Analele ştiinţifice ale Universităţii „Al. I. Cuza" din IAŞI
Tomul XXVI, s. I a, 1980, f. 2

HYPERSTABILITY FOR LINEAR PARABOLIC SYSTEMS

BY

VIOREL HIRIŞ and MIHAIL MEGAN

Introduction. The construction of the Lyapunov functions in order to obtain conditions to ensure the absolut stability for the non-linear feed-back finite-dimensional systems leads to the system [1].

$$\begin{cases} PA+A^{*}P=-Q^{*}Q, \\ C-B\;P=\;\;R^{*}Q, \\ \qquad D=\;\;R^{*}R, \end{cases}$$

with the matrix unknowns P, Q, R, A, B, C, D being given matrices. This system is called the L u r i e s system, because it was Lurie the first who considered these equations [2].

To solve this system in the finite-dimensional case, sufficient conditions were given by V. A. Y a k u b o v i c h [3], R. E. K a l m a n [4], G. S z e g ö and R. E. K a l m a n [5], V. M. P o p o v [6]. With the solutions for this systems, one can characterize the hyperstability property introduced by V. M. P o p o v [7]. Also, V. M. P o p o v showed the way in which we can associate a variational problem to this system.

Infinite-dimensional generalizations were obtained in the papers [8], [9], [10], [11], [12]. The authors used in [11] the bounded operator techniques to give necessary and sufficient conditions for hyperstability of linear systems in Hilbert spaces.

In this paper we attempt to generalize these results for systems governed by abstract evelutions of parabolic type [13].

2. **System description and preliminary results.** Let X and Y be two real Hilbert spaces. The space of all linear maps $M : X\to Y$ will be denoted $L(X, Y)$. When $X=Y$ we sh ll use the notation $L(X)$. If X^{*} and Y^{*} are the topological duals of X and Y, then the transposed operator of M is an element of $L(Y^{*}, X^{*})$ denoted by M^{*} and defined by

$$M^{*}y^{*}, x)=(y^{*}, Mx), \quad y^{*}\in Y^{*}, \quad x\in X,$$

where (,) denotes the duality between X^{*} and X. The inner products and the norms are indexed with the respective spaces.

Let V and H be two real Hilbert spaces. Throughout this paper, we suppose that H^{*} is identified with H and there exists a continuous dense injection π of V into H; also, we denote $J: V\to V^{*}$ the canonical isometry on V.

For each $T>0$, let $W(0, T)$ be the Hilbert space

$$W(0, T)=\left\{x : x\in L^2(0, T ; V) \text{ and } J\frac{dx}{dt}\in L^2(0, T ; V^*)\right\},$$

with norm

$$\|x\|^2_{W(0,T)}=\|x\|^2_{L^2(0,T;V^*)}+\left\|J\frac{dx}{dt}\right\|^2_{L^2(0,T;V^*)}$$

(here the derivative dx/dt is taken in the sense of V^*-valued distribution on $(0, T)$ and $x'(t)$ will denote the classical derivative).

Lemma 2.1. *Suppose that $x, y\in W(0, T)$. Then*

(i) there exists continuous function $\overline{x} : [0, T]\to H$, such that

$$\pi x(t)=\overline{x}(t) \text{ a.e. on } (0, T);$$

(ii) there exists an absolute continuous function $\tilde{x} : [0, T]\to V^*$ such that $d\tilde{x}/dt\in L^2(0, T ; V^*)$ and $\tilde{x}(t)=Jx(t)$ and $\tilde{x}'(t)=Jdx(t)/dt$ a.e. on $(0, T)$;

(iii) if $M\in L(V^*)$, then $f(t)=<M\tilde{x}(t), \tilde{y}(t)>_{V^*}$ is an absolute continuous scalar function on $[0, T]$ and

$$\int_0^T(<M\tilde{x}(t), \tilde{y}(t)>_{V^*})'dt=<M\tilde{x}(T), \tilde{x}(T)>_{V^*}-<M\tilde{x}(0), \tilde{x}(0)>_{V^*}.$$

Proof. For (i) and (ii) see [14]. Because V^* is a reflexive space, by using *(ii)*, we can apply Lemma 2.4, [15] and we obtain *(iii)*.

Now, we are given a continuous bilinear form $a : V\times V\to R$, which is assumed coercive, i.e. there exists $c>0$ such that:

$$a(v, v)>c\,.\,\|v\|^2_V,\quad \forall v\in V.$$

Then it can be written [13]:

$$a(v_1, v_2)=(Av_1, v_2),\quad \forall v_1, v_2\in V,$$

where $A\in L(V, V^*)$. As operator in the Hilbert space H, the operator A is the infinitesimal generator of the C_0-semigroup $\{S(t), t\geq 0\}$ on H, with the domain

$$D(A)=\{v\in V : Av\in H\}.$$

Let U be another real Hilbert space with $U^*=U$ and $B\in L(U, V^*)$; also, we consider the class $\mathcal{U}_T=L^2(0, T ; U)$.

For every $x_0\in H$ and $u\in\mathcal{U}_T$ the equation

$$(A, B)\qquad \frac{dx}{dt}=Ax+Bu$$

has a unique solution $x(t)=x(t, x_0, u)$ with $x\in W(0, T)$, $\overline{x}(0)=x_0$ and

$$\int_0^T(\varphi'(t)x(t)+\varphi(t)\,.\,(Ax(t)+Bu(t)))dt=0,\quad \forall\varphi\in\mathcal{D}(0, T),$$

or, equivalently $x(t)=J^{-1}\tilde{x}(t)$ a.e. on $(0, T)$, where

$$\begin{cases}\tilde{x}'(t)=Ax(t)+Bu(t) \\ \tilde{x}(0)=\pi^*x_0\end{cases},\quad \text{a.e. on } (0, T)$$

Lemma 2.2. *If $x(t)=x(t, x_0, u)$ is a solution of (A, B) with $x_0\in H$, $u\in\mathcal{U}_T$ and $P\in L(V)$ such that $P^*J=JP$ then*

$$2\int_0^T(Ax(t)+Bu(t),\ Px(t))\,dt=<P^*\tilde{x}(T), \tilde{x}(T)>_{V^*}-<P^*\tilde{x}(0), \tilde{x}(0)>_{V^*}.$$

Proof. Because $x\in W(0, T)$ is a solution of the equation (A, B) and by using Lemma 2.1 with $M=P^*$, we obtain:

$$<P^*\tilde{x}(T), \tilde{x}(T)>_{V^*}-<P^*\tilde{x}(0), \tilde{x}(0)>_{V^*}=\int_0^T(<P^*\tilde{x}(t), \tilde{x}(t)>_{V^*})'dt=$$

$$=\int_0^T<P^*\tilde{x}'(t), \tilde{x}(t)>_{V^*}dt+\int_0^T<P^*\tilde{x}(t), \tilde{x}'(t)>_{V^*}dt=\int_0^T(P^*\tilde{x}'(t), J^{-1}\tilde{x}(t))dt+$$

$$+\int_0^T(\tilde{x}'(t), J^{-1}P^*\tilde{x}(t))dt=\int_0^T(P^*\tilde{x}'(t), J^{-1}\tilde{x}(t))dt+\int_0^T(\tilde{x}'(t), PJ^{-1}\tilde{x}(t))dt=$$

$$=2\int_0^T(\tilde{x}'(t), PJ^{-1}\tilde{x}(t))dt=2\int_0^T(Ax(t)+Bu(t), Px(t))\,dt.$$

If $C\in L(V, U)$ and $D\in L(U)$ then we consider the system

$$(S)\qquad \begin{aligned}\frac{dx}{dt}&=Ax+Bu,\\ y&=2Cx+Du,\end{aligned}$$

where the output y corresponding to the input $u\in\mathcal{U}_T$ and the initial data $x_0\in H$ is denoted by $y(t)=y(t, x_0, u)=2Cx(t, x_0, u)+Du(t)$.

In this case, the attached Lurie system is given by

$$(S)\qquad \begin{aligned}A^*P+P^*A&=-Q^*Q,\\ B^*P+R^*Q&=C,\\ D&=R^*R,\end{aligned}$$

with the unknowns $P\in L(V)$, $Q\in L(V, U)$ and $R\in L(U)$.

If $D>0$ then we associate to the system (S), the algebraic Riccati equation

$$\mathcal{R}(S)\quad P^*(A-BD^{-1}C)+(A-BD^{-1}C)^*P+C^*D^{-1}C+P^*BD^{-1}B^*P=0$$

with the unknown $P\in L(V)$.

Lemma 2.3. *(P,Q,R) is a solution of $\mathcal{L}(S)$ if and only if P is a solution of the equation $\mathcal{R}(S)$.*

Proof. If $(P,Q,R)\in L(V)\times L(V,U)\times L(U)$ is a solution for $\mathcal{L}(S)$, then by second equation of the system, it follows, $Q=(R^{-1})^*(C-B^*P)$ and replacing this in the first equation we obtain

$$A^*P+P^*A=-(C^*-P^*B)\,D^{-1}\,(C-B^*P).$$

i.e. P is a solution for $\mathcal{R}(S)$.

Conversely, if P is a solution for the Riccati equation, then

$$R=D^{1/2}, Q=D^{-1/2}\,(C-B^*P)$$

satisfy the last two equations and also :

$$Q^*Q=(C^*-P^*B)\,D^{-1}\,(C-B^*P)=-(A^*P+P^*A)\,.$$

Now we associate to the system (S) the cost functional

$$F(T,x_0,u)=\int_0^T <y(t),u(t)>_U dt,$$

where $0<T<\infty$, $x_0\in H$, $u\in\mathcal{U}_T$ and $y(t)=y(t,x_0,u)$.

Lemma 2.4. *If (P,Q,R) is a solution of the system $\mathcal{L}(S)$, with $P^*J=JP$ then*

$$F(T,x_0,u)=<P^*\tilde{x}(T),\tilde{x}(T)>_{V^*}-<P^*\tilde{x}(0),\tilde{x}(0)>_{V^*}+$$
$$+\int_0^T\|Qx(t)+Ru(t)\|_U^2\,dt.$$

Proof. Firstly, we remark that

$$(P^*Ax(t),\ x(t))=(Ax(t),\ Px(t))\ \text{and}\ <B^*Px(t),u(t)>_U=(Bu(t),Px(t)).$$

Let (P,Q,R) be a solution of $\mathcal{L}(S)$. Then for every $t\in(0,T)$ we have

$$<2Cx(t)+Du(t),u(t)>_U=2\ <(B^*P+R^*Q)x(t),u(t)>_U+<Du(t),u(t)>_U=$$
$$=2\ <Qx(t),\ Ru(t)>_U+2(Bu(t),\ Px(t))+\|\ Ru(t)\ \|_U^2-(Q^*Qx(t),\ x(t))+$$
$$+<Qx(t),\ Qx(t)>_U=((A^*P+P^*A)x(t),\ x(t))+2(Bu(t),\ Px(t))+$$
$$+\|\ Qx(t)+Ru(t)\ \|_U^2=2(Ax(t)+Bu(t),\ Px(t))+\|\ Qx(t)+Ru(t)\ \|_U^2.$$

The desired result follows from Lemma 2.2 by integration.

3. The main results. In this section we suppose that $D>0$ and consider a closed-convex set $K\subset\mathcal{U}_T$.

Definition 3.1. *The system (S) is said to be*

(i) hyperstable, if

$$F(T,0,u)\geq 0\ \text{for all}\ T\geq 0\ \text{and}\ u\in\mathcal{U}_T\,;$$

(ii) uniform hyperstable, if there exists $m>0$ such that $F(T,0,u)\geq m\,\|u\|_T^2$ for all $T\geq 0$ and $u\in\mathcal{U}_T$.

Theorem 3.1. *If the system (S) is an uniform hyperstable system, then for every $T\geq 0$ and any $x_0\in H$ there exists a unique element $u_0\in K$ such that*

$$(i)\qquad F(T,x_0,u_0)=\inf_{u\in K}F(T,x_0,u)\,;$$

$$(ii)\qquad \int_0^T<B^*p(t)+Cx(t,0,u_0)+Du_0(t),\ u(t)-u_0(t)>_U dt\geq 0,\quad \forall u\in K,$$

where $p\in W(0,T)$ is the unique solution of the „adjoint system"

$$(S^*)\qquad \begin{cases}\dfrac{dp}{dt}=-A^*p-C^*u_0\\ p(T)=0.\end{cases}$$

Proof. We denote

$$\pi_T(u_1,u_2)=\int_0^T(<Cx(t,0,u_1),u_2(t)>_U+<Cx(t,0,u_2),u_1(t)>_U+$$
$$+<Du_1(t),u_2(t)>_U)dt$$

and

$$L_{T,x_0}(u)=-\int_0^T<Cx(t,x_0,0),u(t)>_U dt.$$

Since $x(t,x_0,u)=x(t,0,u)+x(t,x_0,0)$, we have

$$F(T,x_0,u)=\pi_T(u,u)-2L_{T,x_0}(u)=F(t,0,u)-2L_{T,x_0}(u).$$

The uniform hyperstability assumption ensures that the continuous symmetrical form π_T is a coercive form on $\mathcal{U}_T$ and thus by the Minimum Theorem [13] one obtains the existence of the unique control $u_0\in K$ with the property (i). The control $u_0\in K$ is characterized by the variational inequalities

$$\pi_T(u_0,u-u_0)\geq L_{T,x_0}(u-u_0)\,,\quad \forall u\in K.$$

In our case this becomes

$$\int_0^T(<Cx(t,0,u_0),u(t)-u_0(t)>_U+<Cx(t,0;u-u_0),u_0(t)>_U+$$
$$+<Du_0(t),u(t)-u_0(t)>_U)dt\geq-\int_0^T<Cx(t,x_0,0),u(t)-u_0(t)>_U dt$$

and by virtue of the „adjoint system" we have

$$\int_0^T \langle Cx(t,0,u-u_0),\, u_0(t)\rangle_U\, dt = \int_0^T (C^* u_0(t),\, x(t,0,u-u_0))\, dt =$$

$$= \int_0^T \left\langle -\frac{d\tilde{p}(t)}{dt} - A^* p(t),\, x(t,0,u-u_0)\right\rangle_{V^*} dt =$$

$$= \int_0^T \left\langle -\frac{d\tilde{p}(t)}{dt},\, \tilde{x}(t,0,u-u_0)\right\rangle_{V^*} dt - \int_0^T (A^* p(t), x(t,0;u-u_0))\, dt =$$

$$= -\int_0^T \left((\langle \tilde{p}(t), \tilde{x}(t,0,u-u_0)\rangle_{V^*})' + \left\langle \tilde{p}(t), \frac{d\tilde{x}(t,0,u-u_0)}{dt}\right\rangle_{V^*} \right) dt -$$

$$- \int_0^T (Ax(t,0,u-u_0), u(t))\, dt = \int_0^T \langle B(u(t)-u_0(t)), \tilde{p}(t)\rangle_{V^*} dt =$$

$$= \int_0^T \langle B^* p(t), u(t)-u_0(t)\rangle_U\, dt.$$

On the other hand, we can write

$$\int_0^T \langle C(x(t,0,u_0) + x(t,x_0,0)), u(t)-u_0(t)\rangle_U\, dt =$$

$$- \int_0^T \langle Cx(t,x_0,u_0), u(t)-u_0(t)\rangle_U\, dt.$$

Hence it follows that u_0 satisfies (i) if and only if

$$\int_0^T \langle B^* p(t) + Cx(t,0,u_0) + Du_0(t), u(t)-u_0(t)\rangle_U\, dt \geq 0,$$

for every $u \in K$.

Corollary 3.2. *In the conditions of the Theorem 3.1 and if $K = \mathcal{U}_T$, the optimal control $u_0 \in \mathcal{U}_T$ is characterized by the formula*

$$u_0(t) = D^{-1}(Bp(t) + Cx(t,x_0,u_0)) \text{ a.e. on } (0,T).$$

Proof. By the same arguments, using the Euler-equation

$$\pi_T(u_0, u) = L_{T,x_0}(u),$$

we obtain

$$B^* p(t) + Cx(t,x_0,u) + Du_0(t) = 0, \text{ a.e. on } (0,T).$$

Corollary 3.3. *In the hypotheses of the preceding corollary the optimal control u_0 is given by the system*

$$\frac{dx}{dt} = (A + C)x - BD^{-1}B^* p,$$

$$\frac{dp}{dt} = (-A^* + C^* D^{-1} B^*)p + C^* D^{-1} Cx,$$

$$x(0) = x_0, \quad p(T) = 0$$

Proof. Is obvious from the preceding results.

Theorem 3.4. *If the system (S) is a uniform hyperstable one and the optimal control u_0 has the „controllability property" :*

$$\forall v \in V, \quad \forall \varepsilon > 0, \quad \forall E \subset [0,T], \quad \mu(E) = 0, \quad \exists t_0 \in [0,T] E,$$
$$\exists x_0 \in H : \| x(t_0, x_0, u_0) - v \|_V < \varepsilon,$$

there exists a solution (P, Q, R) of the system $\mathcal{L}(S)$ such that $P^* J = JP$.

Proof. Let $P_0 : V \to V$ be the operator defined by

$$P_0 x_0 = p(0), \quad x_0 \in H,$$

where p is the solution of the „adjoint system" (S^*).

In the sequel, we proceed in a similar manner as in [10]. Namely, we have $P_0 \in L(H,V)$ with $P_0\, x(t,x_0,u_0) = p(t)$ and

$$\pi P_0 h = P_0^* \pi^* h \text{ for every } h \in H.$$

Thus, for $P = P_0 \pi \in L(V)$ we have $P^* J = JP$ and it follows that the optimal control verifies the relation $u_0(t) = -D^{-1}(B^* P + C)\, x(t,x_0,u_0)$ a.e. Define the operator $M : V \to V^*$ by

$$(Mv_1, v_2) = (P^* A v_1, v_2) + (A^* P v_1, v_2) + (Q^* Q v_1, v_2), \quad \forall v_1, v_2 \in V.$$

By using the definitions of the functions $x_0(t) = (t, x_0, u_0)$ and $p(t)$ after replacing the expression of u_0 and p, it follows that

$$\int_0^T (Mx_0(t), y(t))\, dt = 0 \text{ for every } y \in W(0,T).$$

Particularly, we consider $y(t) = Mx_0(t)$, and then we obtain $Mx_0(t) = 0$ a.e. on $(0,T)$ and therefore, by the „controlability property", we obtain that $Mv = 0$ $\forall v \in V$. This shows that the operator P is a solution for the Riccati equation and thus by Lemma 2.3. it follows that the Lurie's system $\mathcal{L}(S)$ has a solution (P, Q, R) with $P^* J = JP$.

Definition 3.2. *We say that the system (S) has the minimal stability property, if for every $x_0 \in H\, (x_0 \neq 0)$, there exists $u_0 \in \mathcal{U} = L^2(0,\infty; U)$, such that*

(i) $\lim\limits_{t\to\infty} x(t,x_0,u_0) = 0$;

(ii) $F(t,x_0,u_0) \leq 0$ *for all* $t \geq 0$.

Theorem 3.5. *If the system (S) has the minimal stability property and there exists a solution (P, Q, R) of $\mathcal{L}(S)$ with $P^*J = JP$, then $P^* \geq 0$.*

Proof. By Definition 3.2 and Lemma 2.4 we have

$$<P^*\tilde{x}(T), \tilde{x}(T)>_{V^*} - <P^*\tilde{x}(0), \tilde{x}(0)>_{V^*} \leq F(T, x_0, u) \leq 0.$$

Hence

$$<P^*\tilde{x}(0), \tilde{x}(0)>_{V^*} \geq <P^*\tilde{x}(T), \tilde{x}(T)>_{V^*} \to 0 \text{ if } T \to \infty$$

and because $\pi^* : H \to V^*$ is surjective we obtain $P^* \geq 0$.

Theorem 3.6. *If there exists a solution (P, Q, R) for $\mathcal{L}(S)$ such that $P^* \geq 0$ then (S) is an hyperstable system.*

Proof. Is obvious from the Lemma 2.4.

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Received 2.11.1980

Department of Mathematics
University of Timişoara
Romania

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din IAŞI
Tomul XXVI, s. I a, 1980, f. 2

EXISTENCE FOR NONLINEAR DIFFERENTIAL SYSTEMS OF HYPERBOLIC TYPE

BY

G. MOROŞANU

1. Introduction. Our objective is to investigate the existence and uniqueness of solutions to the nonlinear hyperbolic differential system

$$\text{(1.1)} \qquad \begin{aligned} &\alpha(x)\frac{\partial u}{\partial t} - \frac{\partial v}{\partial x} + A(x, u) = f(x, t), \\ &\beta(x)\frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} + B(x, v) = g(x, t), \\ &\text{for } 0 < x < 1,\ 0 < t < T, \end{aligned}$$

with the initial data

$$\text{(1.2)} \qquad u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad 0 \leq x \leq 1,$$

and boundary conditions

$$\text{(1.3)} \qquad (u(0, t) - u(1, t)) \in L\ (v(0, t),\ v(1, t)),\ 0 \leq t \leq T,$$

where A and B are some functions from $]0, 1[\times R$ into R, $(R =]-\infty, \infty[)$ and L is a nonlinear (possibly) multivalued mapping from $R^2 = R \times R$ into itself.

Such problems are very much discussed in the literature. They arise in the theory of electrical transmission lines and in hydraulics. Existence, uniqueness and asymptotic behavior of solutions for problems of this type have been studied by many authors (see e.g. [2], [3], [5—11], [14] and Răsvan's book [13] for further references). Our starting point is the paper of V. Barbu [2], where the mapping L in the boundary conditions (1.3) was supposed to be a subdifferential. In the present paper we shall conditions, assume only that

(H_1) *$L : R^2 \to R^2$ is an arbitrary maximal monotone operator.*

Here are now some other basic hypotheses which will be used in what follows.

(H_2) The functions $r \to A(x, r)$ and $r \to B(x, r)$ are continuous and monotone increasing on R, a.e. $x \in]0, 1[$. Furthermore, the functions $x \to A(x, r)$ and $x \to B(x, r)$ belong to $L^2(0,1)$, for each fixed $r \in R$.