Then the generalized contractive relations can be written as:

$$a_5 d(x, Tx)\, d(y, Ty) + d(x, y) \,.\, L_{3.1} < d(x, y) \,.\, R_{3.1} \text{ where,}$$

(3.4) $$a_5 + a_1 + a_3 > a_4 > a_1 + a_2 \text{ and } 2a_1 + a_2 + a_3 + a_5 = 2a_4,$$
$$(a_4 \neq 0,\ a_1, a_3, a_5 \text{ can not be zero simultaneously});$$

$$a_5 d(x, Tx)\, d(y, Ty) + d(Tx, Ty) \,.\, L_{3.1} < d(Tx, Ty) \,.$$
(3.5) $$R_{3.1}[Tx \neq Ty] \text{ where } a_3 \neq 0,\ a_4 \neq 0,\ a_1 + a_3 > a_4 >$$
$$a_1 + a_2 + a_5 \text{ and } 2a_1 + a_2 + a_3 + a_5 = 2a_4,$$

$$d(x, y)\, L_{3.1} < d(x, y) \,.\, R_{3.1} + a_5 d(x, Tx)\, d(y, Ty).$$
(3.6) $$\text{where } a_3 \neq 0,\ a_4 \neq 0,\ a_1 + a_3 > a_4 + a_5,\ a_4 > a_1 + a_2 \text{ and}$$
$$2a_1 + a_2 + a_3 = 2a_4 + a_5.$$

$$d(Tx, Ty)\, L_{3.1} < d(Tx, Ty) \,.\, R_{3.1} + a_5 d(x, Tx)\, d(y, Ty)\ (Tx \neq Ty)$$
(3.7) $$\text{where } a_3 \neq 0,\ a_1 + a_3 > a_4 > a_1 + a_2 - a_5 \text{ and}$$
$$2a_1 + a_2 + a_3 = 2a_4 + a_5.$$

Similarly, for each of (3.2) and (3.3) we may also define four generalised contractive relations as shown above.

With help of any one of the above contractive relation we may also prove fixed point theorems similar to theorem 3.

4. Concluding Remarks. *Examples of mappings which satisfy the condition* (2.1) but do not satisfy (1.1).

Example 1. Let $M = [0, 1]$ and $Tx = x$, $x \in M$.

Example 2. Let $M = [0, 10]$ and $Tx = x$, $x \in [0, 1) \cup (1, 2) \cup (2, 3) \cup \ldots \cup (9, 10)$. $Tx = x - 1$, $x = 1, 2, 3, \ldots 10$.

Example 3. Let $M = [0, 1, 2, 3, \ldots.. 10]$, $Tx = x - 1$, $x = 1, 2, \ldots.. 10$. $T(0) = 0$.

It is easy to verify that mappings defined in the above examples do not satisfy (1.1) but satisfy (2.1).

Furthermore, in theorem (3) we have assumed that T is *orbitally continuous*. We note that if in Theorem (3) we assume T continuous instead of „orbitally continuous" then Theorem also holds. In example (2) we observe that though T is not „Continous" but it is orbitally continuous, T has fixed points. In example (3), we see that T has an unique fixed but in the other two examples T has many fixed points. So mappings satisfying (2.1) may not have *unique* fixed point.

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SOME CONTRACTIVE MAPPINGS

BY

S. K. CHATTERJEA

1. Introduction. Various contraction type mappings on a complete metric space (x, d) have been found as generalizations of the well-known B a n a c h contraction. For examples, R. K a n n a n [1] has introduced a contraction type mapping $T : X \to X$ by means of the metric relation

(1.1) $$d(Tx, Ty) \leqslant \alpha\,[d(x, Tx) + d(y, Ty)],\ 0 < \alpha < \frac{1}{2}.$$

The author [2] of the present paper has introduced a contraction type mapping $T : X \to X$ by means of the metric relation

(1.2) $$d(Tx, Ty) \leqslant \alpha\,[d(x, Ty) + d(y, Tx)],\ \ 0 < \alpha < 1/2.$$

Subsequently G. H a r d y and T. R o g e r s [3] have introduced a contraction type mapping $T : X \to X$ by means of the metric relation

(1.3) $$d(Tx, Ty) \leqslant \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty) + \alpha_4 d(x, Ty) + \alpha_5 d(y, Tx),$$

where $\alpha_i \geqslant 0$ and $\sum_{i=1}^{5} \alpha_i < 1$. There are several other contraction type mappings for which we refer to the paper of B. E. R h o a d e s [4], where such contraction type mappings are compared in details. In a recent paper [5] B. F i s h e r has proved the following contractive mapping yielding a unique fixed point:

Theorem 1. *If T is continuous mapping of a compact metric space (X, d) into itself such that*

(1.4) $$d(Tx, Ty) < \frac{1}{2}[d(x, Tx) + d(y, Ty)]$$

for all $x \neq y$, $x, y \in X$, then T has a unique fixed point.

Although the assumption of continuity is superfluous [4, p. 285], yet the proof of Fisher seems to be shorter and interesting. Indeed, carrying out the same proof, the following contractive theorems involving the metric relations (1.2) and (1.3) can be proved:

Theorem 2. *If T is a continuous mapping of a compact metric space (X, d) into itself such that*

$$d(Tx, Ty) < \frac{1}{2}[d(x, Ty)+d(y, Tx)] \tag{1.5}$$

for all $x \not\equiv y$, $x, y \in X$, then T has a unique fixed point.

Theorem 3. *If T is a continuous mapping of a compact metric space (X, d) into itself such that*

$$d(Tx, Ty) < \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty) + \alpha_4 d(x, Ty) + \alpha_5 d(y, Tx), \tag{1.6}$$

where $\alpha_i \geqslant 0$ and $\sum_{i=1}^{5} \alpha_i = 1$, *for all distinct $x, y \in X$, then T has a unique fixed point.*

Some particular cases of interest are pointed out from theorem 3. In fact, the theorem of Fisher is a particular case of theorem 3. Finally in a recent work [6] the present author and K. M. Ghosh have proved the following theorem:

Theorem 4. *If S is a compact subset of a Banach space E, T is a continuous self-mapping of S and there exists a family of functions $F \equiv \{f_\alpha\}_{\alpha\in S}$ from $[0, 1/5]$ into S, such that:*

(i) *for each $\alpha \in S$, $f_\alpha(1/5) = \alpha$,*
(ii) *for a function $\Phi : (0, 1/5) \to (0, 1/5)$,*

$$\|f_{Tx}(t) - f_{Ty}(t)\| \leqslant \Phi(t)\{\|x - y\| + \|x - f_{Tx}(t)\| + \|y - f_{Ty}(t)\| + \|x - f_{Ty}(t)\| + \|y - f_{Tx}(t)\|\},$$

for all $x, y \in S$ and for all $t \in (0, 1/5)$,
(iii) *for $t \to t_0$ in $[0, 1/5]$ and $\alpha \to \alpha_0$ in S, $f_\alpha(t) \to f_{\alpha_0}(t_0)$ in S, then T has a fixed point in S.*

Now by virtue of theorem 3, we can extend theorem 4 as follows:

Theorem 5. *If S is a compact subset of a Banach space E, T is a continuous self-mapping of S and there exists a family of functions $F \equiv \{f_\alpha\}_{\alpha\in S}$ from $0, 1/5$ into S such that*

(i) *for each $\alpha \in S$, $f_\alpha(1/5) = \alpha$,*
(ii) *for a function Φ: $[0, 1/5] \to \{1/5\}$*

$$\|f_{Tx}(t) - f_{Ty}(t)\| < \Phi(t)\{\|x - y\| + \|x - f_{Tx}(t)\| + \|y - f_{Ty}(t)\| + \|x - f_{Ty}(t)\| + \|y - f_{Tx}(t)\|\},$$

for distinct $x, y \in S$ and for all $t \in [0, 1/5]$,
(iii) *for $t \to t_0 \in [0, 1/5]$ and $\alpha \to \alpha_0 \in S$, $f\alpha(t) \to f\alpha_0(t_0)$ in S, then T has a fixed point in S.*

2. Proof of proposed theorems.

Proof of theorem 2. Let us define a function f on X by $f(x) = d(x, Tx)$. Now d and T being continuous, it follows that f is a continuous function on X. Thus X being compact, there exists a point $z \in X$ such that $f(z) = \inf\{f(x), x \in X\}$. Let us now suppose that $Tz \neq T^2z$. Thus we have

$$f(Tz) = d(Tz, T^2z) < 1/2\,[d(z, T^2z) + d(Tz, Tz)] \leqslant 1/2\,[d(z, Tz) + d(Tz, T^2z)] = 1/2\,[f(z) + f(Tz)],$$

so that $f(Tz) < f(z) = \inf\{f(x);\ x \in X\}$, which is absurd and so $Tz = T^2z$. In other words $Tz = z$; so that z is a fixed point.

Next suppose z_1 and z_2 are two distinct fixed points. Then

$$d(z_1, z_2) = d(Tz_1, Tz_2) < 1/2\,[d(z_1, Tz_2) + d(z_2, Tz_1)] = d(z_1, z_2)$$

which is absurd and so $z_1 = z_2$. Thus the fixed point is unique.

Proof of theorem 3. As before we define a function f on X by $f(x) = d(x, Tx)$. Likewise if $Tz \neq T^2z$, we get

$$\begin{aligned}
f(Tz) &= d(Tz, T^2z)\\
&< \alpha_1 d(z, Tz) + \alpha_2 d(z, Tz) + \alpha_3 d(Tz, T^2z) + \alpha_4 d(z, T^2z) + \alpha_5 d(Tz, Tz) =\\
&= \alpha_1 d(z, Tz) + \alpha_2 d(z, Tz) + \alpha_3 d(Tz, T^2z) + \alpha_4 d(z, T^2z)\\
&\leqslant \alpha_1 d(z, Tz) + \alpha_2 d(z, Tz) + \alpha_3 d(Tz, T^2z) + \alpha_4[d(z, Tz) + d(Tz, T^2z)]\\
&= (\alpha_1 + \alpha_2 + \alpha_4)\, d(z, Tz) + (\alpha_3 + \alpha_4)\, d(Tz, T^2z)\\
&= (\alpha_1 + \alpha_2 + \alpha_4) f(z) + (\alpha_3 + \alpha_4) f(Tz).
\end{aligned} \tag{1}$$

Again by the symmetry property of a metric, we get

$$f(Tz) = d(T^2z, Tz) < (\alpha_1 + \alpha_3 + \alpha_5) f(z) + (\alpha_2 + \alpha_5) f(Tz). \tag{2}$$

Adding (1) and (2) we obtain

$$(2 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5) f(Tz) < (2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) f(z) = (2 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5) f(z),$$

i.e. $$f(Tz) < f(z) = \inf\{f(x);\ x \in X\},$$

which is absurd and thus $Tz = T^2z$, i.e. $Tz = z$.

Next suppose z_1 and z_2 are two distinct fixed points. Then

$$d(z_1, z_2) = d(Tz_1, Tz_2) < \alpha_1 d(z_1, z_2) + \alpha_2 d(z_1, Tz_1) + \alpha_3 d(z_2, Tz_2) + \alpha_4 d(z_1, Tz_2) + \alpha_5 d(z_2, Tz_1) = (\alpha_1 + \alpha_4 + \alpha_5)\, d(z_1, z_2),$$

which is absurd and thus $z_1 = z_2$.

Remark. If $\alpha_1 = \alpha_4 = \alpha_5 = 0$, then the theorem 2 reduces to the following:

Theorem 1'. *If T is a continuous mapping of a compact metric space (X, d) into itself such that*

$$d(Tx, Ty) < \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty),$$

where $\alpha_2 \geqslant 0$, $\alpha_3 \geqslant 0$ and $\alpha_2 + \alpha_3 = 1$, for all distinct $x, y \in X$, then T has a unique fixed point.

The theorem 1′ may be considered as an extension of theorem 1. Similarly other particular cases like theorem 1′ can be deduced from theorem 3 by putting $\alpha_4 = \alpha_5 = 0$ or $\alpha_2 = \alpha_3 = 0$.

Finally we return to the proof of theorem 5. Let us put $k_n = n/[5(n+1)]$ $n = 1, 2, \ldots$, so that $k_n \to 1/5$. Also let $T_n : S \to S$ be defined by $T_n x = f_{Tx}(k_n)$ for all $x \in S$. Since $T(S) \subset S$ and $0 < k_n < 1/5$, each T_n is well defined and maps S into S. Now for each n, we have for all distinct $x, y \in S$:

$$\|T_n x - T_n y\| = \|f_{Tx}(k_n) - f_{Ty}(k_n)\| < \Phi(k_n)\{\|x - y\| + \|x - f_{Tx}(k_n)\| + \\ + \|y - f + y(k_n)\| + \|x - f_{Ty}(k_n)\| + \|y - f_{Tx}(k_n)\| = 1/5\,\{\|x - y\| + \|x - \\ - T_n(x)\| + \|y - T_n(y)\| + \|x - T_n(y)\| + \|y - T_n(x)\|\}.$$

Thus T_n is a contractive mapping on S in the sense of theorem 3 for $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5$. So each T_n has a unique fixed point $x_n \in S$. Since S is compact, there is a subsequence $\{x_{nj}\}$ of $\{x_n\}$ such that $\{x_{nj}\} \to$ some $x \in S$. Now $T_{nj}\, x_{nj} = x_{nj} \to x \in S$. On the other hand T is continuous and consequently $Tx_{nj} \to Tx$. Now by (iii) and (i) we have

$$T_{nj}\, x_{nj} = f_{Tx_{nj}}(k_n) \to f_{Tx}(1/5) = Tx,$$

so that $Tx = x$.

3. Concluding remark. Theorem 2 and theorem 3 are established for a compact metric space, whereas theorem 5 is established for a compact subset of a Banach space. The object of discussing theorem 5 is to extend compact star-shaped subsets (which include contractive mappings on compact convex subsets as a proper subclass) of a linear space with the help of theorem 3, i.e to discuss contractive mapping on a certain class of non-convex sets, just as we did in [6] for extending contraction mapping on compact star-shaped subsets of a linear space (vide theorem 4 of the present paper).

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SUR UNE ÉQUATION INTÉGRALE DU TYPE VOLTERRA NON-LINÉAIRE

PAR

STIG-OLOF LONDEN

1. Introduction. Dans ce travail on considère le comportement qualitatif des solutions de l'équation du type Volterra,

$$x'(t) + \int_0^t g(x(t-\tau))\, da(\tau) = f(t), \quad t \geqq 0, \tag{1.1}$$

où g, a, f sont des fonctions réelles données et x signifie la solution.

En particulier on démontrera le résultat suivant :

Théorème. *Supposons que*

(1.2) $g \in C(R)$, (1.5) $\mathrm{Re}\,\hat{a}(\omega) \geqq 0, \quad \omega \in R$,

(1.3) $a \in NBV(R^+)$, (1.6) $\hat{a}(\omega) = 0, \ \omega \in S \overset{df}{=} \{\omega \mid \mathrm{Re}\,\hat{a}(\omega) = 0\}$,

(1.4) $\int_t^\infty |da(\tau)| \in L^1(R^+)$, (1.7) $0 \in S$,

(1.8) $x \in L^\infty \cap LAC(R^+)$, *$x$ vérifie* (1.1) *p.p. sur R^+ et soit la fonction f telle que*

(F_∞) $f \in L^\infty(R^+)$, $f(t) \to 0$, $\int_0^t f(s)\, ds \to F$, *pour $t \to \infty$, où $|F| < \infty$.*

Alors ou i) ou bien ii) ci-dessous est vérifié.

$$\text{i)} \quad \lim_{t\to\infty} [x(t) + Ag(x(t))] = x(0) + F, \tag{1.9}$$

$$\limsup_{t\to\infty,\ |s| \leqq T} |x(s+t) - x(t)| = 0, \ \textit{pour tout } T \textit{ positif}. \tag{1.10}$$

ii) Il existent des fonctions $\tilde{f}$, $\tilde{x}$ vérifiant