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# SOLUTIONS OF SOME THIRD-ORDER NON-LINEAR DIFFERENTIAL EQUATIONS: FREQUENCY-DOMAIN METHOD

BY

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**1. Introduction**: The frequency-domain method developed by Popov [3] is largely used in the theory of control systems. It has been successfully applied to non-linear oscillations of certain kinds of second, third and fourth order differential equations. We extend here, the range of applications to some third order differential equations with two non-linearities. Precisely, we shall consider the equations

$$x''' + ax'' + g(x') + h(x) = q(t),$$

and

$$x''' + ax'' + g_1(x)\, x' + h(x) = q(t),$$

for existence, uniqueness of solutions which are bounded in $\mathbf{R}$ and are globally exponentially stable and which are periodic, respectively almost periodic in the same time as $q(t)$.

We shall use the following theorem of Yacubovich:

Consider the system

$$x' = Ax - B\varphi(\sigma) + p(t), \quad \sigma = C^* x \tag{1.1}$$

$$\varphi(\sigma) = \operatorname{col} \varphi_j(\sigma_j) \qquad (j = 1, 2, \ldots, n)$$

$$\varphi(0) = 0,$$

$$0 \leqslant \frac{\varphi_j(\sigma_j) - \varphi_j(\tilde{\sigma}_j)}{\sigma_j - \tilde{\sigma}_j} \leqslant \mu_j;$$

where $A$ is a stable $n \times n$ matrix; $B$ and $C$ are $n \times m$ matrices, $B \neq 0$; $\sup |p(t)| < \infty$ for all $t \in \mathbf{R}$.

If there exist $\tau_1, \tau_2, \ldots, \tau_m$ all strictly positive such that for matrices $L = \operatorname{diag}(\tau_j)$ and $K = \operatorname{diag}(\mu_j)$

$$LK^{-1} + \operatorname{Re}[LC^* (i\omega I - A)^{-1} B] > 0. \tag{1.2}$$

for all $\omega \in \mathbf{R}$, then

(i) the system has a unique bounded solution on $\mathbf{R}$, and
(ii) for any two solutions, the estimation ($\gamma > 0$, $\varepsilon > 0$ depend only on the equation)

$$| x_1(t) - x_2(t) | \leqslant \gamma e^{-\varepsilon(t-t_0)} | x_1(t_0) - x_2(t_0) |$$

holds.
(iii) If $p(t)$ is periodic, respectively almost periodic, so also is the bounded solution.

**2. Results :** The following lemma, the proof of which follows from usual methods, will be applied in the main results.

**Lemma :** *The cubic polynomial*

$$Q(y) = y^3 + k_1 y^2 + k_2 y + k_3$$

*for which $k_1, k_2, k_3$ are real with $k_1 < 0$, and $|3k_2| < k_1^2$, has its minimum value at*

$$\bar{y} = -(k_1/3)\{[2 - (3k_2/2k_1^2)] - 0[(3k_2/k_1^2)^2]\}$$

*and this minimum value will be positive if*

$$-k_1k_2[(2/3) - (1/4)(3k_2/k_1^2)] + (4k_1^3/27) + k_3$$

*is chosen positive.*

**Theorem 1 :** *Consider the equation*

$$x''' + ax'' + g(x') + h(x) = q(t) \tag{2.1}$$

*with $a > 0$, $g, h$ continuous in their arguments and*

$$g(0) = h(0) = 0.$$

*Suppose*

(i) *there exist constants $b, c$ such that $\alpha b > c > 0$ and real numbers $\mu_1, \mu_2$ such that*

$$b \leqslant \frac{g(z) - g(\bar{z})}{z - \bar{z}} \leqslant b + \mu_1,$$

$$c \leqslant \frac{h(z) - h(\bar{z})}{z - \bar{z}} \leqslant c + \mu_2 \text{ for } z \neq \bar{z},$$

(ii)

$$2\alpha c - b^2 > \mu_1 b > \alpha c - 2b^2,$$

$$\mu_2 > |(b^2 - 2\alpha c)/\alpha| - [(\alpha^2 - 2b)/\alpha]^2$$

*and $\mu_2$ satisfies the inequality*

$$\alpha^2\mu_2^2 + \mu_2[(8\alpha/9)(\alpha^2 - 2b)^2 - 2\alpha(b^2 - 2\alpha c) + (4c/3)(\alpha^2 - 2b)] +$$
$$+ [(b^2 - 2\alpha c)^2 + (16/81)(\alpha^2 - 2b)^4 - (8/3)(\alpha^2 - 2b)^2(b^2 - 2\alpha c) +$$
$$+ (4c^2/3)(\alpha^2 - 2b)] > 0.$$



(iii) $\quad q(t)$ *is bounded ;*

*and for some $\lambda$ satisfying*

(iv)
$$\min_{\mu_1,\mu_2}\{(4\alpha/\mu_1) + (8/3\mu_1\mu_2)(\alpha^2 - 2b - \mu_1)^2,$$
$$(\mu_1\mu_2/4)[(4/27)(\alpha^2 - 2b - \mu_1)^3 + c^2 + c\mu_2]\}$$
$$> \lambda > (4\alpha/\mu_1) - (8/3\mu_1\mu_2)(\alpha^2 - 2b - \mu_1)^2$$
$$\text{with } \lambda \neq (4\alpha/\mu_1);$$

*Then*

(A) *there exists a unique solution $x(t)$ for (2.1) such that $x(t), x'(t), x''(t)$ are bounded in $\mathbf{R}$ and globally exponentially stable ;*

(B) *If, in addition, $q(t)$ is periodic (respectively almost periodic), then the same is true for the solution $x(t)$ of (2.1).*

*Proof of Theorem 1 :* Let $g(z) = bz + \bar{g}(z)$ and

$$h(z) = cz + \bar{h}(z).$$

For convenience, we shall reduce (2.1) to the form of (1.1) by putting

$$x_1' = -bx_2 - cx_3 - \bar{g}(x_2) - \bar{h}(x_3) + q(t)$$
$$x_2' = x_1 - \alpha x_2$$
$$x_3' = x_2$$

making

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \quad A = \begin{pmatrix} 0 & -b & -c \\ 1 & -\alpha & 0 \\ 0 & 1 & 0 \end{pmatrix};$$

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}; \text{ and}$$

$$p(t) = \begin{pmatrix} q(t) \\ 0 \\ 0 \end{pmatrix}. \text{ Thus } \sigma = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \text{ and}$$

$$\varphi(\sigma) = \begin{pmatrix} \bar{g}(x_2) \\ \bar{h}(x_3) \end{pmatrix}.$$

Let $\Delta$ denote the determinant of $(i\omega I - A)$, that is $\Delta = (c - \alpha\omega^2) + i\omega(b - \omega^2)$. Hence,

$$G(i\omega) = C^*(i\omega I - A)^{-1}B = \begin{pmatrix} \dfrac{i\omega}{\Delta} & \dfrac{i\omega}{\Delta} \\ \dfrac{1}{\Delta} & \dfrac{1}{\Delta} \end{pmatrix}$$

and

$$G^*(-i\omega) = B^*(-i\omega - A)^{-1}C = \begin{pmatrix} \dfrac{-i\omega}{\bar{\Delta}} & \dfrac{1}{\bar{\Delta}} \\ \dfrac{-i\omega}{\bar{\Delta}} & \dfrac{1}{\bar{\Delta}} \end{pmatrix},$$

where $\bar{\Delta} = (c - \alpha\omega^2) - i\omega(b - \omega^2)$. Thus, the Yacubovich frequency condition (1.2) becomes

$$\pi(\omega) = \begin{pmatrix} \dfrac{\tau_1}{\mu_2} + \dfrac{\omega^2\tau_1(b-\omega^2)}{|\Delta|^2} & \dfrac{\tau_2\Delta + i\omega\tau_1\bar{\Delta}}{2|\Delta|^2} \\ \dfrac{\tau_2\bar{\Delta} - i\omega\tau_1\Delta}{2|\Delta|^2} & \dfrac{\tau_2}{\mu_2} + \dfrac{\tau_2(c-\alpha\omega^2)}{|\Delta|^2} \end{pmatrix} > 0 \tag{2.3}$$

with the notations $L = \operatorname{diag}(\tau_j)$; $K = \operatorname{diag}(\mu_j)$ $(j=1,2)$. Calculation shows that (2.3) can be written in the form

$$\pi(\omega) \equiv \begin{pmatrix} \dfrac{\tau_1}{\mu_1|\Delta|^2}P_1(\omega) & \dfrac{\tau_2\Delta + i\omega\tau_1\bar{\Delta}}{2|\Delta|^2} \\ \dfrac{\tau_2\bar{\Delta} - i\omega\tau_1\Delta}{2|\Delta|^2} & \dfrac{\tau_2}{\mu_2|\Delta|^2}P_2(\omega) \end{pmatrix} > 0, \tag{2.3a}$$

where

$$P_1(\omega) = \omega^6 + \omega^4(\alpha^2 - 2b - \mu_1) + \omega^2(b^2 - 2\alpha c + \mu_1 b) + c^2$$

and

$$P_2(\omega) = \omega^6 + \omega^4(\alpha^2 - 2b) + \omega^2(b^2 - 2\alpha c - \alpha\mu_2) + (c^2 + \alpha\mu_2).$$

Also the determinant of $\pi(\omega)$, when we let $\lambda = (\tau_1/\tau_2)$ is

$$\frac{\tau_1\tau_2}{\mu_1\mu_2|\Delta|^2}P_3(\omega)$$

with

$$P_3(\omega) = \omega^6 + \omega^4(\alpha^2 - 2b - \mu_1) +$$
$$+ \omega^2\left(b^2 - 2\alpha c - \alpha\mu_2 + b\mu_1 - \frac{\mu_1\mu_2\lambda}{4}\right) + \left(c^2 + c\mu_2 - \frac{\mu_1\mu_2}{4\lambda}\right).$$

Polynomials $P_1(\omega)$, $P_2(\omega)$ and $P_3(\omega)$ are cubic in $\omega^2$.



In order that condition (2.3a) be satisfied, we shall use the Sylvester's Criterion that the determinant of the principal minors of the matrix $\pi(\omega)$ be positive. This reduces to either $P_1(\omega)$ and $P_3(\omega)$ be both positive or $P_2(\omega)$ and $P_3(\omega)$ be both positive.

By the Lemma proved above, either of these will be true if conditions (ii) and (iv) of the Theorem are satisfied. All the conditions of Yacubovich Theorem are satisfied and thus our theorem is proved.

In considering the second equation, we shall use the following definition:

Under the frequency-domain method, the system

$$x' = A^*x - C\varphi(\sigma_1) + p_1(t), \quad \sigma_1 = B^*x,$$

will be called the dual of the equation

$$x' = Ax - B\varphi(\sigma) + p(t), \quad \sigma = C^*x,$$

where $A^*$, $B^*$ and $C^*$ are complex conjugate transposed of matrices $A$, $B$ and $C$ respectively. It must be noted that condition (1.2) of Yacubovich's Theorem is true for dual systems.

**Theorem 2**: *Let $b$, $c$ be positive constants such that $\alpha b > c$ and $\mu_1$, $\mu_2$ be real numbers. If condition (i) in Theorem 1 is replaced by*

$$\text{(i')} \qquad b \leqslant \int g_1(x)\,dx \leqslant b + \mu_1.$$
$$c \leqslant \frac{h(x) - h(\bar{x})}{x - \bar{x}} \leqslant c + \mu_2,$$
$$h(0) = 0, \quad x \neq \bar{x},$$

*then the same conclusions as in Theorem 1 are valid for the equation*

$$x''' + \alpha x'' + g_1(x)x' + h(x) = q(t). \tag{2.4}$$

*Proof of Theorem 2*: Let $\int_0^x g_1(x)\,dx = bx + \bar{g}_1(x)$ and $h(x) = cx + \bar{h}(x)$.

Using the equivalent system

$$x_1' = x_2,$$
$$x_2' = -bx_1 - \alpha x_2 + x_3 - \bar{g}_1(x_1),$$
$$x_3' = -cx_1 - \bar{h}(x_1) + q(t),$$

we see that (2.4) is the dual to the equation (2.1) where our corresponding matrices to those in (1.1) are respectively $A^*$, $C$ and $B^*$ with matrices $A$, $B$ and $C$ as in Theorem 1.

Hence under the frequency-domain method the conclusions are valid.

*Remark*: Theorems 1 and 2 have extended the results of [4; pp. 335—336] to the case when $g(x')$ or $g_1(x)\,x'$ are not merely linear functions of $x'$ but quasi-linear functions of $x'$. Exponential stability was established by both methods one via Liapunov function theory and the other by the frequency-domain method. The later has the added advantage of arriving at the result (and more results) of a larger class of equations through simplified inequalities.

## REFERENCES


1. Barbalat I. and Halanay A. — *Applications of the frequency-method to forced non-linear oscillations.* Math. Nachrichten, 1970, 44, 165—179.
2. Barbalat I. — *Conditions pour a dans le comportement de certaines équations différentielles du troisième et du quatrième ordre. Equations différentielles et functionelles non-linéaries*, edited by P. Janssens, J. Mawhin and N. Rouche, Hermann Publishing, Paris, 1973, 80—91.
3. Popov V. M. — *Hyperstability of Control Systems.* Editura Academiei, Bucureşti and Springer-Verlag, Berlin, 1973.
4. Reissig R., Sansone G. and Conti R. — *Non-Linear Differential Equations of Higher Order.* Noordhoff International Publishing, 1974.


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# EIN MAXIMALES PRINZIP FÜR STARK ELLIPTISCHE SYSTEME MIT EINER GERADEN ANZAHL VON PARTIELLEN DIFFERENTIALGLEICHUNGEN

VON

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Der Zweck dieser Arbeit ist algebraische Bedingungen für die Koeffizienten eines stark elliptischen Systems mit einer geraden Anzahl von partiellen Differentialgleichungen II Grades zu bringen, für die das maximale Prinzip gültig ist. Man bringt auch einen Satz über die Eindentigkeit der Lösung des Problems von Dirichlet für diese Art von Systemen.

Wir gehen von der Arbeit [3] aus und werden hier allgemeinere und vervollständigte Resultate in bezug auf die Arbeiten [1] und [2] geben.

1. Es sei ein begrenzter Bereich $D \subset R^n$, mit der Grenze $\Gamma$ gegeben. $M_{nn}(R)$ sei die Menge der quadratischen Matrizen $n$-ter Ordnung mit reellen Elementen. Für $n$ eine gerade Zahl, berichsichtigt man folgendes System mit partiellen Differentialgleichungen:

$$Lu = \sum_{i,j=1}^{n} A_{ij}(x) \frac{\partial^2 u}{\partial x_i\, \partial x_j} + \sum_{i=1}^{n} A_i(x) \frac{\partial u}{\partial x_i} + A_0(x)\, u = B(x), \tag{1}$$

wo

$$A_{ij},\quad A_i,\quad A_0 \in C(\bar{D}, M_{nn}(R)),\quad B \in C(\bar{D}, R^n),\quad x \in R^n. \tag{2}$$

Wir setzen voraus daβ die Matrizen $A_{ij}$ symmetrisch sind, also

$$A_{ij} = A_{ji},\quad i, j = \overline{1, n}. \tag{3}$$

Wir bezeichnen mit $f$ die quadratische Form von $\dfrac{n(n+1)(n+2)}{2}$ Veränderlichen die durch

$$f = \langle e, Le \rangle = \sum_{i,j=1}^{n} \langle e, A_{ij}\, e''_{ij} \rangle + \sum_{i=1}^{n} \langle e, A_i\, e'_i \rangle + \langle e, A_0\, e \rangle, \tag{4}$$

definiert ist, wo $\langle , \rangle$ das Symbol für das skalare Produkt aus $R^n$ ist und

$$e = (y_1, \ldots, y_n)^T \tag{5}$$