Proof. Let $i, j \in \{1, 2, \ldots, N\}$, $i \neq j$. The element $[(P^*)^2]_{ij}$ is obtained by:

$$[(P^*)^2]_{ij} = \sum_{k=1}^{N} p^*_{ik} \cdot p^*_{kj}.$$

Obviously, the product $p^*_{ik} \cdot p^*_{kj}$ is equal to 0 or to 1. It is equal to 1 if there exists a path $\mathfrak{C}[n_i, n_j]$ passing through n_k, and it is equal to 0 otherwise. Hence, $[(P^*)^2]_{ij}$ represents the number of nodes belonging to at least a path from n_i to n_j, that implies that $[(P^*)^2]_{ii}$ is equal to the number of nodes belonging to at least a circuit including n_i. Because the nodes belonging to a circuit including n_i generate $\mathcal{C}_0$, we have: $[(P^*)^2]_{ii} = |\mathcal{C}_0|$. (Q.E.D.).

Theorem 2. *Let $\mathcal{C}_0$ be the strongly connected component that contains the node n_i. The node n_j belongs to $\mathcal{C}_0$ if and only if $[(P^*)^2]_{ij} = [(P^*)^2]_{ii}$.*

Proof. Suppose $[(P^*)^2]_{ij} = [(P^*)^2]_{ii}$. In this case the number of nodes belonging to at least a path $\mathfrak{C}[n_i, n_j]$ is equal to the number of nodes of $\mathcal{C}_0$ (see theorem 1). But every node of $\mathcal{C}_0$ belongs to at least a path from n_i to n_j. Hence, $n_j \in \mathcal{C}_0$. (Q.E.D.).

Reciprocally, suppose that $n_j \in \mathcal{C}_0$. In this case, obviously, we have:

$$[(P^*)^2]_{ij} \leqslant [(P^*)^2]_{ii}. \tag{1}$$

Because every node of $\mathcal{C}_0$ belongs to at least a path from n_i to n_j, we obtain according to theorem 1:

$$[(P^*)^2]_{ij} \geqslant [(P^*)^2]_{ii}. \tag{2}$$

From (1) and (2) it results: $[(P^*)^2]_{ij} = [(P^*)^2]_{ii}$. (Q.E.D.). Theorem 2 suggests a simple algorithm for computing the strongly connected components of D.

Algorithm. 1. $k := 0$, $\mathcal{C}_0 := \varnothing$. 2. $k := k + 1$.

3. If there exists a node $n_{i(k)}$, $i(k) \in \{1, 2, \ldots, N\}$ so that $n_{i(k)} \notin \bigcup_{\alpha=0}^{k-1} \mathcal{C}_\alpha$, then we construct the set $\mathcal{C}_k := \{n_{i(k)}\} \cup \{n_j \mid n_j \notin \bigcup_{\alpha=0}^{k-1} \mathcal{C}_\alpha$ and $[(P^*)^2]_{i(k)j} = [(P^*)^2]_{i(k)i(k)}$, $j = 1, 2, \ldots, N\}$, go to 2; else STOP.

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THE EQUATIONS OF RADIATIVE RELATIVISTIC MAGNETOHYDRODYNAMICS. INVARIANCE GROUPS FOR ONE-DIMENSIONAL FLOW

BY

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1. Introduction. In [1] we have established the equations of radiation field and its interaction with continuous medium in motion. We now derive, in Section 2, the equations which describe the motion of the medium in the presence of the radiation field and of the electromagnetic field, that is the equations of radiative magnetohydrodynamics, making use of the particle, momentum and energy balances. We also allow the medium velocity to be great enough that relativistic effects may become important. However, we also give the nonrelativistic limit of these equations. We use the way followed in [2], but we also take into account the electromagnetic field. We give, in Section 3, the covariant form of the equations of radiative relativistic magnetohydrodynamics, and in Section 4, some considerations about the energy equation and thermodynamics. In Section 5, the equations for a compressible ideal perfectly conducting gas are written.

Following the method given in [1], in Section 6, determine the invariance groups for the system of the hydrodynamics (HD), of the magnetohydrodynamics (MHD), of the radiation hydrodynamics (RHD), and of the radiative magnetohydrodynamics (RMHD), which describe the one-dimensional flow of an ideal perfectly conducting gas. In Section 7, we analyse some group properties of the previous systems.

2. The equations of radiative relativistic magnetohydrodynamics. We consider a compressible ideal fluid, moving with velocity $\mathbf{u}(\mathbf{r}, t)$ with respect to a reference frame R. We introduce a second reference frame R_0, namely the frame moving with the fluid. In this frame, we denote by $\rho(\mathbf{r}, t)$ the mass density of fluid, $p^{(m)}(\mathbf{r}, t)$ the fluid pressure, and $E^{(m)}(\mathbf{r}, t)$ the fluid energy density in excess of the rest energy. If c is the vacuum speed of light, we define

$$\lambda = (1 - u^2/c^2)^{-1/2};$$

then in the reference frame R for which the fluid velocity is $\mathbf{u}$, the mass density of fluid becomes $\lambda\rho$, and the total fluid energy density

$E_t^{(m)}(\mathbf{r}, t)$, the fluid energy flux $S_j^{(m)}(\mathbf{r}, t)$, the fluid stress tensor $\tau_{jk}^{(m)}(\mathbf{r}, t)$ are given by

$$\begin{aligned} E_t^{(m)} &= \lambda^2 [\rho c^2 + E^{(m)} + p^{(m)}] - p^{(m)}, \\ S_j^{(m)} &= \lambda^2 [\rho c^2 + E^{(m)} + p^{(m)}] u_j, \qquad (j,k=1,2,3) \\ \tau_{jk}^{(m)} &= -[p^{(m)}\delta_{jk} + \lambda^2 c^{-2} (\rho c^2 + E^{(m)} + p^{(m)}) u_j u_k]. \end{aligned} \tag{2.1}$$

If **E** is the electric field intensity, **H** the magnetic field intensity, **D** the electric induction vector, **B** the magnetic induction vector, **J** the current density, in the reference frame R, then we define: the electromagnetic energy density $E^{(em)}(\mathbf{r}, t)$, the electromagnetic energy flux $S_j^{(em)}(\mathbf{r}, t)$ and the electromagnetic stress tensor $\tau_{jk}^{(em)}(\mathbf{r}, t)$ by

$$\begin{aligned} E^{(em)} = {}^1\!/_2(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}),\ \mathbf{S}^{(em)} = \mathbf{E} \times \mathbf{H},\ \tau_{jk}^{(em)} = E_j D_k + H_j B_k - \\ - {}^1\!/_2 \delta_{jk} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}). \end{aligned} \tag{2.2}$$

Let $E^{(r)}(\mathbf{r}, t)$, $S_j^{(r)}(\mathbf{r}, t)$, $\tau_{jk}^{(r)}(\mathbf{r}, t)$ also be the corresponding quantities for the radiation field defined as in [1].

From the particle, momentum, and energy balances for an arbitrary volume element fixed in space, in absence of the external forces and of the external sources of energy, we deduce the equations:

$$\frac{\partial}{\partial t}(\lambda\rho) + \frac{\partial}{\partial x_k}(\lambda\rho u_k) = 0, \tag{2.3}$$

$$\begin{aligned} \frac{\partial}{\partial t} [\lambda^2 c^{-2} (\rho c^2 + E^{(m)} + p^{(m)}) u_j + c^{-2} (S_j^{(em)} + S_j^{(r)})] + \frac{\partial p^{(m)}}{\partial x_k} + \\ + \frac{\partial}{\partial x_k} [\lambda^2 c^{-2} (\rho c^2 + E^{(m)} + p^{(m)}) u_j u_k - \tau_{jk}^{(em)} - \tau_{jk}^{(r)}] = 0, \end{aligned} \tag{2.4}$$

$$\begin{aligned} \frac{\partial}{\partial t} [\lambda^2 (\rho c^2 + E^{(m)} + p^{(m)}) - p^{(m)} + E^{(em)} + E^{(r)}] + \frac{\partial}{\partial x_k} [\lambda^2 (\rho c^2 + \\ + E^{(m)} + p^{(m)}) u_k + S_k^{(em)} + S_k^{(r)}] = 0. \end{aligned} \tag{2.5}$$

An alternate form of the energy equation follows by multiplying (2.3) by c^2 and substracting the result from (2.5). We obtain the equivalent equation:

$$\begin{aligned} \frac{\partial}{\partial t} [\lambda (\lambda - 1) \rho c^2 + \lambda^2 (E^{(m)} + p^{(m)}) - p^{(m)} + E^{(em)} + E^{(r)}] + \\ + \frac{\partial}{\partial x_k} [\lambda (\lambda - 1) \rho c^2 u_k + \lambda^2 (E^{(m)} + p^{(m)}) u_k + S_k^{(em)} + \\ + S_k^{(r)}] = 0. \end{aligned} \tag{2.5''}$$

This makes the passage to the nonrelativistic limit more straightforward

The equations (2.3), (2.4), (2.5''), in the nonrelativistic limit, become:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k) = 0, \tag{2.3'}$$

$$\begin{aligned} \frac{\partial}{\partial t} (\rho u_j + c^{-2} S_j^{(em)} + c^{-2} S_j^{(r)}) + \frac{\partial p^{(m)}}{\partial x_j} + \frac{\partial}{\partial x_k} (\rho u_j u_k - \\ - \tau_{jk}^{(em)} - \tau_{jk}^{(r)}) = 0, \end{aligned} \tag{2.4'}$$

$$\begin{aligned} \frac{\partial}{\partial t} ({}^1\!/_2 \rho u^2 + E^{(m)} + E^{(em)} + E^{(r)}) + \frac{\partial}{\partial x_k} [({}^1\!/_2 \rho u^2 + E^{(m)} + \\ p^{(m)}) u_k + S_k^{(em)} + S_k^{(r)}] = 0, \end{aligned} \tag{2.5'}$$

since, in this limit $\lambda (\lambda - 1) c^2 = {}^1\!/_2 u^2$. To these equations, which describe the fluid flow, one adds the equations of the radiation field, the equations of the electromagnetic field, as well as the corresponding constitutive equations.

Equations (2.3) — (2.5) are the so-called *conservative form of the equations of radiative relativistic magnetohydrodynamics* for a volume element fixed in space. These equations can also be written in a form which is also conservative, but for a volume element moving with the fluid velocity, by the replacing the local derivative $\partial/\partial t$ by the material derivative d/dt given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u_k \frac{\partial}{\partial x_k}.$$

We find:

$$\lambda \rho \frac{d}{dt}\left(\frac{1}{\lambda\rho}\right) - \frac{\partial}{\partial x_k} u_k = 0, \tag{2.6}$$

$$\begin{aligned} \lambda \rho \frac{d}{dt}\left\{\frac{1}{\lambda\rho} c^{-2} [\lambda^2 (\rho c^2 + E^{(m)} + p^{(m)}) u_j + S_j^{(em)} + S_j^{(r)}]\right\} + \\ + \frac{\partial p^{(m)}}{\partial x_j} - \frac{\partial}{\partial x_k} [\tau_{jk}^{(em)} + \tau_{jk}^{(r)} + c^{-2} (S_j^{(em)} + S_j^{(r)}) u_k] = 0, \end{aligned} \tag{2.7}$$

$$\begin{aligned} \lambda\rho \frac{d}{dt} \left\{\frac{1}{\lambda\rho} [\lambda (\lambda - 1) \rho c^2 + \lambda^2 (E^{(m)} + p^{(m)}) - p^{(m)} + E^{(em)} + E^{(r)}]\right\} + \\ + \frac{\partial}{\partial x_k} [S_k^{(em)} + S_k^{(r)} + (p^{(m)} - E^{(em)} - E^{(r)}) u_k] = 0; \end{aligned} \tag{2.8}$$

with the nonrelativistic limit

$$\rho \frac{d}{dt}\left(\frac{1}{\rho}\right) - \frac{\partial u_k}{\partial x_k} = 0, \tag{2.6'}$$

$$(2.7') \qquad \rho \frac{d}{dt}\left[u_j + \frac{1}{\rho} c^{-2} (S_j^{(em)} + S_j^{(r)})\right] + \frac{\partial p^{(m)}}{\partial x_j} - \frac{\partial}{\partial x_k} [\tau_{jk}^{(em)} + \tau_{jk}^{(r)} + c^{-2} (S_j^{(em)} + S_j^{(r)}) u_k] = 0,$$

$$(2.8') \qquad \rho \frac{d}{dt}\left[{}^1\!/_2 u^2 + \frac{1}{\rho} (E^{(m)} + E^{(em)} + E^{(r)})\right] + \frac{\partial}{\partial x_k} [S_k^{(em)} + S_k^{(r)} + (p^{(m)} - E^{(em)} - E^{(r)}) u_k] = 0.$$

3. The covariant form of the equations of radiative relativistic magnetohydrodynamics. Let the momentum-energy tensor be defined for an ideal fluid as in [3, § 9)], [4, p. 303] by

$$(3.1) \qquad T_{\alpha\beta}^{(m)} = [\rho + c^{-2} (E^{(m)} + p^{(m)})] U_\alpha U_\beta + p^{(m)} \delta_{\alpha\beta} \quad (\alpha, \beta = 1, 2, 3, 4),$$

where

$$U_\alpha = \lambda \frac{dx_\alpha}{dt}$$

is the velocity quadrivector, while x_α ($\alpha = 1, 2, 3, 4$) are the coordinate of an event in the Minkowski space with $x_4 = i c t$. From (3.1) and (2.1) we find easily that

$$T_{jk}^{(m)} = - \tau_{jk}^{(m)}, \; T_{44}^{(m)} = - E_t^{(m)}, \; T_{j4}^{(m)} = T_{4j}^{(m)} = \frac{i}{c} S_j^{(m)}, \; (j, k = 1, 2, 3).$$

Let $T_{\alpha\beta}^{(em)}$ and $T_{\alpha\beta}^{(r)}$ also be the momentum-energy tensors of the electromagnetic and radiation fields respectively defined as in [5, p. 53], [1]. We can define the momentum-energy tensor [6] for an ideal fluid in the presence of the electromagnetic and radiation fields by $T_{\alpha\beta} = T_{\alpha\beta}^{(m)} - T_{\alpha\beta}^{(em)} - T_{\alpha\beta}^{(r)}$. The equations (2.4), (2.5) take then the form:

$$(3.2) \qquad \frac{\partial T_{\alpha\beta}}{\partial x_\beta} = 0.$$

Equation (3.2) expresses the conservative character of the momentum-energy tensor.

4. The energy equation and the thermodynamics. Let

$$E = E_t^{(m)} + E^{(r)} + E^{(em)}, \; S_j = S_j^{(m)} + S_j^{(r)} + S_j^{(em)}, \; \tau_{jk} = \tau_{jk}^{(m)} + \tau_{jk}^{(r)} + \tau_{jk}^{(em)}$$

be, respectively, the total energy density, the total energy flux, and the total stress tensor. As in the reference frame R_0, the total energy flux S_{0j} is given, in the nonrelativistic case, by $S_{0j} = S_j + \tau_{jk} u_k - E u_j$, if we also denote

$$U = 1/\rho \, (E^{(m)} + E^{(r)} + E^{(em)}),$$

the total internal energy per unit mass, the equations of motion and the equation of energy can be written:

$$\rho \frac{d}{dt} u_j + c^{-2} \frac{\partial}{\partial t} (S_j^{(r)} + S_j^{(em)}) = \frac{\partial}{\partial x_k} \tau_{jk}, \quad \rho \frac{d}{dt} ({}^1\!/_2 u^2 + U) = = \frac{\partial}{\partial x_k} (\tau_{jk} u_j) - \frac{\partial}{\partial x_k} S_{0k}.$$

From here, by eliminating the kinetic energy, we get

$$(4.1) \qquad \rho \frac{dU}{dt} = \tau_{jk} \frac{\partial}{\partial x_k} u_j - \frac{\partial}{\partial x_k} S_{0k} + c^{-2} u_j \frac{\partial}{\partial t} (S_j^{(r)} + S_j^{(em)}).$$

Equation (4.1) can be put in the more useful form:

$$(4.2) \qquad \rho dU = \rho \left[- p d\left(\frac{1}{\rho}\right) + E_k d\left(\frac{D_k}{\rho}\right) + H_k d\left(\frac{B_k}{\rho}\right)\right] + \left[\frac{1}{\sigma} J_0^2 - \frac{\partial}{\partial x_k} S_{0k}^{(r)}\right] dt,$$

where p is the total pressure $p = p^{(m)} + p^{(r)} + p^{(em)}$. In the equation (4.2)

$$dL^{(rev)} = - pd\left(\frac{1}{\rho}\right) + E_k d\left(\frac{D_k}{\rho}\right) + H_k d\left(\frac{B_k}{\rho}\right)$$

represents the reversible work given (per unit mass) to the fluid, in the time dt,

$$dL^{(ir)} = \frac{1}{\rho\delta} J_0^2 \, dt,$$

the energy (per unit mass) dissipated in the interior, due to the electric viscosity σ, in the time dt, and

$$dQ^{(r)} = - \frac{1}{\rho} \frac{\partial}{\partial x_k} S_{0k}^{(r)} \, dt$$

the energy radiated (per unit mass) in the interior, in the time dt. Therefore, the equation (4.2) has the form:

$$dU = dL^{(rev)} + dL^{(ir)} + dQ^{(r)},$$

but $d Q^{(r)} + d L^{(ir)}$ is the heat received by the element of fluid and which it does not give up. From the second principle of the thermodynamics, we have $d Q^{(r)} + d L^{(ir)} = T \, d S$, where T (r, t) is the fluid temperature, and S (r, t) is its total entropy: $S = S^{(m)} + S^{(r)} + S^{(em)}$.

Thus

$$(4.3) \qquad TdS = dU - dL^{(rev)}$$

and therefore

$$\rho T \frac{dS}{dt} = \frac{1}{\delta} J_0^2 \frac{\partial}{\partial x_k} S_{0k}^{(r)}.$$

or, using the equations of the radiation field:

$$(4.4) \qquad \rho T \frac{dS}{dt} = \frac{1}{\sigma} J_0^2 + \frac{\partial E^{(r)}}{\partial t} + P_0^{(r)}.$$

From (4.4), for $S^{(r)}$, we have

$$T \frac{dS^{(r)}}{dt} = d\left(\frac{E^{(r)}}{\rho}\right) + p^{(r)} d\left(\frac{1}{\rho}\right),$$

or, with (2.6'):

$$\rho T \frac{dS^{(r)}}{dt} = \frac{dE^{(r)}}{dt} + \frac{4}{3} E^{(r)} \frac{\partial}{\partial x_k} u_k,$$

which replaced in (4.4), gives

$$(4.5) \quad \rho T \frac{d}{dt}(S^{(m)} + S^{(em)}) + \frac{4}{3} E^{(r)} \frac{\partial}{\partial x_k} u_k + u_k \frac{\partial}{\partial x_k} E^{(r)} = \frac{1}{\delta} J_0^2 + P_0^{(r)}$$

for the equation of the energy.

5. The equations of radiative magnetohydrodynamics for an ideal perfectly conducting gas. For an ideal gas:

$$p^{(m)} = \rho R T, \; S^{(m)} = S_0^{(m)} - R \ln \left(\frac{\rho}{\rho_0}\right) \left(\frac{T_0}{T}\right)^{\frac{1}{\gamma - 1}},$$

where $R = c_p - c_v$, $\gamma = c_p/c_v$, c_p and c_v being the specific heats at constant pressure and volume, respectively; $S_0^{(m)}$, ρ_0, T_0 are constants.

For a perfectly conducting fluid, Ohm's law requires $\mathrm{E} + \mathrm{u} \times \mathrm{B} = 0$. So that, the equations of radiative magnetohydrodynamics for an ideal perfectly conducting gas become:

$$(5.1) \quad \frac{\partial \rho}{\partial t} + \rho \frac{\partial}{\partial x_k} u_k + u_k \frac{\partial \rho}{\partial x_k} = 0,$$

$$(5.2) \quad \rho \frac{\partial}{\partial t} u_j + \rho u_k \frac{\partial}{\partial x_k} u_j + \frac{\partial p^{(m)}}{\partial x_j} + \frac{1}{3} \frac{\partial E^{(r)}}{\partial x_j} + c^{-2} \frac{\partial}{\partial t} S_j^{(r)} + \frac{1}{\mu} (\mathrm{B} \times \mathrm{rot}\, \mathrm{B})_j = 0,$$

$$(5.3) \quad \begin{aligned} &\frac{1}{\gamma - 1} \frac{\partial p^{(m)}}{\partial t} + \frac{1}{\gamma - 1} u_k \frac{\partial p^{(m)}}{\partial x_k} + \frac{\gamma}{\gamma - 1} p^{(m)} \frac{\partial}{\partial x_k} u_k + \\ &+ \frac{4}{3} E^{(r)} \frac{\partial}{\partial x_k} u_k + u_k \frac{\partial E^{(r)}}{\partial x_k} = (E^{(r)} - a\, T^4), \end{aligned}$$

$$(5.4) \quad \frac{\partial}{\partial t} B_j + [\mathrm{rot}\, (\mathrm{B} \times \mathrm{u})]_j = 0, \text{ with } \frac{\partial}{\partial x_k} B_k = 0,$$

$$(5.5) \quad \frac{\partial E^{(r)}}{\partial t} + \frac{\partial}{\partial x_k} S_k^{(r)} = -\alpha (E^{(r)} - a\, T^4),$$

$$(5.6) \quad \frac{\partial}{\partial t} S_j^{(r)} + \frac{c^2}{3} \frac{\partial E^{(r)}}{\partial x_j} = -\alpha \left(S_j^{(r)} - \frac{1}{3} E^{(r)} u_j - a\, T^4 u_j\right),$$

which represents a system of 12 quasilinear partial differential equations with 12 unknown functions, in the variables t, x_j.

6. The invariance groups for the equations of radiative magnetohydrodynamics in the one-dimensional flow. We assume that the functions u_j, ρ, $p^{(m)}$, B_j, $E^{(m)}$, $S_j^{(r)}$ depend only on a single spatial variable x and on the time t. In this case, from (5.4) it follows $B_1 = \text{const.}$, which we shall denote in the following by B, and therefore, from (5.1) — (5.6) we obtain the system:

$$(6.1) \quad \begin{aligned} &\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \frac{1}{\rho} \frac{\partial p^{(m)}}{\partial x} + \frac{B_2}{\mu \rho} \frac{\partial B_2}{\partial x} + \frac{B_3}{\mu \rho} \frac{\partial B_3}{\partial x} = \frac{1}{\rho} c^{-2} [\alpha (S_1^{(r)} - \\ &\qquad - \frac{4}{3} E^{(r)} u_1) + P^{(r)} u_1], \end{aligned}$$

$$(6.2) \quad \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} - \frac{B}{\mu \rho} \frac{\partial B_2}{\partial x} = \frac{1}{\rho} c^{-2} [\alpha (S_2^{(r)} - \frac{4}{3} E^{(r)} u_2) + P^{(r)} u_2],$$

$$(6.3) \quad \frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x} - \frac{B}{\mu \rho} \frac{\partial B_3}{\partial x} = \frac{1}{\rho} c^{-2} [\alpha (S_3^{(r)} - \frac{4}{3} E^{(r)} u_3) + P^{(r)} u_3],$$

$$(6.4) \quad \frac{\partial \rho}{\partial t} + u_1 \frac{\partial \rho}{\partial x} + \rho \frac{\partial u_1}{\partial x} = 0,$$

$$(6.5) \quad \begin{aligned} &\frac{\partial p^{(m)}}{\partial t} + u_1 \frac{\partial p^{(m)}}{\partial x} + [\gamma p^{(m)} + \frac{4}{3} (\gamma - 1) E^{(r)}] \frac{\partial u_1}{\partial x} + \\ &+ (\gamma - 1) u_1 \frac{\partial E^{(r)}}{\partial x} = (\gamma - 1) P^{(r)}, \end{aligned}$$

$$(6.6) \quad \frac{\partial B_2}{\partial t} + u_1 \frac{\partial B_2}{\partial x} + B_2 \frac{\partial u_1}{\partial x} - B \frac{\partial u_2}{\partial x} = 0,$$

$$(6.7) \quad \frac{\partial B_3}{\partial t} + u_1 \frac{\partial B_3}{\partial x} + B_3 \frac{\partial u_1}{\partial x} - B \frac{\partial u_3}{\partial x} = 0,$$

$$(6.8) \quad \frac{\partial E^{(r)}}{\partial t} + \frac{\partial}{\partial x} S_1^{(r)} = -P^{(r)},$$

$$(6.9) \quad \frac{\partial}{\partial t} S_1^{(r)} + \frac{c^2}{3} \frac{\partial E^{(r)}}{\partial x} = -\alpha (S_1^{(r)} - \frac{4}{3} E^{(r)} u_1) - P^{(r)} u_1,$$

$$(6.10) \quad \frac{\partial}{\partial t} S_2^{(r)} = -\alpha (S_2^{(r)} - \frac{4}{3} E^{(r)} u_2) - P^{(r)} u_2,$$

$$(6.11) \quad \frac{\partial}{\partial t} S_3^{(r)} = -\alpha (S_3^{(r)} - \frac{4}{3} E^{(r)} u_3) - P^{(r)} u_3,$$

where $\alpha = \text{const.}$, $p^{(m)} = \rho R T$, $P^{(r)} = \alpha (E^{(r)} - a\, T^4)$.

To unify the notation for a further discussion, we put: $x^0 = t$, $x^1 = x$, $u^1 = u_1$, $u^2 = u_2$, $u^3 = u_3$, $u^4 = \rho$, $u^5 = p^{(m)}$, $u^6 = B_2$ $u^7 = B_3$, $u^8 = E^{(r)}$, $u^9 = S_1^{(r)}$, $u^{10} = S_2^{(r)}$; $u^{11} = S_3^{(r)}$; $\partial u^a / \partial x^0 = -p_0^a$, $\partial u^a / \partial x^1 = p^a$ $(a = \overline{1, 11})$.

The system (6.1) — (6.11) takes then the form:

$$(S) \qquad p^a_0 = F^a_{p^b}\, b \cdot p^b + F^a_0, \quad (a, b = \overline{1,11}),$$

where $F^a_{p^b}$ is the coefficient of the derivative p^b in the equation (6. a) of the system (6.1)—(6.11) and F^a_0 is the column of the righthand sides of the same system.

We apply here the procedure from [1] for the determination of the invariance groups for the system (S) by means of Lie algebra of its infinitesimal operators.

Let

$$X = \xi^\alpha \frac{\partial}{\partial x^\alpha} + \eta^a \frac{\partial}{\partial u^a} = \xi^\alpha p_\alpha + \eta^a q_a \tag{6.12}$$

be an infinitesimal operator which acts on analytical functions of the form $f(x, u)$.

A necessary and sufficient condition that the system (S) be invariant with respect to the operator X is that the functions $\xi^\alpha(x, u)$ and $\eta^a(x, u)$ be solution of the definition equations:

$$(A) \qquad \xi^0_{u^d} F^a_{p^{(b}} F^d_{p^{c)}} + \xi^1_{u^d} \delta^a_b F^d_{p^{c)}} - F^a_{p^d} F^d_{p^{(b}} \xi^0_{u^{c)}} - F^a_{p^{(b}} \xi^1_{u^{c)}} = 0,$$

$$(B) \qquad \begin{aligned} &\eta^a_{u^c} F^c_{p^b} - \delta^a_b(\xi^1_{x^0} + \xi^1_{u^c} F^c_0) - F^a_{p^b}(\xi^0_{x^0} + \xi^0_{u^c} F^c_0) - F^a_0 \xi^0_{u^c} F^c_{p^b} - \xi^\alpha F^a_{p^b, x^\alpha} - \\ &- \eta^c F^a_{p^b, u^c} - F^a_{p^c} \eta^c_{u^b} + F^a_{p^c} F^c_{p^b} \xi^0_{x^1} + F^a_{p^c} F^c_0 \xi^0_{u^b} + F^a_{p^b} \xi^1_{x^1} = 0, \end{aligned}$$

$$(C) \qquad \eta^a_{x^0} + \eta^a_{u^c} F^c_0 - F^a_0 \xi^0_{x^0} - F^a_0 \xi^0_{u^c} F^c_0 - \xi^\alpha F^a_{0, x^\alpha} - \eta^c F^a_{0, u^c} - \eta^b_{x^1} F^a_{p^b} + F^a_{p^b} F^b_0 \xi^0_{x^1} = 0,$$

where the subscripts x^α, u^a at ξ, η and after the comma at F^a_0 and $F^a_{p^b}$ denote partial derivatives, and ($b\,c$) is the symmetrization of the pair b, c.

The set of operators, with ξ and η solutions of the system (A) — (C), forms a subalgebra $L^N_r(S)$ of the Lie algebra L^N of all infinitesimal operators X which act in the N-dimensional space V_N of the variables x^α, u^a. $L^N_r(S)$ is called the invariance Lie algebra of the system (S). The Lie group $G^N_r(S)$ of the local transformations, associated to the Lie algebra $L^N_r(S)$, will be the invariance Lie group of the system (S).

6.1. The groups G^5_6(HD) and $G^5_7(H\,D)$ of hydrodynamics. We first assume that in (S), (A) — (C) the indices a, b, c, d = 1, 4, 5. In this case, which was already treated in [7, cap. I] from (A) we find $\xi^\alpha_{u^a} = 0$, and further from (B) we obtain $\xi^0_{x^1} = 0$ and

$$\eta^1 = (\xi^1_{x^1} - \xi^0_{x^0})u^1 + \xi^1_{x^0}, \; \eta^4 = [a(x^0, x^1) - 2(\xi^1_{x^1} - \xi^0_{x^0})]\, u^4, \; \eta^5 = \\ = a(x^0, x^1)u^5.$$

Since $F^a_0 = 0$, the last expressions replaced in (C), give for $\gamma \neq 3$:

$$\xi^0 = a_1 + a_6 x^0, \quad \xi^1 = a_2 + a_3 x^0 + (a_4 + a_6)x^1,$$

$$\eta^1 = a_3 + a_4 u^1, \quad \eta^4 = (-2a_4 + a_5)u^4, \quad \eta^5 = a_5 u^5,$$

and, for $\gamma = 3$:

$$\xi^0 = a_1 + a_6 x^0 + a_7(x^0)^2, \quad \xi^1 = a_2 + a_3 x^0 + (a_4 + a_6)x^1 + a_7 x^0 x^1,$$

$$\eta^1 = a_3 + (a_4 - a_7 x^0)u^1 + a_7 x^1, \; \eta^4 = (a_5 - 2a_4 - a_7 x^0)u^4, \; \eta^5 = \\ = (a_5 - 3a_7 x^0)u^5,$$

where a_i are constants.

One remarks that for $a_7 = 0$ we find the case $\gamma \neq 3$.

The Lie algebra $L^5_6(H\,D)$ ($\gamma \neq 3$) is determined by the basic operators of the form (6.12):

$$X_1 = p_0, X_2 = p_1, X_3 = x^0 p_1 + q_1, X_4 = x^1 p_1 + u^1 q_1 - 2u^4 q_4, \\ X_5 = u^4 q_4 + u^5 q_5, \; X_6 = x^0 p_0 + x^1 p_1,$$

and for $L^5_7(H\,D)$ ($\gamma \neq 3$) we must add

$$X_7 = (x^0)^2 p_0 + x^0 x^1 p_1 + (x^1 - x^0 u^1)\, q_1 - x^0 u^4 q_4 - 3x^0 u^5 q_5$$

to the previous operators.

The constants of structure of the algebra $L^5_7(H\,D)$ are

$$c^2_{13} = c^1_{16} = c^7_{17} = c^2_{26} = c^2_{24} = c^7_{27} = c^3_{34} = 1, \; c^4_{17} = -1, \; c^5_{17} = -3, c^6_{17} = 2,$$

the others being zero.

6.2. The group G^8_9(MHD) of the magnetohydrodynamics. We now suppose a, b, c, d = $\overline{1,7}$ in (S), (A) — (C). We have again from (A), $\xi^\alpha_{u^a} = 0$, and (B) give $\xi^0_{x^1} = 0$ and

$$\eta^1 = -(\xi^0_{x^0} - \xi^1_{x^1})u^1 + \xi^1_{x^0}, \; \eta^2 = -(\xi^0_{x^0} - \xi^1_{x^1})u^2 + a\, u^3 + b, \; \eta^3 = \\ = -(\xi^0_{x^0} - \xi^1_{x^1})u^3 - au^2 + c,$$

$$\eta^4 = 2(\xi^0_{x^0} - \xi^1_{x^1})u^4, \; \eta^5 = 0, \; \eta^6 = au^7, \; \eta^7 = -au^6,$$

where a, b, c are arbirary functions of x^α. By substituting the above expressions into (C), since $F^a_0 = 0$, we get the general solution of the definition equations:

$$\xi^0 = a_1 + (-a_6 + a_8)x^0, \; \xi^1 = a_2 + a_3 x^0 + a_8 x^1, \; \eta^1 = a_3 + a_6 u^1,$$

$$\eta^2 = a_4 + a_6 u^2 + a_7 u^3, \; \eta^3 = a_5 + a_6 u^3 - a_7 u^2, \; \eta^4 = -2a_6 u^4,$$

$$\eta^5 = 0, \qquad \eta^6 = a_7 u^7, \qquad \eta^7 = -a_7 u^6,$$

so that the Lie algebra $L^9_8(MHD)$ is given by the basic operators:

$$X_1 = p_0, X_2 = p_1, X_3 = x^0 p_1 + q_1, X_4 = q_2, X_5 = q_3, \\ X_6 = -x^0 p_0 + u^1 q_1 + u^2 q_2 + u^3 q_3 - 2\, u^4 q_4, \\ X_7 = u^3 q_2 - u^2 q_3 + u^7 q_6 - u^6 q_7, X_8 = x^0 p_0 + x^1 p_1,$$

and the constants of structure are:

$$c^2_{13} = -c^1_{16} = c^1_{18} = c^2_{28} = c^3_{36} = c^4_{46} = -c \;\; = c^5_{56} = c^4_{57} = 1,$$

the others being zero.

6.3. The groups G_r^N(RHD) of the radiation hydrodynamics. We take in (S), $(A)-(C)$, $a, b, c, d = 1, \ldots, 5, 8, \ldots, 11$. From (A) it results $\xi^{\alpha}_{u^i} = 0$, and then (B) give:

$$\xi^0_{x^0} = \xi^1_{x^1},\ \xi^0_{x^1} = \xi^1_{x^0} = 0,\ \eta^1 = 0,$$

$$\eta^2 = f^2(x^0, x^1;\ u^2, u^3), \quad \eta^3 = f^3(x^0, x^1; u^2. u^3),$$

$$\eta^4 = a(x^0, x^1)u^4, \qquad \eta^5 = a(x^0, x^1)u^5 - \frac{\gamma-1}{\gamma} b(x^0, x^1), \tag{6.13}$$

$$\eta^8 = a(x^0, x^1)u^8 + b(x^0, x^1),\ \eta^9 = a(x^0, x^1)u^9 + c(x^0, x^1),$$

where f^2, f^3, a, b, c are arbitrary functions of the indicated arguments and η^{10}, η^{11} are arbitrary functions of all arguments.

Taking into account the form of the right-side members of (S), in what follows, we consider three cases.

6.3.1. The group G_6^7(RHD1) for a perfect transparent fluid. In this case (cf. [8]) $\alpha = 0$ and therefore $F_0^a = 0$. By substituting ξ^{α} and $\eta^a(a = 1, 4, 5, 8, 9)$ from (6.13) into (C), we get the general solution

$$\xi^0 = a_1 + a_6 x^0,\quad \xi^1 = a_2 + a_6 x^1,\quad \eta^1 = 0,\quad \eta^4 = a_4 u^4,$$

$$\eta^5 = \frac{4}{3}(\gamma-1)a_3 + a_4 u^5,\quad \eta^8 = -\gamma a_3 + a_4 u^8,\quad \eta^9 = a_4 u^9 + a_5.$$

The Lie algebra $L_6^7(RHD1)$ has then the basis:

$$X_1 = p_0, X_2 = p_1, X_3 = \frac{4}{3}(\gamma-1)q_5 - \gamma q_8, X_4 = u^4 q_4 + u^5 q_5 + u^8 q_8 + u^9 q_9,$$

$$X_5 = q_9, X_6 = x^0 p^0 + x^1 p_1.$$

The constants of structure of the algebra $L_6^7(RHD1)$ are:

$$c_{16}^1 = c_{26}^2 = c_{34}^3 = -c_{45}^5 = 1,$$

the others are zero.

6.3.2. The group G_{10}^{11}(RHD²) for the uncoupled equations. We replace the expressions (6.13) for ξ^{α} and $\eta^a(a = 1, \ldots, 5, 8, \ldots, 11)$ into the equations (C), where in F_0^a we neglect the terms depending on the velocity, which introduce the coupling. The general solution of the definition equations is then:

$$\xi^0 = a_1,\ \xi^1 = a_2,\ \eta^1 = \eta^4 = \eta^5 = \eta^8 = \eta^9 = 0.$$

$$\eta^2 = a_3\frac{1}{\alpha} + a_5 e^{-\alpha x^0} + a_7 u^2 + a_8 u^3,\ \eta^3 = a_4\frac{1}{\alpha} + a_6 e^{-\alpha x^0} + a_9 u^2 + a_{10} u^3,$$

$$\eta^{10} = -a_5 c^2 e^{-\alpha x^0} u^4 + a_7 u^{10} + a_8 u^{11},\quad \eta^{11} = -a_6 c^2 e^{-\alpha x^0} u^4 +$$
$$+ a_9 u^{10} + a_{10} u^{11}.$$

The Lie algebra $L_{10}^{11}(RHD2)$ has thereby the basic operators:

$$X_1 = p_0, X_2 = p_1, X_3 = \frac{1}{\alpha}q_2, X_4 = \frac{1}{\alpha}q_3,$$

$$X_5 = e^{-\alpha x^0}(q_2 - c^2 u^4 q_{10}), X_6 = e^{-\alpha x^0}(q_3 - c^2 u^4 q_{11}), X_7 = u^2 q_2 +$$
$$+ u^{10} q_{10},$$

$$X_8 = u^3 q_2 + u^{11} q_{10}, X_9 = u^2 q_3 + u^{10} q_{11}, X_{10} = u^3 q_3 u^{11} q_{11}.$$

The constants of structure are:

$$c_{15}^5 = c_{16}^6 = -\alpha, c_{37}^3 = c_{39}^4 = c_{48}^3 = c_{4,10}^4 = c_{57}^5 = c_{59}^6 = c_{68}^5 = c_{6,10}^6 =$$
$$= -c_{78}^8 = -c_{89}^7 = c_{89}^{10} = c_{79}^9 = -c_{8,10}^8 = c_{9,10}^9 = 1;$$

the others are zero.

6.3.3. The group G_2^{11}(RHD3) for the coupled equations. We now substitute (6.13) for ξ^{α} and $\eta^a(a = 1, \ldots, 5, 8, \ldots, 11)$ into the equations (C), with F_0^a of the complete form. We get:

$$\xi^0 = a_1,\quad \xi^1 = a_2,\quad \eta^a = 0.$$

The Lie algebra $L_2^{11}(RHD3)$ has the basis

$$X_1 = p_0,\quad X_2 = p_1,$$

and consequently

$$c_{12}^1 = c_{12}^2 = 0.$$

6.4. The groups G_r^N(RMHD) of the radiative magnetohydrodynamics. We consider in (S), $(A)-(C)$, $a, b, c, d = \overline{1, 11}$. From (A), it results $\xi^{\alpha}_{u^i} = 0$, and thereby from (B), if $(u^1)^2 - \frac{c^2}{3} \neq 0$, it follows:

$$\xi^0_{x0} = \xi^1_{x1},\ \xi^0_{x1} = \xi^1_{x0} = 0,\ \eta^1 = 0,\ \eta^2 = au^4 + b,\ \eta^3 = -au^2 + c,$$

$$\eta^4 = 0,\ \eta^5 = \frac{4}{3}(\gamma-1)k,\ \eta_6 = au^7,\ \eta^7 = -au^6,\ \eta^8 = -\gamma k, \tag{6.14}$$

$$\eta^9 = h,\ \eta^{10} = f^{10}(x^0, x^1;\ u^{10}, u^{11}),\ \eta^{11} = f^{11}(x^0, x^1;\ u^{10}, u^{11}),$$

where a, b, c, k, h are arbitrary functions of x^0, x^1, and f^{10}, f^{11} are arbitrary functions of the indicated arguments.

We have again the three cases.

6.4.1. The group G_8^{11}(RMHD1) for a perfect transparent fluid. We have $\alpha = 0$ and consequently $F_0^a = 0$. On substituting ξ^{α} and $\eta^a(a = \overline{1,9})$ from (6,14) into (C) we obtain the general solution of the definition; equations:

$$\xi^0 = a_1 + a_8 x^0,\ \xi^1 = a_2 + a_8 x^1,\ \eta^1 = 0,\ \eta^2 = a_3 + a_6 u^3,\ \eta^3 =$$
$$= a_4 - a_6 u^2,$$

$$\eta^4 = 0,\ \eta^5 = \frac{4}{3}(\gamma - 1)a_6,\ \eta^6 = a_6 u^7,\ \eta^7 = -a_6 u^6,\ \eta^8 = -\gamma a_6,\ \eta^9 = a_7.$$

The Lie algebra L_8^{11} (*RMHD*1) is given by

$$X_1 = p_0,\ X_2 = p_1,\ X_3 = q_2,\ X_4 = q_3,\ X_5 = \frac{4}{3}(\gamma - 1)q_5 - \gamma q_8,$$

$$X_6 = u^3 q_2 - u^2 q_3 + u^7 q_6 - u^6 q_7,\ X_7 = q_9,\ X_8 = x^0 p_0 + x^1 p_1,$$

the constants of structure are:

$$c_{18}^1 = c_{28}^2 = -c_{36}^4 = c_{46}^3 = 1,$$

the others are zero.

6.4.2. The group G_5^{13}(RMHD2) for the uncoupled equations. We substitute ξ^α and $\eta^a(a = \overline{1,11})$ given by (6.14) into (C) with F_0^a as in 6.3.2. and we find:

$$\xi^0 = a_1,\ \xi^1 = a_2,\ \eta^1 = \eta^4 = \eta^5 = \eta^8 = \eta^9 = 0,\ \eta^2 = a_3 + a_5 u^3,$$

$$\eta^3 = a_4 - a_5 u^2,\ \eta^6 = a_5 u^7,\ \eta^7 = -a_5 u^6,\ \eta^{10} = a_5 u^{11},\ \eta^{11} = -a_5 u^{10}.$$

The Lie algebra L_5^{13}(RMHD2) has the basic operators:

$$X_1 = p_0,\ X_2 = p_1,\ X_3 = q_2,\ X_4 = q_3,$$

$$X_5 = u^3 q_2 - u^2 q_3 + u^7 q_6 - u^6 q_7 + u^{11} q_{10} - u^{10} q_{11},$$

and the constants of structure

$$-c_{35}^4 = c_{45}^3 = 1$$

the others being zero.

6.4.3. The group G_2^{13}(RMHD3) for the coupled equations. By the substitution of ξ^α and $\eta^a(a = \overline{1,11})$ into (C), when F_0^a has the complete form, we have:

$$\xi^0 = a_1,\ \xi^1 = a_2,\ \eta^a = 0.$$

The Lie algebra $L_2^{13}(R\,M\,H\,D\,3)$ is given by:

$$X_1 = p_0,\ X_2 = p_1,$$

with, $$c_{12}^1 = c_{12}^2 = 0.$$

7. Some properties of the groups G_r^N. Consider the matrix

(7.1) $$M = \| \xi_i^\alpha,\ \eta_i^a \|$$

$\alpha = 1, 2$; $a = \overline{1, N-2}$, $i = \overline{1, r}$, of the components of the vectors of a group G_r^N. Let q be the generic rank of M [9, § 13].

7.1. The groups G_6^5(HD) and G_7^5(HD) have $q = 5$. A point $P_0(x_0^0, x_0^1;\ u_0^1, u_0^4, u_0^5) \in V_5$ for which u_0^4 and u_0^5 are different from zero is an ordinary

point. The points for which $u^4 = u^5 = 0$ are singular points of order three. For $G_6^5(HD)$, the subgroup of stability of an ordinary point P_0 is generated by the operator.

$$Y_1 = (x^0 - x_0^0)\,p_0 + (x^1 - x_0^1)\,p_1,$$

and, for $G_7^5(HD)$ by Y_1 and

$$Y_2 = [(x^0)^2 - (x_0^0)^2]\,p_0 + [x^0(x^1 - x_0^1) + x_0^0(x^1 - x_0^1)]\,p_1 + [(x^1 - x_0^1) - (x^0 - x_0^0)\,u^1]\,q_1 - (x^0 - x_0^0)\,u^4 q_4 - 3(x^0 - x_0^0)\,u^5 q_5.$$

The groups $G_6^5\,(HD)$ and $G_7^5\,(HD)$ are transitive. There are not therefore absolute invariants. The minimum invariant variety is all the space V_5.

7.2. The group G_8^9(MHD) has $q = 7$. If $u_0^4 \neq 0$ and u_0^6 or u_0^7 are different from zero, the point $P_0(x_0;\ u_0^a) \in V_9$ is an ordinary point. The points for which $u^4 = u^6 = u^7 = 0$ are singular points of order five.

The subgroup of stability of an ordinary point P_0 is generated by the operator

$$Y_1 = (x^0 - x_0^0)\,p_0 - (x^1 - x_0^1)\,p_1.$$

$G_8^9\,(MHD)$ is an intransitive group. If P_0 is an ordinary point, there are two absolute invariants and the minimum invariant variety for P_0 is given by the equations:

$$u^5 = u_0^5,\ (u^6)^2 + (u^7)^2 = (u_0^6)^2 + (u_0^7)^2.$$

7.3.1. The group G_6^7(RHD1) has $q = 5$. If u_0^4 and $\gamma u_0^5 + \frac{4}{3}(\gamma - 1)u_0^8$ are different from zero, the point $P_0(x_0^0, x_0^1;\ u_0^1, u_0^4, u_0^5, u_0^8, u_0^9) \in V_7$ is an ordinary point. The points for which $u^4 = 0$ and $\gamma u^5 + \frac{4}{3}(\gamma - 1)\,u^8 = 0$ are singular points of order four. The subgroup of stability of an ordinary point P_0 is given by

$$Y_1 = (x^0 - x_0^0)\,p_0 + (x^1 - x_0^1)\,p_1.$$

$G_7^6\,(RHD1)$ is an intransitive group. If P_0 is an ordinary point, there are two absolute invariants and the minimum invariant variety for P_0 has the equations:

$$u^1 = u_0^1,\quad \frac{\gamma u^5 + \frac{4}{3}(\gamma - 1)u^8}{u^4} = \frac{\gamma u_0^5 + \frac{4}{3}(\gamma - 1)u_0^8}{u_0^4}.$$

7.3.2. The group G_{10}^{11}(RHD2) has $q = 6$. If one of u_0^4, u_0^{10}, u_0^{11} is different from zero, the point $P_0(x_0^0, x_0^1;\ u_0^1, \dots, u_0^5, u_0^8, \dots, u_0^{11}) \in V_{11}$ is an ordinary point, and the points for which $u^4 = u^{10} = u^{11} = 0$ are singular points of order four.

The subgroup of stability of an ordinary point P_0 is of order four and it is generated by the operators:

$$Y_1 = \left[(u^2 - u_0^2) + \frac{u_0^{10}}{c^2 u_0^4}(e^{-\alpha(x^0 - x_0^0)} - 1)\right] q_2 + \left[u^{10} - \frac{u_0^{10}}{u_0^4} u^4 e^{-\alpha(x^0 - x_0^0)}\right] q_{10},$$

$$Y_2 = \left[(u^3 - u_0^3) + \frac{u_0^{11}}{c^2 u_0^4}(e^{-\alpha(x^0 - x_0^0)} - 1)\right] q_2 + \left[u^{11} - \frac{u_0^{11}}{u_0^4} u^4 e^{-\alpha(x^0 - x_0^0)}\right] q_{10},$$

$$Y_3 = \left[(u^2 - u_0^2) + \frac{u_0^{10}}{e^2 u_0^4}(e^{-\alpha(x^0 - x_0^0)} - 1)\right] q_3 + \left[u^{10} - \frac{u_0^{10}}{u_0^4} u^4 e^{-\alpha(x^0 - x_0^0)}\right] q_{11},$$

$$Y_4 = \left[(u^3 - u_0^3) + \frac{u_0^{11}}{c^2 u_0^4}(e^{-\alpha(x^0 - x_0^0)} - 1)\right] q_3 + \left[u^{11} - \frac{u_0^{11}}{u_0^4} u^4 e^{-\alpha(x^0 - x_0^0)}\right] q_{11}.$$

G_{10}^{11} $(RHD2)$ is an intransitive group. For each ordinary point P_0 there are five absolute invariants. The minimum invariant variety for P_0 is given by the equations:

$$u^1 = u_0^1,\ u^4 = u_0^4,\ u^5 = u_0^5,\ u^8 = u_0^8,\ u^9 = u_0^9.$$

7.3.3. **The group G_2^{11}(RHD3)** has $q = 2$. Any point $P_0(x_0^0, x_0^1; u_0^1, \ldots, u_0^5, u_0^8, \ldots, u_0^{11}) \in V_{11}$ is an ordinary point, and it does not admit a subgroup of stability other than the identity.

G_2^{11} $(RHD3)$ is an intransitive group. There are for P_0 nine absolute invariantes:

$$u^a = u_0^a\ (a = 1, \ldots, 5,\ 8, \ldots, 11).$$

7.4.1. **The group G_8^{11}(RMHD1)** has $q = 7$. If either u_0^6 or u_0^7 is different from zero, $P_0(x_0, u_0^a) \in V_{11}$ is an ordinary point. The points for which $u^6 = u^7 = 0$ are singular points of order six.

The subgroup of stability of an ordinary point P_0 is generated by the operator

$$Y_1 = (x^0 - x_0^0)\, p_0 + (x^1 - x_0^1)\, p_1.$$

G_8^{11} $(RMHD1)$ is an intransitive group. For each point P_0 there are four absolute invariants. The minimum invariant variety is given by the equations:

$$u^1 = u_0^1,\ u^4 = u_0^4,\ \gamma u^5 + \frac{4}{3}(\gamma - 1)u^8 = \gamma u_0^5 + \frac{4}{3}(\gamma - 1)u_0^8,\ (u^6)^2 +$$

$$+ (u^7)^2 = (u_0^6)^2 + (u_0^7)^2.$$

7.4.2. **The group G_5^{13}(RMHD2)** has $q = 5$. If at least one of $u_0^6, u_0^7, u_0^{10}, u_0^{11}$ is different from zero, the point $P_0(x_0, u_0^a) \in V_{13}$ is an ordinary point. The points for which $u_6 = u_7 = u^{10} = u^{11} = 0$ are singular points of order four. Every ordinary point, it does not admit a subgroup of stability other than the identity.

G_5^{13} $(RMHD2)$ is an intransitive group. There are eight absolute invariants. The minimum invariant variety is given by

$$u^1 = u_0^1,\ u^4 = u_0^4,\ u^5 = u_0^5,\ u^8 = u_0^8,\ u^9 = u_0^6,$$

$$(u^6)^2 + (u^7)^2 = (u_0^6)^2 + (u_0^7)^2,\ (u^{10})^2 + (u^{11})^2 = (u_0^{10})^2 + (u_0^{11})^2$$

$$\frac{u^6 u^{11} - u^7 u^{10}}{u^6 u^{10} - u^7 u^{11}} = \frac{u_0^6 u_0^{11} - u_0^7 u_0^{10}}{u_0^6 u_0^{10} - u_0^7 u_0^{11}}.$$

7.4.3. **The group G_2^{13}(RMHD3)** has $q = 2$. Any point $P_0(x_0, u_0^a) \in V_{13}$ is an ordinary point, and it does not admit a subgroup of stability other than the identity. The group is intransitive and for P_0 there are eleven absolute invariants

$$u^a = u_0^a\ (a = 1, 11).$$

7.5. The algebras L_8^9 (MHD), L_8^{11} $(RMHD1)$ and $L_5^{13}(RMHD2)$ are resoluble.

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