Théorème 3. *Si*

$$\alpha > 0 \text{ et } \int_s^t \mu(G(u))\, du \leqslant -\alpha(t-s), \quad \forall t \geqslant s,$$

alors les hypothèses 3° des théorèmes 1 et 2 sont vraies.

La démonstration de ce théorème résulte aisément, puisqu'on a :

$$|B(t, s)| \leqslant e^{-\alpha(t-s)}, \quad \forall t \geqslant s.$$

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ON ABSTRACT MEASURE DELAY DIFFERENTIAL EQUATIONS

BY

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1. **Introduction.** Recently, Haimovici [4] and R.R. Sharma [6] have considered differential equations in which an ordinary derivative is replaced by a derivative of a set function, and as a result, the solutions happen to be signed measures. Such differential equations may be called abstract measure differential equations (*AMDE*).

In the present paper we deal with an *AMDE* involving a delay and study some basic problems, such as existence and uniqueness of solutions, extension of solutions, maximal and minimal solutions. The main tool is the application of fixed point theorems due to Banach, Schauder and Tarski. We also obtain a differential inequality and a local stability result. Examples are given to illustrate some of the ideas involved in the theory.

2. **Preliminaries.** Let X be a real Banach space with any convenient norm $\|.\|$. For any two points x, y in X, the segment $\overline{xy}$ is defined by

$$\overline{xy} = \{z \in X : z = x + r(y-x),\ 0 \leqslant r < 1\}.$$

Let x_0 and y_0 be two fixed points in X such that $x_0 \neq 0$, $y_0 \neq 0$, and $\overline{Oy_0} \subset \overline{Ox_0}$ where 0 is the zero vector in X. Let z be a point of X, such that $\overline{Ox_0} \subset \overline{Oz}$. For this z and $x \in \overline{y_0z}$, define the sets $\overline{S}_x$ and S_x as follows :

$$\overline{S}_x = \{rx : -\infty < r \leqslant 1\},\ S_x = \{rx : -\infty < r < 1\}.$$

For $x_1, x_2 \in \overline{y_0z}$, we write $x_1 \leqslant x_2$ (or $x_2 \geqslant x_1$) if $\overline{y_0x_1} \subset \overline{y_0x_2}$. Let the positive number $\|x_0 - y_0\|$ be denoted by w. For each $x \in \overline{x_0z}$, $z \geqslant x_0$, let x_W denote, that element of $\overline{y_0z}$ for which the relations $x_W \leqslant x$ and $\|x - x_w\| = w$ hold. Note that x_w and wx are not the same unless $w = 0$ and $x = 0$.

Let M_0 be the smallest ∂-algebra on $\overline{S}_{x_0}$ containing $\{x_0\}$ and the sets $\overline{S}_x$, $x \in \overline{y_0x_0}$. For any $z \geqslant x_0$, let M_z denote the smallest ∂-algebra defined on S_z, containing M_0 and the sets $\overline{S}_x$, $x \in \overline{x_0z}$. For any set S in X and a ∂-algebra M on S, let the symbol $ca(S, M)$ denote the space of all finite signed measures. It is known that $ca(S, M)$ is a Banach space, with respect to the norm $\|.\|$ given by

$$\|p\| = |p|(S),\ p \in ca(S, M),$$

where $|p|$ is the total variation measure of p [3, p. 240]. For any positive number H, define sets B_H and C_H as follows:

$$B_H=\{t\in R:\ -H<t<H\},\ C_H=\{u\in ca(\bar{S}_{x_0},\ M_0):\ \|u\|<H\}.$$

In what follows, μ denotes a ∂-finite positive measure and q a given element of C_H.

Consider now the following $AMDE$ involving a delay w,

$$\begin{cases}\dfrac{dp}{d\mu}=f(x,p(\bar{S}_x),\ p(\bar{S}_{x_w})),\\ p(E)=q(E),\ E\in M_0\end{cases}\tag{2.1}$$

where $\dfrac{dp}{d\mu}$ is the Radon-Nikodym derivative of p with respect to μ $f(x,s,t)$ is a real function defined on $S_z\times B_H\times B_H$ such that for $p\in ca(S_z,M_z)$, $f(x,p(\bar{S}_x),\ p(\bar{S}_{x_w}))$ is μ-integrable and $q\in C_H$.

Difinition. *For an initial measure $q\in C_H$, a real measure $p\in ca(S_z,M_z)$ (for some $z\geqslant x_0$) is said to be a solution of* (2.1) *if*

(i) $p(E)=q(E),\ E\in M_0$;

(ii) $p(E)\in B_H,\ E\in M_z$;

(iii) $p\ll\mu$ on $\overline{x_0z}$;

(iv) *p satisfies* (2.1) *a.e.* $[\mu]$, *on* $\overline{x_0z}$.

Remark. The conditions (iii) and (iv) together are equivalent to

$$p(E)=\int_E f(x,p(\bar{S}_x),\ p(\bar{S}_{x_w}))\,d\mu,\ E\in M_z,\ E\subset\overline{x_0z}.$$

Whenever a solution p exists for some $z\geqslant x_0$, we say that p exists on $\overline{x_0z}$ and denote it by $p(\bar{S}_{x_0},\ q)$.

3. **Existence and extension of a solution.** We need the following assumptions:

(A_1) For any $z\geqslant x_0$ the ∂-algebra M_z on S_z is compact, with respect to the topology generated by

$$d(E_1,E_2)=\mu(E_1-E_2)+\mu(E_2-E_1);$$

(A_2) $\mu(\{x_0\})=0$;

(A_3) $f(x,s,t)$ is continuous in s and t for each $x\in S_z$;

(A_4) There exists a μ-integrable, non-negative function $W(x)$ satisfying the condition

$$|f(x,s,t)|\leqslant W(x),$$

uniformly in s and t, for $x\in S_z,\ s\in B_H,\ t\in B_H$;

(A_5) q is continuous on M with respect to the metric d defined in (A_1).

Theorem 1. *Let $q\in C_H$ be a given initial measure and let the assumptions* (A_1) *to* (A_5) *hold. Then there exists a point $x_1\geqslant x_0$ for which there is a solution $p(\bar{S}_{x_0},\ q)$ of* (2.1) *existing on $\overline{x_0x_1}$.*

Proof. Let $\{r_n\}$ be a decreasing sequence of real numbers such that $r_n\to1$ and $S_{r_1x_0}\supset S_{r_2x_0}\supset\ldots\supset S_{x_0}$. Then the set $(S_{r_nx_0}-\bar{S}_{x_0})$ is empty as $n\to\infty$ and hence

$$\lim_{n\to\infty}(S_{r_nx_0}-\bar{S}_{x_0})=0.$$

We can therefore choose a number r and the corresponding point $x_1=rx_0$, such that $\bar{S}_{x_0}\supset\bar{S}_{x_1}$ and

$$\int_{\overline{x_0x_1}} W(x)\,d\mu<H-\|q\|.\tag{3.1}$$

Consider the subset D of $ca(S_{x_1},\ M_{x_1})$ defined by

$$D=\{p:p(E)=q(E)\text{ if }E\in M_0\text{ and }\|p\|\leqslant K=q+\int_{\overline{x_0x_1}}W(x)\,d\mu\}$$

It can be observed that D is a closed, convex and bounded subset of $ca(S_{x_1},\ M_{x_1})$. Define an operator T on D as follows:

$$Tp(E)=q(E)\text{if }E\in M_0,$$

$$Tp(E)=\int_E f(x,p(\bar{S}_x),\ p(\bar{S}_{x_w}))\,d\mu,\ \text{if }E\in M_{x_1}\text{ and }E\subset\overline{x_0x_1}.\tag{3.2}$$

From the definitions of D and T, the assumptions of the theorem, the relation (3.1) and an application of Ascoli's theorem, it can be shown that T is a completely continuous operator on D. Hence an application of Schauder's fixed point theorem shows that there is a solution of (2.1) existing on $\overline{x_0x_1}$.

Remarks 1) The sets $\bar{S}_x$ and S_x involved in the above theorem may also be replaced by other suitable sets so that there should be a one to one correspondence between these sets and the points $x\in\overline{y_0z}$;

2) The uniqueness of the solution of (2.1) can also be established by using assumptions (A_2), (A_4) and Lipschitz condition on f of the following form:

$$|f(x,s_1,t_1)-f(x,s_2,t_2)|\leqslant L(|s_1-s_2|+|t_1-t_2|),$$

where L is a positive constant, $x\in S_z$ and $s_1,t_1,s_2,t_2\in B_H$.

3) Theorem 1 is a local existence theorem. If the function f is defined on $S_z\times R\times R$ where S_z is any given set containing $\bar{S}_{x_0}$ and if all the conditions of Theorem 1 are satisfied with (A_4) replaced by (A_4') where (A_4') is as follows:

(A_4') There exists a μ-integrable function $W(x)$ defined on S_z satisfying $|f(x,s.t)|\leqslant W(x)$ uniformly in s and t and

$$\int_{\overline{x_0z}} W(x)d\mu=K_0,$$

then there exists a solution $p(\bar{S}_{x_0},\ q)$ of (2.1) existing on $\overline{x_0z}$.

We now state and prove a theorem on extension of a solution of (2.1).

Theorem 2. *Let a solution $p(\bar{S}_{x_0}, q)$ of (2.1) exist on $\overline{x_0x_1}$ under the assumptions of Theorem 1. Let for any z satisfying $x_1 \leq z$, $\mu(\{x\})=0$ for all $x \in \overline{x_1z}$, then the solution $p(\bar{S}_{x_0}, q)$ can be extended to the whole set $\overline{x_0z}$.*

Proof. Since $\mu(\{x_1\})=0$, we may consider the restriction of $p(\bar{S}_{x_0}, q)$ on the segment x_{1w}, x_1 as the initial measure. Let it be q_1. By Theorem 1 there exists a point x_2 and a solution $p_1(\bar{S}_{x_1}, q_1)$ where p_1 is defined on the smallest ∂ algebra M_{x_2} containing M_{x_1}, and the sets $\bar{S}_x$, $x \in \overline{x_1x_2}$. Define now p on M_{x_2} as follows:

$$p(E)=p(\bar{S}_{x_0}, q)(E), \text{ if } E \in M_{x_1},$$

$$p(E)=p_1(\bar{S}_{x_1}, q_1)(E), \text{ if } E \in M_{x_2} \text{ and } E \subset \overline{x_1x_2}.$$

It can be observed that p is the desired extension of $p(\bar{S}_{x_0}, q)$ on $\overline{x_0x_2}$ satisfying (2.1).

4. Special case. In this section we shall show that an certain situation the AMDE (2.1) can be reduced to an ordinary differential—difference equation

$$\text{(4.1)} \qquad \begin{aligned} &\frac{dy}{dx}=f(x, y(x), y(x-w)), \text{ if } x \in [x_0, \infty) \\ &y(x)=g(x), \text{ if } x \in [x_0-w, x_0]. \end{aligned}$$

Assume that

(i) $X=R$;

(ii) $\mu=m$ (the Lebesgue measure on R);

(iii) $S_x=(-\infty, x)$, and $\bar{S}_x=(-\infty, x]$, $x \in R$;

(iv) q is a given initial real Borel measure defined on $\bar{S}_{x_0}$.

Under these assumptions, equation (2.1) takes the form

$$\text{(4.2)} \qquad \begin{aligned} &\frac{dp}{dm}=f(x, p((-\infty, x]), p((-\infty, x-w])) \\ &p(E)=q(E), \ E \in M_0. \end{aligned}$$

Theorem 3. *Let the assumptions (i), (ii), (iii) and (iv) hold. Then corresponding to each solution $p(\bar{S}_{x_0}, q)$ of (4.2) existing on $[x_0, x_1)$, there exists a solution $y(x_0, g)$ of (4.1) on $[x_0, x_1)$ whenever $q(\{x\})=0$ for each $x \in [x_0-w, x_0]$. Conversely, if g is a continuous function of bounded variation on $[x_0-w, x_0]$, then to every solution $y(x_0, g)$ of (4.1) there corresponds a solution $p(\bar{S}_{x_0}, q)$ of (4.2) existing on $[x_0, x_1)$ with a suitable initial measure q.*

Proof. Let $p(\bar{S}_{x_0}, q)=p$ be a solution of (4.2) existing on $[x_0, x_1)$. Define a Borel measure p_1 on R as follows:

$$p_1((-\infty, x))=\begin{cases} 0, \text{ for } x \leq x_0-w, \\ p((-\infty, x])-p((-\infty, x_0-w]), \text{ for } x \in (x_0-w, x_1), \\ p((-\infty, x_1)), \text{ for } x \geq x_1. \end{cases}$$

and $p_1(E)=p(E)$, for $E \subset [x_0-w, x_1)$.

Further, define functions $y_1(x)$, $y(x)$ and $g(x)$ as follows:

$$\begin{aligned} &y_1(x)=p_1((-\infty, x)), \text{ for } x \in R, \\ &y(x)=y_1(x)+p((-\infty, x_0-w]), \text{ for } x \in [x_0-w, x_1), \\ &g(x)=y(x), \text{ for } x \in [x_0-w, x_0]. \end{aligned}$$

These definitions and the assumptions on q imply that $p_1(\{x\})=0$, for $x \in [x_0-w, x_0]$ and hence g is continuous on $[x_0-w, x_0]$ (see Theorem 8.14 on page 163 of [5]). Now from the above definitions we conclude that

$$\text{(4.3)} \qquad y(x)=p((-\infty, x])=p(\bar{S}_x), \text{ for } x \in [x_0-w, x_1).$$

p being a solution of (4.2), we have $p \ll m$ on $[x_0, x_1)$. Hence $y(x)$ is an absolutely continuous function on $[x_0, x_1)$. Finally, from (4.2) and (4.3), we get

$$p([x_0, x])=\int_{[x_0, x]} \left(\frac{dp}{dm}\right) dm, \ x \in [x_0, x_1),$$

$$\text{that is } p((-\infty, x])-p((-\infty, x_0]=\int_{[x_0, x]} f(x, p(\bar{S}_x), p(\bar{S}_{x-w}))\, dm.$$

This further implies that

$$y'(x)=f(x, y(x), y(x-w)) \text{ a.e. } [m], \text{ for } \in [x_0, x_1).$$

Hence $y(x)$ is a solution of (4.1) on $[x_0, x_1)$.

The converse part of the theorem can also be proved on similar lines. Here we assumed that g is a function of bounded variation on $[x_0-w, x_0]$ in order to construct an initial measure q on $\bar{S}_{x_0}$. The details are omitted.

Below, we give two examples which are illustrative.

Examples 1. Let $X=R$, $S_x=(-\infty, x)$, $x_0=0$ and $w=2$. Define an initial measure q on M_0 as follows:

$$q(E)=\begin{cases} -n, \text{ if } n \in E \cap \{-1, -2\} \\ 0, \text{ otherwise.} \end{cases}$$

Define a measure μ on R as follows:

$$\mu(E)=\begin{cases} 2^{-n}, \text{ if } n \in E \cap \{1, 2, \ldots\} \\ 0, \text{ otherwise.} \end{cases}$$

Consider the AMDE

$$\frac{dp}{d\mu} = p(\bar{S}_{x-2}),\ x \in R$$

$$p(E) = q(E),\ E \in M_0,$$

which is equivalent to

$$p(E) = q(E),\ E \in M_0,$$

(4.4)
$$p(E) = \int_E p(\bar{S}_{x-2})\, d\mu,\ E \subset (0, \infty).$$

Since the function f under consideration satisfies the Lipschitz condition (4.4) has a unique solution, existing on $[0, \infty)$. In fact, we observe from the definitions of q, μ and (4.4) that $p(S_1) = 9/2$, $p(S_2) = 21/4$, $p(S_3) = 93/16$. It can be verified that the solution p of (4.4) is a finite positive measure concentrated at 0, 1, 2,... .

Example 2. Let $X = R$, μ the Lebesgue measure on R, $S_t = [0, t)$, $t \in (0, \infty)$ and $q(E) = \mu(E)$ for $E \subset [0, 1]$, and $w = 1$. Consider the *AMDE*

(4.5)
$$\frac{dp}{d\mu} = 6\, p(\bar{S}_{t-1}),\ p(E) = q(E),\ E \subset [0, 1].$$

It is clear that for any $t \in [0, 1]$, we have

(4.6)
$$p(\bar{S}_t) = p([0, t]) = q([0, t]) = \int_0^t 1\, ds = t.$$

For $t \in [1, 2]$, we have by (4.6)

(4.7)
$$p(\bar{S}_t) = q(\bar{S}_1) + \int_{[1, t]} 6p(S_{t-1})\, d\mu = 1 + 3(t-1)^2.$$

Again when $t \in [2, 3]$, (4.7) yields

$$p(\bar{S}_t) = p(\bar{S}_2) + \int_2^t [6 + 16(s - 2^2]\, ds = 4 + 6t^2 + 6(t-2)^3 - 12.$$

Thus the solution of (4.5) can be determined on $[1, \infty)$ stepwise.

5. **Extremal solutions and stability.** In this section we shall prove the existence of maximal and minimal solutions of (2.1) using lattice fixed point theorem due to T a r s k i [7], which not only establishes the existence of a solution, but also the existence of extremal solutions at the same time. In proving the results of this section, we require the following lemma.

Lemma. *Let $S_0 = ca(X, M)$ denote the set of all finite signed measures defined on the σ-algebra M. Then S_0 is a complete lattice with respect to the order relation $\leqslant$ defined by*

$$p_1 \leqslant p_2 \text{ if } p_1(E) \leqslant p_2(E),\ E \in M.$$

Proof. S_0 is a vector lattice ([8], p. 391—392). If $p_1, p_2 \in S_0$ and if $|p_1| \leqslant |p_2|$, then clearly $\|p_1\| \leqslant \|p_2\|$. This implies that S_0 is a Banach-lattice ([1], p. 246). Also if $p_1, p_2 \geqslant 0$, then $\|p_1 + p_2\| = \|p_1\| + \|p_2\|$. This shows that S_0 is an abstract L-space ([1], p. 254), which further, implies that S_0 is a complete lattice and the proof of the lemma is complete.

Definition. *A solution $p_M = p_M(\bar{S}_{x_0}, q)$ of (2.1) existing on $\overline{x_0 z}$ is said to be a maximal solution of (2.1) if for every solution $p = p(\bar{S}_{x_0}, q)$ of (2.1) existing on $\overline{x_0 z}$, we have*

$$p(E) \leqslant p_M(E),\ E \in M_z.$$

A minimal solution $p = p_m(\bar{S}_{x_0}, q)$ of (2.1) can be defined similarly. The following theorem establishes the existence of maximal and minimal solutions of (2.1).

Theorem 5. *Assume that*

(B_1) μ is a positive measure such that $\mu(\{x_0\}) = 0$;

(B_2) the function $f(x, s, t)$ defined on $S_z \times B_H \times B_H$ is nondecreasing in s and t for each $x \in S_z$;

(B_3) there exists a non-negative function $W(x)$ on S_z such that $|f(x, s, t)| \leqslant W(x)$, uniformly in s and t. Then there exists a unique maximal solution p_M and a unique minimal solution p_m of (2.1) on $\overline{x_0 x_1}$ for some $x_1 \geqslant x_0$.

Proof. Choose a point $x_1 \geqslant x_0$ and the subset D of $ca(\bar{S}_{x_1}, M_{x_1})$ as in Theorem 1. Then from the definition of D it is clear that D is bounded. Also, by the above lemma, $ca(\bar{S}_{x_1}, M_{x_1})$ is a complete lattice. Hence D is also a complete lattice. Define an operator T as in Theorem 1 (given by (3.2)). Let $p \in D$ and $E \in M_{x_1}$. Then there exist sets F and G in M_{x_1} such that $F \in M_0$, $G \subset \overline{x_0 x_1}$, $E = F \cup G$, and $F \cap G$ is a null set. It follows from the definition of T, that

$$Tp(E) = q(E) + \int_G f(x, p(\bar{S}_x), p(\bar{S}_{x_w}))\, d\mu,$$

which by virtue of the assumption (B_3) implies that

$$\|Tp\| \leqslant \|q\| + \int_{\overline{x_0 x_1}} W(x)\, d\mu = K,$$

K being the positive number occuring in the definition of D. This proves that T maps D into itself. Further, the assumption (B_2) implies that T is an isotone operator on D. Hence by an application of the lattice fixed point theorem due to T a r s k i [7], we conclude that the set $F_0 = \{p \in D : Tp = p\}$ is non-empty and is a complete lattice. From the definition of T, it follows that each member of F_0 is a solution of (2.1). Since F_0 is a complete lattice, we conclude that the maximal and the minimal solutions $p_M = \text{lub}\, F_0$ and $p_m = \text{glb}\, F_0$ exist on $\overline{x_0 x_1}$, and the proof of the theorem is complete.

We are now in a position to obtain a fundamental inequality of Gronwall-type. It is known that the inequalities of this type play a vital role in nonlinear analysis.

Theorem 6. *Let all the assumptions of Theorem 4 be satisfied. Suppose further that $\Phi \in D$ satisfies the inequalities*

$$\Phi(E) \leqslant q(E), \ E \in M_0$$

(5.1)
$$\frac{d\Phi}{d\mu} \leqslant f(x, \Phi(\bar{S}_x), \Phi(\bar{S}_{x_w})), \ x \in \overline{x_0 x_1}.$$

Then

$$\Phi(E) \leqslant p_M(E), \ E \in M_{x_1},$$

where $p_M = p_M(\bar{S}_{x_0}, q)$ is the maximal solution of (2.1) on $\overline{x_0 x_1}$.

Proof. Let $p_0 = \sup D$ and consider the lattice interval $[\Phi, p_0]$ which is obviously a complete lattice. Define an operator T on D as in Theorem 1. Then T is isotone on S and maps S into itself. We show that T maps $[\Phi, p_0]$ into itself. To do this, it is enough to show that $p \in D$, $\Phi \leqslant p$ implies that $\Phi \leqslant Tp$. Let $E \in M_{x_1}$, $E \subset \overline{x_0 x_1}$. Then, by using the inequality (5.1) and the assumption (B_2), we obtain

$$\Phi(E) \leqslant \int_E f(x, \Phi(\bar{S}_x), \Phi(\bar{S}_{x_w}))\, d\mu \leqslant \int_E f(x, p(\bar{S}_x), p(\bar{S}_{x_w}))\, d\mu = Tp(E).$$

An application of lattice fixed point theorem shows that the maximal solution $p_M(\bar{S}_{x_0}, q)$ of (2.1) lies in $[\Phi, p_0]$. Hence the uniqueness of p_M implies that $\Phi(E) \leqslant p_M(E)$, $E \in M_{x_1}$. The proof is complete.

Below, we prove a theorem which can be viewed as a local stability result. Assume that

(C_1) $$f(x, 0, 0) = 0 \text{ for all } x \in S_z, \ z \geqslant x_0;$$

(C_2) Given a positive number ε, there exists a number $\eta = \eta(\varepsilon) \geqslant 0$ such that

$$|f(x, s_1, t_1) - f(x, s_2, t_2)| \leqslant \varepsilon \max(|s_1 - s_2|, |t_1 - t_2|),$$

whenever $|s_i|$, $|t_i| \leqslant \eta$ and $(x, s_i, t_i) \in S_z \times R \times R$, for $i = 1, 2$.

Theorem 6. *Let the assumptions (C_1) and (C_2) hold. Then there exists a number $\varepsilon_0 > 0$ such that for any ε in $(0, \varepsilon_0)$ and a fixed $b \in (0, 1)$, there is a unique solution $p(\bar{S}_{\cdot}, q)$ of (2.1) existing on $\overline{x_0 z}$, satisfying $\| p \| \leqslant \varepsilon$, whenever $\| q \| \leqslant b\varepsilon$.*

Proof. Corresponding to the positive number $(1-b)/L_0$, where $L_0 = \mu(\overline{x_0 z})$, there exists by ($C_2$) a number $\varepsilon_0 > 0$ such that

(5.2) $$|f(x, s_1, t_1) - f(x, s_2, t_2)| \leqslant \frac{1-b}{L_0} \max(|s_1 - s_2|, |t_1 - t_2|)$$

whenever $|s_i|$, $|t_i| \leqslant \varepsilon_0$, $i = 1, 2$.

Defining $\bar{S}(\varepsilon) = \{p : p \in ca(S_z, M_z), \| p \| \leqslant \varepsilon\}$ and using (C_1) and (5.2), we get

(5.3) $$|f(x, p(\bar{S}_x), p(\bar{S}_{x_w}))| \leqslant (1-b)\, \varepsilon / L_0.$$

Define an operator T on $\bar{S}(\varepsilon)$ as in Theorem 4. Then it can be shown by using (5.2) and (5.3) and the assumption on q, that T maps $\bar{S}(\varepsilon)$ into itself and also that T is a contraction on $\bar{S}(\varepsilon)$. Hence there exists a unique solution $p(\bar{S}_{x_0}, q)$ of (2.1) existing on $\overline{x_0 z}$ such that $\| p \| \leqslant \varepsilon$. This completes the proof of the theorem.

Remarks: (i) Theorem 6 is a kind of local stability result in the sense of the following definition.

Definition. *A solution $p(\bar{S}_{x_0}, q)$ of (2.1) is said to be locally stable with respect to the initial measure q if given $\varepsilon > 0$ there exists a number $\eta = \eta(\varepsilon) > 0$ such that whenever $\| q \| \leqslant \eta$, then $\| p \| \leqslant \varepsilon$.*

(ii) Theorem 6 also gives an upper bound of the unique solution of (2.1).

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