H on TM. If $f: M \to R$ is a convex function, then by the Propositions 5 and 2, it follows that the restriction f^v/M_0 is a convex function on M_0. For an Euclidian space this result has an elementary interpretation.

Remark. If in the Proposition 5 the manifold M is a locally Euclidian space, then the horizontal distribution H is completely integrable and in this case the proof of the proposition follows from the Proposition 3.

Remark. The Propositions 4 and 5 show that the vertical lift of a convex function on a Riemannian manifold is both a vertical and a horizontal convex function on the tangent bundle TM. But, generally, this lift is not a convex function.

Now we want to give sufficient conditions for a distribution $\tilde{D}$ on TM in order that the vertical lift of a certain convex function $f: M \to R$ be a $\tilde{D}$-convex function.

Let $\tilde{X}$ be a vector field on the tangent bundle TM of a Riemannian manifold M.

In local coordinates we can write:

$$\tilde{X} = \sum_i \left[X^i \left(\frac{\partial}{\partial x^i}\right)^h + Y^i \left(\frac{\partial}{\partial x^i}\right)^v \right], \tag{8}$$

where $X^i, Y^i \in C^\infty(TM, R)$. We calculate $\operatorname{Hess}(f^v)_z(\tilde{X}, \tilde{X})$, for a point $z \in TM$.

$$\operatorname{Hess}(f^v)_z(\tilde{X}, \tilde{X}) = \sum_{i,j} X^i X^j \operatorname{Hess}(f^v)_z\left(\left(\frac{\partial}{\partial x^i}\right)^h, \left(\frac{\partial}{\partial x^j}\right)^h\right) +$$

$$+ 2 \sum_{i,j} X^i Y^j \operatorname{Hess}(f^v)_z\left(\left(\frac{\partial}{\partial x^i}\right)^h, \left(\frac{\partial}{\partial x^j}\right)^v\right) + \sum_{i,j} Y^i Y^j \operatorname{Hess}(f^v)_z\left(\left(\frac{\partial}{\partial x^i}\right)^v, \left(\frac{\partial}{\partial x^j}\right)^v\right),$$

We have:

$\operatorname{Hess}(f^v)_z\left(\left(\frac{\partial}{\partial x^i}\right)^h, \left(\frac{\partial}{\partial x^j}\right)^h\right) = \operatorname{Hess}(f)_{\pi(z)}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$, by the formula (3);

$\operatorname{Hess}(f^v)_z\left(\left(\frac{\partial}{\partial x^i}\right)^v, \left(\frac{\partial}{\partial x^j}\right)^v\right) = 0$, by the Remark for the Proposition 4;

$$\operatorname{Hess}(f^v)_z\left(\left(\frac{\partial}{\partial x^i}\right)^h, \left(\frac{\partial}{\partial x^j}\right)^v\right) = \left(\frac{\partial}{\partial x^i}\right)^h \left(df\left(\pi_*\left(\frac{\partial}{\partial x^j}\right)^v\right) \circ \pi\right) -$$

$$- df\left(\pi_*\left(\tilde{\nabla}_{\left(\frac{\partial}{\partial x^i}\right)^h}\left(\frac{\partial}{\partial x^j}\right)^v\right)\right) = - df\left(\pi_*\left[\left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\right)^v + \frac{1}{2} hR\left(\cdot, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^i}\right]\right) =$$

$$= -\frac{1}{2} df\left(\pi_* hR\left(\cdot, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^i}\right).$$

If we consider the tensor field of type (1, 1),

$$R\left(\cdot, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^i} = R^m_{ikj} \frac{\partial}{\partial x^m} \otimes dx^k, \text{ then:}$$

$$hR\left(\cdot, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^i} = \sum_{m,k} R^m_{ikj}\, y^k \left(\frac{\partial}{\partial x^m} - \Gamma^p_{mn}\, y^n \frac{\partial}{\partial y^p}\right), \text{ and therefore}$$

$$\operatorname{Hess}(f^v)_z\left(\left(\frac{\partial}{\partial x^i}\right)^h, \left(\frac{\partial}{\partial x^j}\right)^v\right) = -\frac{1}{2} R^m_{ikj}\, y^k \frac{\partial f}{\partial x^m}.$$

Finally, we obtain:

$$\operatorname{Hess}(f^v)_z(\tilde{X}, \tilde{X}) = \operatorname{Hess}(f)_{\pi(z)}(\pi_* X, \pi_* X) - \sum_{i,j} X^i Y^j R^m_{ikj}\, y^k \frac{\partial f}{\partial x^m}. \tag{9}$$

From this formula we can state:

Proposition 6. *Let $f: M \to R$ be a convex function on a Riemannian manifold. Let $\tilde{D}$ be a distribution on the tangent bundle TM such that $\tilde{X} \in \tilde{D}$ implies the conditions:*

$$\sum_{i,j,m} R^m_{ikj} \frac{\partial f}{\partial x^m} X^i Y^j = 0, \ \forall k,$$

for the expression (8) of the vector field $\tilde{X}$.

Then the vertical lift f^v of the function f is a $\tilde{D}$-convex function.

Particularly, if M is a locally Euclidian space, f^v is a convex function on TM.

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A METHOD FOR FINDING THE STRONGLY CONNECTED COMPONENTS OF A DIGRAPH

BY

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In [2], [3], [4] and [5] there are four algorithms for computing the strongly connected components of a digraph. Our paper gives a new method similar to that included in [2]. Let $D = <\mathcal{N}, \mathcal{A}>$ be a *digraph* [1] with $N = \{n_1, n_2, ..., n_N\}$ the set of *nodes* and $\mathcal{A}$ the set of *arcs*. If certain elements of $\mathcal{A}$ can be placed in a sequence of the form $<n_{i_1}, n_{i_2}>$, $<n_{i_2}, n_{i_3}>, ..., <n_{i_{s-1}}, n_{i_s}>$, then, the set $\{<n_{i_1}, n_{i_2}>, <n_{i_2}, n_{i_3}>, ..., <n_{i_{s-1}}, n_{i_s}>\}$ is a *path* from n_{i_1} to n_{i_s} in D, a path denoted by $\mathfrak{C}[n_{i_1}, n_{i_s}]$.

For a digraph we consider the *path matrix*

$$P = (p_{ij}), \ i, \ j = 1, 2, ..., N, \ \text{where}:$$

$$p_{ij} = \begin{cases} 1, & \text{if there exists a path } \mathfrak{C}[n_i, n_j], \\ 0, & \text{otherwise.} \end{cases}$$

In [6] we have suggested a very simple algorithm for computing the path matrix of a digraph from his *adjacency matrix* $A = (a_{ij})$ $i, j = 1, 2, ..., N$ defined as follows:

$$a_{ij} = \begin{cases} 1, & \text{if } <n_i, n_j> \in \mathcal{A}, \\ 0, & \text{if } <n_i, n_j> \notin \mathcal{A}. \end{cases}$$

Two nodes n and m are *mutually strongly connected* $(n \sim m)$ if and only if there exist the paths $\mathfrak{C}[n, m]$ and $\mathfrak{C}[m, n]$. Obviously, „$\sim$" is an equivalence and induces a partition on $\mathcal{N}$ in the classes: $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_k$, called *strongly connected components*. Our problem is to find an algorithm for computing $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_k$. Having this purpose, we consider the matrix $P^* = (p^*_{ij})$ $i, j = 1, 2, ..., N$, where:

$$p^*_{ij} = \begin{cases} p_{ij}, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

Theorem 1. (see [2], p. 28, exercice 9). *If $\mathcal{C}_0$ is the strongly connected component that contains the node n_i, then $[(P^*)^2]_{ii} = |\mathcal{C}_0|$.*