rate on each bidimensional section of G). By taking into account Remark 1, it will be enough to show that the differential of f at any point $x \in G$ is injective. If no, let $u_0 \in X$, $\|u_0\| = 1$, such that $Df(x; u_0) = 0$ and let $u_1 \in X$, $\|u_1\| = 1$, be a vector linearly independent from u_0. Consider the bidimensional linear manifold $\Pi = x + \mathrm{sp}\{u_0, u_1\}$, obtained by a translation of the linear subspace $\mathrm{sp}\{u_0, u_1\}$ generated by u_0 and u_1. By Remark 1, condition (ii') applied for Π has the following form:

$$\sup\{\|Df(x; u)\|,\ \ u \in \Pi - x,\ \ \|u\| = 1\} \leqslant K \inf\{\|Df(x; u)\|,\ \ u \in \Pi - x,\ \ \|u\| = 1\}.$$

Since $u_1 \in \Pi - x$, $u_0 \in \Pi - x$, we get $\|Df(x; u_1)\| \leqslant K\|Df(x; u_0)\|$, whence $Df(x; u_1) = 0$. Therefore we have obtained $Df(x; u) = 0$ for any vector $u \in X$ and that is impossible by (i'') and Remark 1. Since the condition (i') is satisfied, we may argue as in [3] (Proposition 1) in order to obtain:

$$\delta^{\Pi}(x) = \frac{D^{+}_{\Pi} f(x)}{D^{-}_{\Pi} f(x)},$$

for any bidimensional linear manifold Π, such that $x \in G \cap \Pi$. From here, by applying the condition (ii') we get: $\delta^{\Pi}(x) \leqslant K$, for any Π and $x \in G \cap \Pi$.

Theorem 2. *A bidimensionally regular mapping $f: G \to X$ is K-quasiconformal in the sense of Definition 1', if and only if it is K-quasiconformal in the sense of Definition 2'.*

Proof. If f is K-quasiconformal in the sense of Definition 1' then, by Proposition 2, the condition (ii') is satisfied. Since the differential of f is different from zero, the condition (i'') is satisfied. Then, as was proved above, the conditions (i'') and (ii') imply the condition (i').

The other implication follows simply from Proposition 3.

Remark 3. Definition 2' is equivalent to modified Definition 2' for bidimensionally regular mappings and, in particular, for regular mappings.

Corollary. *Definitions 1 and 2 are equivalent for regular mappings.*

Proof. We already know that Definitions 1 and 1' respectively 2 and 2' are equivalent for regular mappings. Since any regular mapping is also bidimensionally regular, we can apply Theorem 2.

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SOME TYPES OF CONTRACTIVE MAPPINGS AND FIXED POINTS

BY

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1. Introduction: Let (M, d) be a metric space. A mapping $T: M \to M$ is called contractive if for every x, y $(x \neq y)$ in M,

$$d(Tx, Ty) < d(x, y) \tag{1.1}$$

holds.

In a paper [1], M. Edelstein proved a theorem which runs as follows.:

Theorem 1. *Let (M, d) be a metric space, T be a mapping of M into itself. If for every x, y $(x \neq y)$, T satisfies (1.1.) and for every $x \in M$*

$$\{T^{n_i}x\} \subset \{T^n x\} \text{ with } \lim_{i\to\infty} T^{n_i}x = u \in M, \tag{1.2}$$

holds, then u is a unique fixed point of T.

In a recent paper [2], the present author has proved the analogous more general

Theorem 2. *Let T be a continuous mapping of a metric space into itself. If for every x, y $(x \neq y)$ in M there exist non-negative real numbers a_i $(i = 1, 2, \ldots 6)$ and at least one of $a_i \neq 0$ $(i = 1, \ldots 5)$ such that*

$$\begin{aligned} d(Tx, Ty)\, d(x, y) < d(x, y)\{a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + \\ + a_4 d(x, Ty) + a_5 d(y, Tx)\} + a_6 d(x, Tx)\, d(y, Ty), \end{aligned} \tag{1.3}$$

where $\sum_{i=1}^{6} a_i = 1$, and for every $x \in M$ satisfies (1.2), then T has a unique fixed point.

The main purpose of the present work is to notice different types of contractive relations similar to (1.3) and to prove some theorems involving these contractive relations.

2. General Contractive Mappings. At first we are going to prove the following two theorems

Theorem 3. *Let T be an orbitally continuous mapping [3] of a metric space M into itself. If for every x, y $(x \neq y)$ in M there exist non-negative real numbers a_i $(i = 1, \ldots, 5)$ such that*

$$(2.1)\qquad \begin{aligned} d(x, Tx)\, d(y, Ty) < d(x, y)\{&a_1 d(x, y) + a_2 d(x, Tx) + \\ &+ a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx)\} \end{aligned}$$

where $\sum_{i=1}^{5} a_i = 1$ *and for every* $x \in M$, *T satisfies* (1.2) *then it has a fixed point.*

Proof of Theorem 3. Let $x_0 \in M$. Define $Tx_0 = x_1$, $Tx_1 = x_2 = T^2x_0$, .., $Tx_{n-1} = x_n = T^n x_0$,... Suppose $\{T^{n_i} x_0\}$ is a convergent subsequence of $\{T^n x_0\}$ and $\lim_{i\to\infty} T^{n_i} x_0 = u$. Now we may suppose that $Tx_n \neq x_n$ for any finite n.

For, if $Tx_n = x_n$ then x_n is a fixed point of T. Set $x = x_0$ and $y = x_1$ in (2.1), then we have

$$\begin{aligned} d(x_0, Tx_0)\, d(x_1, Tx_1) < d(x_0, x_1)\{&a_1 d(x_0, x_1) + a_2 d(x_0, Tx_0) + \\ &+ a_3 d(x_1, Tx_1) + a_4 d(x_0, Tx_1) + a_5 d(x_1, Tx_0)\} \end{aligned}$$

i.e.

$$\begin{aligned} d(x_0, x_1)\, d(x_1, x_2) < d(x_0, x_1)\{&a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + \\ &+ a_3 d(x_1, x_2) + a_4 d(x_0, x_2) + a_5 d(x_1, x_1)\}. \end{aligned}$$

Since $d(x_0, x_1) \neq 0$, we have

$$d(x_1, x_2) < a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_3 d(x_1, x_2) + a_4 d(x_0, x_2),$$

i.e.

$$(A)\qquad (1 - a_3 - a_4)\, d(x_1, x_2) < (a_1 + a_2 + a_4)\, d(x_0, x_1).$$

Again put $x = x_1$ and $y = x_0$ in (2.1). Then by symmetric property we have

$$(B)\qquad (1 - a_2 - a_5)\, d(x_1, x_2) < (a_1 + a_2 + a_4)\, d(x_0, x_1).$$

Summing (A) and (B),

$$d(x_1, x_2) < \frac{2a_1 + a_2 + a_3 + a_4 + a_5}{2 - (a_2 + a_3 + a_4 + a_5)}\, d(x_0, x_1).$$

Hence, $d(x_1, x_2) < d(x_0, x_1)$. $\left(\because \sum_{i=1}^{5} a_i = 1\right)$.

Similarly, $d(x_2, x_3) < d(x_1, x_2)$.

By induction, it can be shown that

$$d(x_{n+1}, x_n) < d(x_n, x_{n-1}) \quad \text{i.e. } d(T^{n+1}x_0, T^n x_0) < d(T^n x_0, T^{n-1} x_0).$$

So $\{d(T^{n+1}x_0, T^n x_0)\}$ is a decreasing sequence of positive real numbers. Hence $\{d(T^{n+1}x_0, T^n x_0)\}$ is convergent. On the other hand T being orbitally continuous, $\lim_{i\to\infty} T(T^{n_i} x_0) = T(\lim_{i\to\infty} T^{n_i} x_0) = Tu$. Thus, $\lim_{i\to\infty} d(T^{n_i+1}x_0, T^{n_i} x_0) = d(Tu, u)$ and $\to \lim_{i\to\infty} d(T^{n_i+2}x_0, T^{n_i+1}x_0) = d(Tu^2, Tu)$.

Therefore, $d(T^2u, Tu) = d(Tu, u)$ as limits of subsequences of convergent sequence $\{d(T^{n+1}x_0, T^n x_0)\}$. Now let us suppose that $d(Tu, u) \neq 0$, then from (2.1) it can be shown that $d(Tu^2, Tu) < d(Tu, u)$ which is a contradiction. Hence $d(Tu, u) = 0$ i.e. $Tu = u$.

Theorem 4 : *Let T be a selfmapping of a metric space M. If* (2.1) *is satisfied then all periodic points are fixed points.*

Proof : Let x_0 be any point in M and k the least integer such that $T^k x_0 = x_0$. Now we have to prove that $Tx_0 = x_0$.

Suppose $k \neq 1$. Let i, j be positive integers such that $0 < i < j \leqslant k$. Further, let $T^i x_0 = T^j x_0$. i.e. $T^i_{x_0} = T^{i+(j-i)} x_0$, i.e., $T^i x_0 = T^i(T^{j-i} x_0)$.

Hence, $T^{j-i} x_0 = x_0$. This is impossible, since k is the least integer such that $T^k x_0 = x_0$. So $T^i x_0 \neq T^j x_0$. Let $j = i + 1$ $(i = 1, 2, 3, \ldots, k)$. Then for $T^i x_0$, $T^{i+1} x_0$ we have from (2.1), $d(T^i x_0, T^{i+1} x_0) < d(T^{i-1} x_0, T^i x_0)$. For $i = 1, 2, \ldots, k$; we have

$$\begin{aligned} d(x_0, Tx_0) = d(T^0 x_0, Tx_0) > d(Tx_0, T^2 x_0) > d(T^2 x_0, T^3 x_0) > \ldots > \\ > d(T^k x_0, T^{k+1} x_0) = d(x_0, Tx_0), \end{aligned}$$

which is impossible. Hence k must be equal to 1 and $Tx_0 = x_0$.

3. Some variants of Theorem 3: In this section we simply mention some variants of contractive relations for fixed point theorems, the proofs of these being routine calculations and so they are omitted in order to reduce the length of the paper.

Let (M, d) be a metric space and T be a mapping of M into itself. T will be called *generalised contractive mapping*, if for every distinct x, y $(ux \neq y)$ in M there exist real numbers $a_i \geqslant 0$ $(i = 1, 2, \ldots, 6)$ such that T satisfies any one of the foollowing relations :

$$(3.1)\qquad \begin{aligned} &a_1 (d(x, Tx) + d(y, Ty)) + a_2 d(x, y) + a_3 d(Tx, Ty) < \\ &a_4 (d(x, Ty) + d(y, Tx)) \text{ where } a_1 + a_3 > a_4 > a_1 + a_2 \text{ and} \\ &\qquad 2a_1 + a_2 + a_3 = 2a_4 \ (a_3 \neq 0 \text{ and } a_4 \neq 0); \end{aligned}$$

$$(3.2)\qquad \begin{aligned} &a_1 (d(x, Tx) + d(y, Ty)) < a_2 d(x, y) + a_3 d(Tx, Ty) + \\ &a_4 (d(y, Tx) + d(x, Ty)) \text{ where } a_2 + a_4 > a_1 > a_4 + a_3 \text{ and} \\ &\qquad 2a_1 = a_2 + a_3 + 2a_4.\ (a_2 \neq 0); \end{aligned}$$

$$(3.3)\qquad \begin{aligned} &a_1 (d(x, Tx) + d(y, Ty)) + a_3 d(Tx, Ty) < a_2 d(x, y) + \\ &a_4 (d(x, Ty) + d(y, Tx)) \text{ where, } a_1 + a_3 - a_4 > 0,\ a_2 + a_4 - a_1 > 0, \end{aligned}$$

and $2a_1 + a_3 = a_2 + 2a_4$ (a_2 and a_3 can not be equal to zero simultaneously).

Furthermore, the contractive relation (3.1) can be further generalized in the following way : Put

$$L_{3.1} = a_1 (d(x, Tx) + d(y, Ty)) + a_2 d(x, y) + a_3 d(Tx, Ty),$$

$$R_{3.1} = a_4 (d(x, Ty) + d(y, Tx)).$$

Then the generalized contractive relations can be written as:

$$a_5 d(x, Tx)\, d(y, Ty) + d(x, y) \,.\, L_{3.1} < d(x, y) \,.\, R_{3.1} \text{ where,}$$

(3.4) $a_5 + a_1 + a_3 > a_4 > a_1 + a_2$ and $2a_1 + a_2 + a_3 + a_5 = 2a_4$, ($a_4 \neq 0$, a_1, a_3, a_5 can not be zero simultaneously);

$$a_5 d(x, Tx)\, d(y, Ty) + d(Tx, Ty) \,.\, L_{3.1} < d(Tx, Ty) \,.$$

(3.5) $R_{3.1}[Tx \neq Ty]$ where $a_3 \neq 0$, $a_4 \neq 0$, $a_1 + a_3 > a_4 > a_1 + a_2 + a_5$ and $2a_1 + a_2 + a_3 + a_5 = 2a_4$,

$$d(x, y)\, L_{3.1} < d(x, y) \,.\, R_{3.1} + a_5 d(x, Tx)\, d(y, Ty).$$

(3.6) where $a_3 \neq 0$, $a_4 \neq 0$, $a_1 + a_3 > a_4 + a_5$, $a_4 > a_1 + a_2$ and $2a_1 + a_2 + a_3 = 2a_4 + a_5$.

$$d(Tx, Ty)\, L_{3.1} < d(Tx, Ty) \,.\, R_{3.1} + a_5 d(x, Tx)\, d(y, Ty)\ (Tx \neq Ty)$$

(3.7) where $a_3 \neq 0$, $a_1 + a_3 > a_4 > a_1 + a_2 - a_5$ and $2a_1 + a_2 + a_3 = 2a_4 + a_5$.

Similarly, for each of (3.2) and (3.3) we may also define four generalised contractive relations as shown above.

With help of any one of the above contractive relation we may also prove fixed point theorems similar to theorem 3.

4. Concluding Remarks. *Examples of mappings which satisfy the condition* (2.1) but do not satisfy (1.1).

Example 1. Let $M = [0, 1]$ and $Tx = x$, $x \in M$.

Example 2. Let $M = [0, 10]$ and $Tx = x$, $x \in [0, 1) \cup (1, 2) \cup (2, 3) \cup \ldots \cup (9, 10)$. $Tx = x - 1$, $x = 1, 2, 3, \ldots 10$.

Example 3. Let $M = [0, 1, 2, 3, \ldots 10]$, $Tx = x - 1$, $x = 1, 2, \ldots 10$. $T(0) = 0$.

It is easy to verify that mappings defined in the above examples do not satisfy (1.1) but satisfy (2.1).

Furthermore, in theorem (3) we have assumed that T is *orbitally continuous*. We note that if in Theorem (3) we assume T continuous instead of „orbitally continuous" then Theorem also holds. In example (2) we observe that though T is not „Continous" but it is orbitally continuous, T has fixed points. In example (3), we see that T has an unique fixed but in the other two examples T has many fixed points. So mappings satisfying (2.1) may not have *unique* fixed point.

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SOME CONTRACTIVE MAPPINGS

BY

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1. Introduction. Various contraction type mappings on a complete metric space (x, d) have been found as generalizations of the well-known B a n a c h contraction. For examples, R. K a n n a n [1] has introduced a contraction type mapping $T : X \to X$ by means of the metric relation

$$d(Tx, Ty) \leqslant \alpha\,[d(x, Tx) + d(y, Ty)],\ 0 < \alpha < \frac{1}{2}. \tag{1.1}$$

The author [2] of the present paper has introduced a contraction type mapping $T : X \to X$ by means of the metric relation

$$d(Tx, Ty) \leqslant \alpha\,[d(x, Ty) + d(y, Tx)],\ 0 < \alpha < 1/2. \tag{1.2}$$

Subsequently G. H a r d y and T. R o g e r s [3] have introduced a contraction type mapping $T : X \to X$ by means of the metric relation

$$d(Tx, Ty) \leqslant \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty) + \alpha_4 d(x, Ty) + \alpha_5 d(y, Tx), \tag{1.3}$$

where $\alpha_i \geqslant 0$ and $\sum_{i=1}^{5} \alpha_i < 1$. There are several other contraction type mappings for which we refer to the paper of B. E. R h o a d e s [4], where such contraction type mappings are compared in details. In a recent paper [5] B. F i s h e r has proved the following contractive mapping yielding a unique fixed point:

Theorem 1. *If T is continuous mapping of a compact metric space (X, d) into itself such that*

$$d(Tx, Ty) < \frac{1}{2}[d(x, Tx) + d(y, Ty)] \tag{1.4}$$

for all $x \neq y$, $x, y \in X$, then T has a unique fixed point.

Although the assumption of continuity is superfluous [4, p. 285], yet the proof of Fisher seems to be shorter and interesting. Indeed, carrying