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ON SEMI-SIMPLE INFRA-NEAR RINGS (II)

BY

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In a previous paper, we have studied some properties of the simple and semi-simple infra-near rings. The notion of infra-near ring has been introduced in [4]. Our definition for semi-simplicity in [5] was suggested by that given for semi-simple near-rings by D.W. Blackett [2]. The whole theory presents an analogy with the theory of semi-simplicity of the rings and near-rings. In spite of this analogy, there are simple left infra-near rings for which the associated left distributive multiplication ($x \circ y = x \cdot y - x \cdot 0$, for all elements x, y) is a near-ring multiplication, but the so-obtained near-ring is not simple in the Blackett's sense (see examples in [5]). The deep reason of the possibility to build such a theory in the case of left infra-near rings is the assumption that there exists a left distributive element d_M in each minimal nonzero right N-subgroup M of the infra-near ring N, such that $d_M \cdot M \neq \{0\}$. As one can see, in all the statements and arguments, we need only the left distributivity of such an element, the other elements being only infra-distributive.

Many properties of semi-simple near-rings have natural generalizations for semi-simple infra-near rings. We have already shown it in [5]. Now we give some other properties of semi-simple infra-near rings with a left identity. We prove that the right ideals of a semi-simple left infra-near ring satisfy the ascending chain condition. We give necessary and sufficient condition for a left infra-near ring to be semi-simple, when it has a left identity.

1. Preliminaries. First we recall some useful definitions and properties.

Definition 1.1. *A left infra-near ring* (shorter, *l.i.n.r.*) *is a triple* $(N, +, .)$, *where* $(N, +)$ *is a group,* $(N, .)$ *is a semigroup, satisfying the following identity*:

(1.1) $$x \cdot (y+z) = x \cdot y - x \cdot 0 + x \cdot z, \text{ for all } x, y, z \in N.$$

Assume that

(1.2) $$0 \cdot x = 0, \text{ for any } x \in N.$$

Definition 1.2. *A right N-group is an additive group* $(M, +)$, *together with a multiplication* $\mu : M \times N \to M$*-denote it by* $\mu(m, x) = m \cdot x$, *for any* $m \in M$, $x \in N$*-such that the following axioms are fulfilled*:

(1.3) $$m \cdot (x+y) = m \cdot x - m \cdot 0 + m \cdot y, \text{ for all } m \in M,\ x, y \in N;$$

(1.4) $$m \cdot (x \cdot y) = (m \cdot x) \cdot y, \text{ for all } m \in M,\ x, y \in N;$$

(1.5) $$0\cdot x=0, \text{ for all } x\in N.$$

Obviously, N and $\{0\}$ are right N-groups.

Definition 1.3. *A right ideal of N is a normal subgroup R of N satisfying the property:*

(1.6) $$(r+x)\cdot y-x\cdot y\in R, \text{ for all } r\in R, \ x, \ y\in N.$$

N and $\{0\}$ are also right ideals of N. A right ideal of N is a right N-subgroup of N (take $x=0$ in (1.6)), but generally the converse is not true. The property (1.6) can be written under one of the forms:

(1.7) $$(x+r)\cdot y-x\cdot y\in R, \text{ for all } r\in R, \ x, \ y\in N,$$

or

(1.8) $$-x\cdot y+(x+r).y\in R, \text{ for all } r\in R, \ x, \ y,\in N,$$

because $(R,+)$ is a normal subgroup of $(N,+)$.

Proposition 1.4. ([5], Proposition 1.2) *If A is a right N-subgroup (a right ideal), and B is a right ideal of the l.i.n.r. N, then $A+B=\{a+b| a\in A, \ b\in B\}$ is a right N-subgroup (a right ideal) of N.*

Definition 1.5. *Let M and M' be two right N-groups. A mapping φ: $M\to M'$ is called a morphism of right N-groups, or a N-homomorphism if $\varphi(m_1+m_2)=\varphi(m_1)+\varphi(m_2)$ and φ $(m\cdot x)=\varphi(m)\cdot x$ for any m_1, m_2, $m\in M$ and $x\in N$.*

Im $\varphi=\varphi(M)=\{\varphi(m) \ / \ m\in M\}$ is a right N-subgroup of M' and Ker $\varphi= =\{m_0|m_0\in M, \ \varphi(m_0)=0\}$ is a normal right N-subgroup of M, having the property:

(1.9) $$(m_0+m)\cdot x-m\cdot x\in \operatorname{Ker}\varphi, \text{ for all } m_0\in\operatorname{Ker}\varphi, \ m\in M, \ x\in N.$$

If $M=N$, Ker φ is a right ideal of N.

Definition 1.6. *A nonzero right N-subgroup (right ideal) M is called minimal, if there is no other nonzero right N-subgroup (right ideal) of N, different from M, which is properly included into M.*

Definition 1.7. *A l.i.n.r. N is said to satisfy the descending chain condition (the ascending chain condition) for right N-subgroups, if any descending (resp. ascending) chain of right N-subgroups $M_1\geqslant M_2\geqslant\cdots$ (resp. $M_1\leqslant M_2\leqslant\cdots$) terminates (i.e. there exists a positive integer p, such that $M_h=M_p$, whenever $h\geqslant p$).* This condition is equivalent to the minimal (resp. maximal) condition for right N-subgroups, i.e. every nonempty collection of right N-subgroups of N has a minimal (maximal) element. One can easily verify:

Proposition 1.8. Ann$_r$ $(T)=\{x|x\in N, \ t\cdot x=t\cdot 0, \ \forall t\in T\}$, *the right annihilator of T, with T a nonempty subset of a right N-group M, is a right ideal of N.*

Definition 1.9. *A l.i.n.r. N is called semi-simple, if the following two conditions are satisfied:*

(1.10) *the descending chain condition for right N-subgroups;*

(1.11) *each minimal nonzero right N-subgroup M of N contains a left distributive element d_M, such that $d_M\cdot M\neq\{0\}$.*

The following statements were proved in [5], Corollaries 2.1 and 2.2, Lemma 2.3, Proposition 2.2 for a semi-simple l.i.n.r. N:

Proposition 1.10. (i) *N can be written as a finite direct sum of minimal nonzero right N-subgroups (minimal nonzero right ideals);*

(ii) *There exists a left identity in N;*

(iii) *Each minimal nonzero right ideal of N is a minimal nonzero right N-subgroup of N;*

(iv) *Each right ideal J has a left identity and J is generated by a left distributive idempotent ($J=e\cdot N$).*

Proposition 1.11. ([5], Proposition 3.1). *A l.i.n.r. N is semi-simple, if and only if it satisfies (1.10) and $S=\cap\{\operatorname{Ann}_r(D(M))|M - a$ minimal nonzero right N-subgroup of $N\}$, where $D(M)=\{d|d\in M, \ d\cdot 0=0\}$, is the ideal $\{0\}$.*

Definition 1.12. *A right N-group M is called cyclic, if there exists a left distributive element $d\in M$, such that $M=d\cdot N$.*

We have shown ([5], Lemma 2.1) that in a semi-simple l.i.n.r. each minimal nonzero right N-subgroup M is cyclic and generated by a left distributive idempotent, which is a left identity of M.

Definition 1.13. *An exact sequence of right N-groups $0\to M\xrightarrow{f}M'$ (i.e. a sequence of N-groups and N-homomorphisms such that Ker $f=\{0\}$) splits, if there exists a N-homomorphism $g: M'\to M$, called the splitting N-homomorphism, such that $gof=1_M$.*

Lemma 1.14. *If the exact sequence $0\to M\xrightarrow{f}M'$ splits, then there exists a normal right N-subgroup R' of M', such that $M'=f(M)+R'$ and $f(M)\cap \cap R'=\{0\}$.*

Proof. Indeed, take $R'=\operatorname{Ker} g$, where $g: M'\to M$ is the splitting N-homomorphism, hence $gof=1_M$. We verify easily that $x'=fog(x')+ +(-fog(x')+x')$, where $fog(x')=f(g(x'))\in f(M)$ and $-fog(x')+x'\in$ Ker g. Take $z\in f(M)\cap R'$, hence $z=f(x)\in R'$, for an element $x\in M$. But $g(z)= =0=g(f(x))=(gof)$ $(x)=1_M(x)=x$, therefore $z=f(0)=0$.

Proposition 1.15. *The right ideals of a semi-simple l.i.n.r. N satisfy the ascending chain condition.*

Proof. Let $A_1\leqslant A_2\leqslant\cdots$ be an ascending chain of right ideals of N. Let $e_i\in A_i$ be a left distributive idempotent generating A_i, $A_i=e_i\cdot A_i= =e_i\cdot N$, $i=1, 2,\cdots$ Denote $A=\bigcup\limits_i A_i$. We can easily verify that A is a right ideal. Take $A=A_i\oplus A_i'$, where $A_i'=\operatorname{Ker}\varphi_{e_i}\cap A$, with $\varphi_{e_i}: N\to N$ the N-homomorphism defined by $\varphi_{e_i}(x)=e_i\cdot x, x\in N$. But $A_i'=(A_{i+1}\cap\operatorname{Ker}\varphi_{e_i})\oplus(A_{i+2}\cap \cap\operatorname{Ker}\varphi_{e_{i+1}})\oplus\cdots$ (a direct sum of subgroups), because the only element contained simultaneously in two right ideals $A_{j+1}\cap\operatorname{Ker}\varphi_{e_j}$, $j=i, \ i+1,\cdots$ is 0 and we have: $A_{i+1}=A_i\oplus(A_{i+1}\cap\operatorname{Ker}\varphi_{e_i})$, $A_{i+2}=A_{i+1}\oplus(A_{i+2}\cap\operatorname{Ker}\varphi_{e_{i+1}})$, etc. We obtain a descending chain of right ideals: $A_1'\geqslant A_2'\geqslant\cdots$, which terminates finitely (N is semi-simple), hence there exists n, such that $A_{n+k}'=A_n'=\{0\}$, for any $k\geqslant 1$. Therefore $A_{n+k}=A_n, k\geqslant 1$, and the ascending chain is finite.

Lemma 1.16. *Let N be a l.i.n.r. If $N=A+B$, $A\cap B=\{0\}$, A is a right N-subgroup of N, B is a right of N and N has a left identity 1, then A is cyclic and idempotently generated.*

Proof. Let $1=e+b$, $e\in A$, $b\in B$ be the decomposition of the left identity of N as sum of two elements in A and B. For any $a\in A$, $a-e\cdot a=1\cdot a-e\cdot a=(e+b)\cdot a-e\cdot a\in A\cap B$, therefore $a-e\cdot a=0$ and $e\cdot a=a$. We have: $e\cdot 0=0$, $e\cdot e=e$, $A=e\cdot A\leqslant e\cdot N\leqslant A$, hence $A=e\cdot A=e\cdot N$.

2. Characterizations of semi-simplicity. Now we are able to prove the main theorem of the paper.

Theorem 2.1. *Let N be a l.i.n.r. with a left identity, 1. The following statements are equivalent:*

(i) *$N=\bigoplus\limits_{i=1}^{s} A_i$, with A_i, $i=1,2,\cdots,s$, minimal nonzero right ideals of N;*

(ii) *If M is a cyclic right N-group, then for any right N-subgroup M_1 of M, there exists a normal right N-subgroup M_2 of M, such that $M=M_1+M_2$, $M_1\cap M_2=\{0\}$ and the relation:*

$$(^{*})\qquad (m_2+m)\cdot x-m\cdot x\in M_2,$$

holds for all $m_2\in M_2$, $m\in M$, $x\in N$;

(iii) *For any right N-subgroup A_1 of N, there exists a right ideal A_2 of N such that $N=A_1+A_2$, $A_1\cap A_2=\{0\}$;*

(iv) *Any exact sequence $0\to P_1\xrightarrow{f}P_2$, with P_1 and P_2 two right N-subgroups of N, splits;*

(v) *If P_1, P_2 are two right N-subgroups of N with $P_1\leqslant P_2$, then any N-homomorphism $h: P_1\to M$, where M is a right N-group, can be extended to a N-homomorphism $\overline{h}: P_2\to M$;*

(vi) *N is semi-simple in the sense of Definition 1.10;*

(vii) *N satisfies the descending chain condition for right ideals and $S=\{0\}$, with S given in Proposition 1.11.*

Proof. (i)⇒(ii): Let M be generated by m, i.e. $M=m.N$, with $m\in D(M)$. We have, by (i), that $N=\bigoplus\limits_{i=1}^{s}A_i$, where A_i, $i=1,2,\dots,s$, are minimal nonzero right ideals of N and 1 is the left identity of N. Therefore $M=\sum\limits_{i=1}^{s} m.A_i$, because any $a\in N$ is uniquely expressed under the form $a=\sum\limits_{i=1}^{s} a_i$, $a_i\in A_i$, $i=1,2,\dots,s$ and $m.a=\sum\limits_{i=1}^{s} m.a_i$, with $m.a_i\in m.A_i$, $i=1,2,\dots,s$. But $m.A_i$, $i=1,2,\dots,s$, are normal right N-subgroups of M, as we can easily verify. Indeed, any $m_1\in M$ is of the form $m_1=m.x_1$, with $x_1\in N$, and we have: $m.a_i-m.a_i'=m.(a_i-a_i')\in m.A_i$, $m_1+m.a_i-m_1=m.x_1+m.a_i-m.x_1=m.(x_1+a_i-x_1)\in m.A_i$, $(m.a_i).x=m.(a_i.x)\in m.A_i$, for all $m_1\in M$, $m.a_i\in m.A_i$, $x\in N$ and $i=1,2,\dots,s$. Now for a nonzero right N-subgroup M_1 of M, consider the collection $\mathcal{P}=\{P \mid P$ is a normal right N-subgroup of M, such that $(p+m).x-m.x\in P$, for all $p\in P$, $m\in M$, $x\in N$ and $M_1\cap P=\{0\}\}$. Because $\{0\}\in\mathcal{P}$, $\mathcal{P}$ is nonempty. $\mathcal{P}$ is partially ordered by set-inclusion. Every totally ordered subset of $\mathcal{P}$ has an upper bound,

hence, by applying Z o r n's lemma, we obtain the existence of a maximal element of $\mathcal{P}$, let us denote it by M_2. We claim that $M=M_1+M_2$, $M_1\cap M_2=\{0\}$. The last assertion is true by the definition of $\mathcal{P}$. For the first one, assume that there exists a $k\in\{1,2,\dots,s\}$, such that $m.A_k\not\subset M_1+M_2$ and take $\theta: A_k\to m.A_k$, $\theta(a_k)=m.a_k$, for any $a_k\in A_k$. θ is a surjective N-homomorphism, since m is left distributive. As Ker θ is a right ideal of N, A_k is a minimal right ideal and $\operatorname{Im}\theta=m.A_k$ is different from $\{0\}$, we obtain Ker $\theta=\{0\}$. Therefore θ is a N-isomorphism and $m.A_k$ is a minimal nonzero right N-subgroup of M, since the N-isomorphism θ induces a bijection between the set of ideals contained in A_k and the set of normal right N-subgroups of $m.A_k$ having the property (*). Then $m.A_k\cap(M_1+M_2)=\{0\}$, hence $m.A_k\cap M_1=\{0\}$. We have also $M_1\cap(m.A_k+M_2)=\{0\}$ and $M_2'=m.A_k+M_2$ is a normal right N-subgroup of M with the property (*). Indeed, M_2' is a normal subgroup of M as a sum of two normal subgroups of M. It is a right N-subgroup, since $m_2'.x=(m.a_k+m_2).x=(m.a_k+m.x_2).x=m.[(a_k+x_2).x-x_2.x]+m.(x_2.x)=m.a_k'+(m.x_2).x=m.a_k'+m_2.x\in M_2'$, for any $m_2'\in M_2'$ and $x\in N$. This happens because A_k is a right ideal, hence $a_k'=(a_k+x_2).x-x_2.x\in A_k$, $M_2\leqslant M$ and $M=m.N$, hence $m_2'=m.a_k+m_2=m.a_k+m.x_2=m(a_k+x_2)$, with $x_2\in N$. Moreover $(m_2'+m).x-m.x=[(m.a_k+(m_2+m)).x-(m_2+m).x]+[(m_2+m).x-m.x]\in M_2'$ since the two elements contained into bracketts belong to $m.A_k$ and M_2. If $m_1=m.a_k+m_2\in M_1\cap(m.A_k+M_2)$, then $m.a_k=m_1-m_2\in(M_1+M_2)\cap m.A_k$, hence $m.a_k=0=m_1-m_2$, $m_1=m_2=0$. Therefore $M_1\cap(m.A_k+M_2)=\{0\}$ and M_2' contradicts the maximality of M_2 in $\mathcal{P}$. Then $M_1+M_2\geqslant m.A_k$, for any $k=1,2,\dots,s$ and $M=M_1+M_2$.

(ii) ⇒(iii): Since N has a left identity, 1, we have $N=1.N$, hence N is a cyclic N-group. Note that a right N-subgroup of N having the property (*) is also a right ideal of N. Now apply (ii) to the cyclic right N-subgroup N.

(iii) ⇒ (iv): Taking $A_1=f(P_1)$ in (iii), we find a right ideal A_2 of N, such that $A_1\cap A_2=\{0\}$ and $N=A_1+A_2$, a semidirect sum of subgroups. Let $\pi: N\to A_1$ be the projection of N onto A_1, given by $\pi(a_1+a_2)=a_1$, for any $a_1=f(p_1)\in A_1$, $a_2\in A_2$. We verify that π is a N-homomorphism: $\pi((a_1+a_2)+(a_1'+a_2'))=\pi((a_1+a_1'+((-a_1'+a_2+a_1')+a_2')))=a_1+a_1'=\pi(a_1+a_2)+\pi(a_1'+a_2')$ and $\pi((a_1+a_2).x)=\pi(((a_1+a_2).x-a_1.x)+a_1.x)=\pi(a_1.x+a_2'')=a_1.x=\pi(a_1+a_2).x$, for any a_1+a_2, $a_1'+a_2'\in N$, $x\in N$. We have applied here the fact that A_2 is a right ideal and the sum is semi-direct. Restricting π to P_2, $\pi/P_2=\pi_2$, we obtain a N-homomorphism $g=f^{-1}\circ\pi_2: P_2\to P_1$, for which $g\circ f(p_1)=(f^{-1}\circ\pi_2)(f(p_1))=f^{-1}(\pi_2(f(p_1)))=f^{-1}(f(p_1))=p_1=1_{P_1}(p_1)$, for any $p_1\in P_1$ and the given exact sequence splits.

(iv)⇒(v): Using the exact sequence $0\to P_1\xrightarrow{i}P_2$ of right N-subgroups, where i is the inclusion of P_1 into P_2, we obtain, in view of (iv), the splitting N-homomorphism $\theta: P_2\to P_1$, with $\theta\circ i=1_{P_1}$. Take the N-homomorphism $\overline{h}=h\circ\theta: P_2\to M$, for a N-homomorphism $h: P_1\to M$. We have $\overline{h}(p_1)=(h\circ\theta)(p_1)=(h\circ\theta)(i(p_1))=h((\theta\circ i)(p_1))=h(p_1)$, for any $p_1\in P_1$, hence $\overline{h}/P_1=h$.

(v) ⇒ (iv): Let $0\to P_1\xrightarrow{f}P_2$ be an exact sequence of right N-groups. Take $f(P_1)=A_1\leqslant P_2$. Then $i: A_1\to A_1$ (the identity mapping) can be „lifted" to a N-homomorphism $h : P_2 \to A_1$. Obviously $g = f^{-1} \circ h : P_2 \to P_1$ is a N-homomorphism and $g \circ f = 1_{P_1}$, hence g is a splitting N-homomorphism.

(iv)$\Rightarrow$(vi) : Let $N \geqslant L_1 \geqslant L_2 \geqslant \ldots$ be a descending chain of right N-subgroups of N. Consider the exact sequences : $0 \to L_1 \xrightarrow{i_1} N$, $0 \to L_2 \xrightarrow{i_2} L_1, \ldots$, $0 \to L_{j+1} \xrightarrow{i_{j+1}} L_j, \ldots$ with i_j — the natural injection. In view of (iv), these sequences split, and, by using Lemma 1.14, we obtain : $N = L_1 + K_1$, $L_1 \cap K_1 = \{0\}$, $L_j = L_{j+1} + K_{j+1}$, $L_{j+1} \cap K_{j+1} = \{0\}$, with K_1 and K_{j+1}, $j \geqslant 1$, right ideals of N. The subset of N given by :

$K = \sum\limits_{j \geqslant 1}^{\cdot} K_j = \{ \sum\limits_{j \geqslant 1}^{\cdot} k_j \mid k_j \in K_j, j \geqslant 1$, all excepting a finite number of them are zero$\}$, is a right ideal of N. To prove it, we show that K is a sum of a finite number of K_j. Indeed, we have $K_j + K_i = K_i + K_j$, for $i < j$, since K_j is a right N-subgroup of $L_{j-1} < L_{i-1}$ $(j-1 > i-1)$ and K_i is a normal right N-subgroup in L_{i-1}. If we take $i_1 < i_2 < \ldots < i_t$ and $K' = \sum\limits_{h=1}^{t} K_{i_h}$, we obtain a right N-subgroup. Indeed, $(k_{i_1} + \ldots + k_{i_t}) - (k'_{i_1} + \ldots + k'_{i_t}) = k''_{i_1} + \ldots + k''_{i_t}$, with $k_{i_h}, k'_{i_h}, k''_{i_h} \in K_{i_h}$, $h = 1, 2, \ldots, t$, because of the associativity and the commutativity of sums of K_j. Then $(k_{i_1} + \ldots + k_{i_t}) \cdot x = ((k_{i_1} + \ldots + k_{i_t}) \cdot x - (k_{i_2} + \ldots + k_{i_t}) \cdot x) + ((k_{i_2} + \ldots + k_{i_t}) \cdot x - (k_{i_3} + \ldots + k_{i_t}) \cdot x) + \ldots + ((k_{i_{t-1}} + k_{i_t}) \cdot x - k_{i_t} \cdot x) + k_{i_t} \cdot x \in K'$, for all elements of K' and N, since each K_{i_h} is a right ideal and consequently a right N-subgroup of N. If $K \neq N$, then, by considering the exact sequence $0 \to K' \xrightarrow{i_{t+1}} N$ and in view of (iv), there exists a right ideal R, such that $N = K + R$, $K \cap R = \{0\}$. But N has a left identity, 1. Then $1 = k + r$, with $k \in K$ (assume $k = k_{i_1} + \ldots + k_{i_t}$) and $r \in R$. Take $0 \neq k_{i_{t+1}} \in K_{i_{t+1}}$. We have $i_1 < i_2 < \ldots < i_t < i_t + 1$, $k_{i_{t+1}} = 1 \cdot k_{i_{t+1}} = ((k_{i_1} + \ldots + k_{i_t} + r) \cdot k_{i_{t+1}} - (k_{i_1} + \ldots + k_{i_t}) \cdot k_{i_{t+1}}) + (k_{i_1} + \ldots + k_{i_t}) \cdot k_{i_{t+1}}$. The element $(k_{i_1} + \ldots + k_{i_t} + r) \cdot k_{i_{t+1}} - (k_{i_1} + \ldots + k_{i_t}) \cdot k_{i_{t+1}}$ belongs to R, therefore $k_{i_{t+1}} - (k_{i_1} + \ldots + k_{i_t}) \cdot k_{i_{t+1}} \in K \cap R = \{0\}$. We have $k_{i_{t+1}} = (k_{i_1} + \ldots + k_{i_t}) \cdot k_{i_{t+1}} \in \sum\limits_{h=1}^{t} K_{i_h}$, hence $k_{i_{t+1}} = k'_{i_1} + \ldots + k'_{i_t}$ or $k'_{i_1} = k_{i_{t+1}} - k'_{i_t} - \ldots - k'_{i_2} \in K_{i_1} \cap L_{i_2 - 1}$. From the relations $i_1 < i_2$ and $i_1 \leqslant i_2 - 1$, it follows that $L_{i_2-1} \leqslant L_{i_1}$, therefore $k'_{i_1} \in L_{i_1} \cap K_{i_1} = \{0\}$ and $k'_{i_1} = 0$. Similarly, $k'_{i_2} = \ldots = k'_{i_t} = 0$, hence $k_{i_{t+1}} = 0$, a contradiction. We obtained that $K_{i_{t+1}} = \{0\}$, $K = K'$, $L_{i_{t+1}} = L_{i_t}$ and the descending chain is finite (we prove that $L_p = L_{i_t}$ for any $p \geqslant i_t$, by the same argument). Now let M be a minimal nonzero right N-subgroup of N. By using Lemma 1.14, (iv) and the Lemma 1.16, we obtain the existence of a right ideal R, such that $N = M + R$, $M \cap R = \{0\}$, and $M = e \cdot M = e \cdot N$, with e a left distributive idempotent.

The statements : (vi) $\Rightarrow$ (vii) and (vii) $\Rightarrow$ (i) were proved in [5] (we recalled these results in Section 1, Propositions 1.11, 1.10, (i)). The theorem is proved.

Proposition 2.2. *Let N be a semi-simple l.i.n.r. If G is a right N-group without any proper right N-subgroup and having an element $g \in G$ such that $g \cdot 0_N = 0_G$, $g \cdot N \neq \{0\}$, then G is N-isomorphic with a minimal nonzero right N-subgroup of N.*

Proof. Let $N = \bigoplus\limits_{i=1}^{t} A_i$ be a complete decomposition of N as a sum of minimal nonzero right N-subgroups. Because $g \cdot N \neq \{0\}$, there exists a N-subgroup of N, A_i, such that $g \cdot A_i \neq \{0\}$. Define the mapping $\tau : A_i \to G$, by $\tau(a_i) = g \cdot a_i$, for all $a_i \in A_i$. Obviously τ is a N-homomorphism, $\mathrm{Im}\, \tau = g \cdot A_i$ is a right N-subgroup of G, different from $\{0\}$, hence $\mathrm{Im}\, \tau = G$. $\mathrm{Ker}\, \tau$ is a right ideal of N, hence a right N-subgroup included into A_i, different from A_i, and A_i is minimal. Therefore $\mathrm{Ker}\, \tau = \{0\}$ and τ is the required N-isomorphism.

This proposition shows — as well as the examples in [5] — that the idea of semi-simplicity applied to left infra-near rings is more general than the same idea in the case of the left near-rings. Its consequences are weaker and the proofs of the theorems ask — in the case of the left infra-near rings — some stronger conditions, as the existence of a left distributive element in each minimal nonzero right N-subgroup of the left infra-near ring. But, of course, it is interesting to see how far the theory goes.

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