of G. We claim that A is the only proper pseudo-ideal of G. Let there be another pseudo-ideal B of G. Since $1 \in B$, $A \subseteq B$. Now if $A \neq B$ then B contains at least one element g^r where r is an odd integer. In this case $B = G$. Hence the proposition.

Proposition 10. *Every finite cyclic group of odd order does not contain any proper pseudo-ideal.*

Proof. Let G be a finite cyclic group of odd order and A be a pseudo-ideal of G. Then the factor group G/A is also of odd order. But in that case G/A cannot have elements of order 2. Hence G does not contain any proper pseudoideal.

Now using the proposition 9 we shall prove the following known result.

Proposition 11. *In a finite field, every element can be expressed as a sum of two squares.*

Proof. Let F be a finite field. Then $F \cong GF(p^n)$ for some prime integer p. Let $\Gamma = GF(p^n)$. Now the nonzero elements of Γ forms a multiplicative cyclic group of even order. Hence, by proposition 9, the set of all nonzero elements of Γ contains one and only one proper pseudo-ideal. So Γ, which may be considered as a multiplicative cyclic group with externally adjoined zero, contains one and only one proper nonzero pseudo-ideal given by $\overline{\Gamma}$ ($\{x^2 \mid x \in \Gamma\}$. Let $\Gamma^* = \{x^2 + y^2 \mid x, y \in \Gamma\}$. Then Γ^* is also a pseudo-ideal of Γ containing $\overline{\Gamma}$. Hence either $\Gamma^* = \overline{\Gamma}$ or $\Gamma^* = \Gamma$. If $\Gamma^* = \overline{\Gamma}$, then $\overline{\Gamma} = \Gamma^*$ is a subfield of Γ. Hence $\overline{\Gamma} = \Gamma^* = GF(p^m)$ for some prime integer m which divides n. Let $n = mk$. But Γ contains $(p^n + 1)/2$ elements. Thus $(p^n + 1) = 2p^m$ or $1 = p^m - p^n = p^m\{2 - p^{m(k-1)}\}$ which is a contradiction unless $p = 1$. Hence $\overline{\Gamma} \neq \Gamma^*$. So $\Gamma^* = \Gamma$. Hence the proposition.

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ON AN ANALYTIC CHARACTERIZATION OF QUASICONFORMALITY IN NORMED SPACES

BY

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1. **Introduction.** There are several possibilities to define quasiconformality on infinite dimensional normed spaces. The simplest one consists in using conditions of a metric character such as that of Gehring ([6], [2]). These conditions may be imposed either globally or on each bidimensional section. In [3] we have suggested a simple condition of analytic character which is equivalent with quasiconformality on finite dimensional spaces. In the present paper we deal with the same problem on general normed spaces.

2. **Definitions and notations.** Let X be a real normed space and $f: G \to X$ be a homeomorphic mapping of a domain G of X into X. The *linear dilatation* of f in a point $x \in G$ is defined as:

$$\delta_L(x) = \delta(x) = \lim_{r \to 0} \sup \frac{L(x, r)}{l(x, r)},$$

where for $r > 0$ small enough:

$$L(x, r) = \sup \{\|f(y) - f(x)\|, \|y - x\| = r\},$$
$$l(x, r) = \inf \{\|f(y) - f(x)\|, \|y - x\| = r\}.$$

If Π is a bidimensional linear manifold which meets G and $x \in G \cap \Pi$, then $\delta_L^{\Pi}(x) = \delta^{\Pi}(x)$ is defined as above by replacing f by its restriction to $G \cap \Pi$.

Let $K \geq 1$ be a fixed real number.

Definition 1. ([4], [6]) *A homeomorphic mapping $f: G \to X$ of a domain $G \subset X$ into X is called K-quasiconformal in the sense of Gehring if*

$$\delta(x) \leq K, \quad \forall x \in G.$$

Definition 1′. *A homeomorphic mapping $f: G \to X$ of a domain $G \subset X$ into X is called bidimensionally K-quasiconformal in the sense of Gehring if*

$$\delta^{\Pi}(x) \leq K, \quad \forall x \in G \cap \Pi,$$

for any bidimensional linear manifold Π which meets G.

Our analytic characterization of quasiconformality will be formulated in terms of scalar derivatives.

The scalar derivatives of f in a point $x \in G$ are defined as in [5] by:

$$D^{+}f(x) = \limsup_{y \to x} \frac{\|f(y) - f(x)\|}{\|y - x\|},\quad D^{-}f(x) = \liminf_{y \to x} \frac{\|f(y) - f(x)\|}{\|y - x\|}$$

$D^{+}f(x)$ is called the *upper* and $D^{-}f(x)$ the *lower scalar derivative* of f at the point x. We have generally:

$$0 \leqslant D^{-}f(x) \leqslant D^{+}f(x) \leqslant +\infty.$$

If $D^{-}f(x) = D^{+}f(x) = 0$ or $D^{-}f(x) = D^{+}f(x) = +\infty$, we say that f is *degenerate* at x. If Π is a bidimensional linear manifold which meets G, then $D^{+}_{\Pi}f(x)$ and $D^{-}_{\Pi}f(x)$ are defined as above by replacing f by the restriction of f to $G \cap \Pi$. It is obvious that:

(1) $\qquad D^{+}_{\Pi}f(x) \leqslant D^{+}f(x),\quad D^{-}f(x) \leqslant D^{-}_{\Pi}f(x),\quad x \in G \cap \Pi.$

Using these derivatives, we will give two new definitions for K-quasiconformality and bidimensional K-quasiconformality and prove their equivalence with Definitions 1 and 1′.

Definition 2. *A homeomorphic mapping $f: G \to X$ of a domain $G \subset X$ into X is called K-quasiconformal, if:*

(i) *f is non-degenerate at any point $x \in G$,*

(ii) $D^{+}f(x) \leqslant KD^{-}f(x),\ \forall x \in G.$

Definition 2′. *A homeomorphic mapping $f: G \to X$ of a domain $G \subset X$ into X is called bidimensionally K-quasiconformal, if for any bidimensional linear manifold Π which meets G:*

(i′) *the restriction of f to $G \cap \Pi$ is non-degenerate at any point $x \in G \cap \Pi$,*

(ii′) $D^{+}_{\Pi}f(x) \leqslant KD^{-}_{\Pi}f(x),\qquad \forall x \in G \cap \Pi.$

We shall use a notion of bidimensional differentiability intermediate between that of Gâteaux and that of Aronszajn [1].

Definition 3. *A mapping $f: G \to X$ of a domain $G \subset X$ into X is called bidimensionally Fréchet-differentiable at $x \in G$, if f is Fréchet-differentiable on each bidimensional section $G \cap \Pi$ of G through x.*

We say that f is Fréchet-differentiable at $x \in G$ on a bidimensional section $G \cap \Pi$ of G through x if there exists a continuous linear mapping $B: \Pi - x \to X$ such that:

$$\lim_{\substack{h \to 0 \\ h \in \Pi - x}} \frac{\|f(x+h) - f(x) - Bh\|}{\|h\|} = 0.$$

It is easy to see that if f is bidimensionally Fréchet-differentiable at $x \in G$, then it is Gâteaux-differentiable at x and $Bu = Df(x; u)$, $u \in \Pi - x$. Therefore if $G \cap \Pi$ and $G \cap \Pi'$ are two bidimensional sections of G through x, then the corresponding bidimensional differentials B and B' coincide with Gâteaux differential on $(\Pi - x) \cap (\Pi' - x)$ and consequently we can define a global linear mapping on X which is equal to the Gâteaux differential of f at x.

3. **Results.** In this section we shall study the relations between the definitions of quasiconformality considered in the preceding section.

P. Caraman [2] has proved that the K-quasiconformality according to Definition 1 implies that in the sense of Definition 1′. We shall prove (Proposition 1) that the K-quasiconformality according to Definition 2 implies that in the sense of Definition 2′. It is not to expect that Definition 1′ be actually equivalent to Definition 1 and Definition 2 to Definition 2′. However it seems difficult to get some counterexamples. P. Caraman [2] has proved that Definitions 1 and 1′ are equivalent for Fréchet-differentiable homeomorphic mappings with differential different from zero. In what follows, these mappings will be called *regular mappings*.

Our first result (Theorem 1) is that Definitions 2 and 2′ are equivalent for Fréchet-differentiable mappings.

The second result (Theorem 2) is that Definitions 1′ and 2′ are equivalent for *bidimensionally regular mappings* (that means bidimensionally Fréchet-differentiable mappings with differential different from zero). In fact for such mappings Definition 2′ may be phrased in an equivalent way by requiring that f be non-degenerate at least on a bidimensional section instead of being non-degenerate on each bidimensional section.

As a Corollary, we get that Definitions 1 and 2 are equivalent for regular mappings.

Proposition 1. *If a homeomorphic mapping $f: G \to X$ is K-quasiconformal in the sense of Definition 2, then it is K-quasiconformal in the sense of Definition 2′.*

Proof. Indeed, from condition (ii) and inequalities (1) it follows:

(2) $\qquad D^{+}_{\Pi}f(x) \leqslant D^{+}f(x) \leqslant KD^{-}f(x) \leqslant KD^{-}_{\Pi}f(x),\quad x \in G \cap \Pi,$

whence condition (ii′) is verified.

Moreover, condition (i) implies:

$$0 < D^{+}f(x) \text{ and } D^{-}f(x) < +\infty,\quad x \in G,$$

which combined with (2) give respectively:

$$0 < D^{-}_{\Pi}f(x) \text{ and } D^{+}_{\Pi}f(x) < +\infty,\quad x \in G \cap \Pi,$$

that is condition (i′).

Lemma. *If $f: G \to X$ is a Fréchet-differentiable mapping at a point $x \in G$, then:*

(3) $\qquad D^{+}f(x) = \sup\{\|Df(x; u)\|, \|u\| = 1\},\ D^{-}f(x) = \inf\{\|Df(x; u)\|, \|u\| = 1\}.$

Proof. Let us denote $m = \inf\{\|Df(x; u)\|, \|u\| = 1\}$ and $M = \sup\{\|Df(x; u)\|, \|u\| = 1\}$.

We have the following inequalities:

$$(4)\quad \frac{\|Df(x;h)\|}{\|h\|} - \frac{\|f(x+h)-f(x)-Df(x;h)\|}{\|h\|} \leqslant \frac{\|f(x+h)-f(x)\|}{\|h\|} \leqslant$$

$$\leqslant \frac{\|Df(x;h)\|}{\|h\|} + \frac{\|f(x+h)-f(x)-Df(x;h)\|}{\|h\|},$$

whence we get:

$$(5)\qquad D^+f(x) \leqslant M,\quad m \leqslant D^-f(x).$$

On the other hand, if $\varepsilon > 0$ is an arbitrary positive number, we can find a unitary vector u such that $M - \varepsilon < \|Df(x;u)\|$, and consequently, by the left side of (4):

$$M - \varepsilon < \frac{\|f(x+tu)-f(x)\|}{t} + \frac{\|f(x+tu)-f(x)-Df(x;tu)\|}{t},\quad t > 0,$$

whence, by letting $t \to 0$, we get: $M - \varepsilon \leqslant D^+f(x)$. Since ε is arbitrary we obtain: $M \leqslant D^+f(x)$. Analogously one obtains: $D^-f(x) \leqslant m$.

Remark 1. By following the same lines of reasoning one can prove that if f is bidimensionally Fréchet-differentiable at x, then:

$$(6)\qquad \begin{aligned} D^+_\Pi f(x) &= \sup\{\|Df(x;u)\|,\ u \in \Pi - x,\ \|u\| = 1\},\\ D^-_\Pi f(x) &= \inf\{\|Df(x;u)\|,\ u \in \Pi - x,\ \|u\| = 1\},\end{aligned}$$

for any bidimensional linear manifold Π through x.

Theorem 1. *A homeomorphic Fréchet-differentiable mapping $f: G \to X$ is K-quasiconformal in the sense of Definition 2 if and only if it is K-quasiconformal in the sense of Definition 2′.*

Proof. By Proposition 1, the K-quasiconformality according to Definition 2 implies that in the sense of Definition 2′, so that, we have to prove only the converse. Firstly, from the inequalities (1), it follows that f is non-degenerate at x, and consequently, the condition (i) is verified. In order to verify the condition (ii), we shall use the relations (3). Let $\varepsilon > 0$ be an arbitrary positive number. Then there exists two unitary vectors u_1 and u_2 such that:

$$(7)\qquad D^+f(x) - \varepsilon < \|Df(x;u_1)\| \text{ and } \|Df(x;u_2)\| < D^-f(x) + \varepsilon.$$

Denote by Π a bidimensional linear manifold which contains $x + u_1$ and $x + u_2$. By combining (ii′), (6) and (7) we get:

$$D^+f(x) - \varepsilon < \|Df(x;u_1)\| \leqslant \sup\{\|Df(x;u)\|,\ u \in \Pi - x,\ \|u\| = 1\} \leqslant$$

$$\leqslant K \inf\{\|Df(x;u)\|,\ u \in \Pi - x,\ \|u\| = 1\} \leqslant K\,\|Df(x;u_2)\| <$$

$$< K\,[D^-f(x) + \varepsilon].$$

whence: $D^+f(x) - \varepsilon < K\,[D^-f(x) + \varepsilon]$. By letting $\varepsilon \to 0$, we obtain the condition (ii).

Remark 2. For Fréchet-differentiable mappings, Definition 2′ may be phrased in an equivalent way, by requiring that f be non-degenerate at any point $x \in G$ instead of being non-degenerate on each bidimensional section through x.

Proposition 2. *Let $f: G \to X$ be a mapping which is Fréchet-differentiable at x on a bidimensional section $G \cap \Pi$ of G through x. If*

$$(8)\qquad \delta^\Pi(x) \leqslant K,$$

then

$$(9)\qquad D^+_\Pi f(x) \leqslant K D^-_\Pi f(x).$$

Proof. From the definition of $\delta^\Pi(x)$, it follows that for any $\varepsilon > 0$, there exists $r_0 = r_0(\varepsilon) > 0$, such that

$$(10)\qquad \frac{L^\Pi(x,r)}{l^\Pi(x,r)} < \delta^\Pi(x) + \varepsilon,$$

whenever $0 < r < r_0$. On the other hand, since f is Fréchet-differentiable at x on $G \cap \Pi$, there exists $r_0' = r_0'(\varepsilon)$ such that:

$$\|Df(x;h)\| - \varepsilon\|h\| < \|f(x+h) - f(x)\| < \|Df(x;h)\| + \varepsilon\|h\|,$$

whenever $x + h \in G \cap \Pi$ and $\|h\| < r_0'$. Consequently, if $\|h\| = r$ and $\|h'\| = r$ where $0 < r < r_0'$, we have:

$$(11)\qquad \|Df(x;h)\| - \varepsilon r < L^\Pi(x,r) \text{ and } l^\Pi(x,r) < \|Df(x;h')\| + \varepsilon r.$$

By combining (11), (10) and (8) we get:

$$(12)\qquad \|Df(x;h)\| - \varepsilon r < (K + \varepsilon)\,[\|Df(x;h')\| + \varepsilon r],$$

whenever $\|h\| = \|h'\| = r$ and $0 < r < \min(r_0, r_0')$. The inequality (12) is equivalent to the following:

$$\|Df(x;u)\| - \varepsilon < (K + \varepsilon)\,[\|Df(x;u')\| + \varepsilon],$$

for any two vectors u and $u' \in \Pi - x$ such that $\|u\| = \|u'\| = 1$. By letting $\varepsilon \to 0$ and taking into account that u and u' are independent, we get:

$$\sup\{\|Df(x;u)\|,\ u \in \Pi - x,\ \|u\| = 1\} \leqslant K \inf\{\|Df(x;u)\|,\ u \in \Pi - x,\ \|u\| = 1\}.$$

From here, by applying (6) we get (9). The proof is finished.

Proposition 2 will be used to show that the K-quasiconformality by Definition 1′ implies that in the sense of Definition 2′, for bidimensionally regular mappings. The following proposition contains the converse implication. In fact the statement is more precise:

the K-quasiconformality by a modified form of Definition 2′ implies that in the sense of Definition 1′. The modification consists in replacing (i′) by

(i″) *for any $x \in G$, there exists a bidimensional section $G \cap \Pi$ of G through x, such that the restriction of f to $G \cap \Pi$ is non-degenerate at x.*

Proposition 3. *If a bidimensionally Fréchet-differentiable mapping $f: G \to X$ is K-quasiconformal by modified Definition 2′, then it is K-quasiconformal by Definition 1′.*

Proof. We shall prove firstly that if f is bidimensionally Fréchet-differentiable, then the conditions (i″) and (ii′) imply (i′) (i.e. f is non-degenerate on each bidimensional section of G). By taking into account Remark 1, it will be enough to show that the differential of f at any point $x \in G$ is injective. If no, let $u_0 \in X$, $\|u_0\| = 1$, such that $Df(x; u_0) = 0$ and let $u_1 \in X$, $\|u_1\| = 1$, be a vector linearly independent from u_0. Consider the bidimensional linear manifold $\Pi = x + \mathrm{sp}\{u_0, u_1\}$, obtained by a translation of the linear subspace $\mathrm{sp}\{u_0, u_1\}$ generated by u_0 and u_1. By Remark 1, condition (ii') applied for Π has the following form:

$$\sup\{\|Df(x; u)\|,\quad u \in \Pi - x,\quad \|u\| = 1\} \leqslant K \inf\{\|Df(x; u)\|,\ u \in \Pi - x,\ \|u\| = 1\}.$$

Since $u_1 \in \Pi - x$, $u_0 \in \Pi - x$, we get $\|Df(x; u_1)\| \leqslant K\|Df(x; u_0)\|$, whence $Df(x; u_1) = 0$. Therefore we have obtained $Df(x; u) = 0$ for any vector $u \in X$ and that is impossible by (i'') and Remark 1. Since the condition (i') is satisfied, we may argue as in [3] (Proposition 1) in order to obtain:

$$\delta^{\Pi}(x) = \frac{D^{+}_{\Pi} f(x)}{D^{-}_{\Pi} f(x)},$$

for any bidimensional linear manifold Π, such that $x \in G \cap \Pi$. From here, by applying the condition (ii') we get: $\delta^{\Pi}(x) \leqslant K$, for any Π and $x \in G \cap \Pi$.

Theorem 2. *A bidimensionally regular mapping $f: G \to X$ is K-quasiconformal in the sense of Definition 1', if and only if it is K-quasiconformal in the sense of Definition 2'.*

Proof. If f is K-quasiconformal in the sense of Definition 1' then, by Proposition 2, the condition (ii') is satisfied. Since the differential of f is different from zero, the condition (i'') is satisfied. Then, as was proved above, the conditions (i'') and (ii') imply the condition (i').

The other implication follows simply from Proposition 3.

Remark 3. Definition 2' is equivalent to modified Definition 2' for bidimensionally regular mappings and, in particular, for regular mappings.

Corollary. *Definitions 1 and 2 are equivalent for regular mappings.*

Proof. We already know that Definitions 1 and 1' respectively 2 and 2' are equivalent for regular mappings. Since any regular mapping is also bidimensionally regular, we can apply Theorem 2.

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SOME TYPES OF CONTRACTIVE MAPPINGS AND FIXED POINTS

BY

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1. Introduction: Let (M, d) be a metric space. A mapping $T: M \to M$ is called contractive if for every $x, y\ (x \neq y)$ in M,

$$d(Tx, Ty) < d(x, y) \tag{1.1}$$

holds.

In a paper [1], M. Edelstein proved a theorem which runs as follows.:

Theorem 1. *Let (M, d) be a metric space, T be a mapping of M into itself. If for every $x, y\ (x \neq y)$, T satisfies (1.1.) and for every $x \in M$*

$$\{T^{n_i}x\} \subset \{T^n x\} \text{ with } \lim_{i \to \infty} T^{n_i}x = u \in M, \tag{1.2}$$

holds, then u is a unique fixed point of T.

In a recent paper [2], the present author has proved the analogous more general

Theorem 2. *Let T be a continuous mapping of a metric space into itself. If for every $x, y\ (x \neq y)$ in M there exist non-negative real numbers a_i ($i = 1, 2, \ldots 6$) and at least one of $a_i \neq 0$ ($i = 1, \ldots 5$) such that*

$$d(Tx, Ty)\, d(x, y) < d(x, y)\{a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx)\} + a_6 d(x, Tx)\, d(y, Ty), \tag{1.3}$$

where $\sum_{i=1}^{6} a_i = 1$, and for every $x \in M$ satisfies (1.2), then T has a unique fixed point.

The main purpose of the present work is to notice different types of contractive relations similar to (1.3) and to prove some theorems involving these contractive relations.

2. General Contractive Mappings. At first we are going to prove the following two theorems

Theorem 3. *Let T be an orbitally continuous mapping [3] of a metric space M into itself. If for every $x, y\ (x \neq y)$ in M there exist non-negative real numbers a_i ($i = 1, \ldots, 5$) such that*