6. Brayton R.K. and Miranker W.L. — *A stability theory for nonlinear mixed initial boundary-value problems*, Arch. Rational Mech. Anal., 17 (1964), 358 — 376.
7. Brayton R.K. and Moser J.K. — *A theory of nonlinear networks*, Quart. Appl. Math., 22 (1964), 1 — 33.
8. Cooke K.L. and Krumme D.W. — *Differential-difference equations and nonlinear initial boundary value problems for linear hyperbolic partial differential equations*, J. Math. Anal. Appl., 24 (1968), 372 — 387.
9. Halanay A. and Răsvan Vl. — *Frequency domain conditions for forced oscillations in difference systems*, Revue Roum. Sc. Techn., ser. Electrotechn. Energ. 24 (1979), 141 — 148.
10. Iftimie V. — *Sur un problème non linéaire pour un système hyperbolique*, Revue Roum. Math. Pures Appl. 20 (1976), 303 — 328.
11. Moroşanu G. — *Stability of solutions of nonlinear boundary-value problems for hyperbolic systems*, J. Nonlinear Analysis, Theory Meth. Appl. (to appear).
12. Pavel N. — *Ecuaţii diferenţiale asociate unor operatori neliniari pe spaţii Banach*, Editura Academiei, Bucureşti, 1977.
13. Răsvan Vl. — *Stabilitatea absolută a sistemelor automate cu întîrziere*, Editura Academiei, Bucureşti, 1975.
14. Slemrod M. — *Nonexistence of oscillations on a nonlinear distributed network*, J. Math. Anal. Appl. 36 (1971), 22 — 40.

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AN INTEGRODIFFERENTIAL EQUATION WITH A DISCONTINUOUS NONLINEARITY

BY

SERGIU AIZICOVICI

1. Let Ω be a bounded domain in R^N with smooth boundary Γ. Let $a : [0, \infty) \to R$, $f : R \to R$, $h : \Omega \times R \to R$ and $u_0 : \Omega \to R$ be given functions. In this paper we will be concerned with the existence of solutions to the initial boundary value problem

$$
\text{(P)} \quad
\begin{aligned}
& u_t(t, x) + \int_0^t a(t-s)\,(-\Delta_x u(s, x) + f(u(s, x)))\,ds = h(t, x), \quad t \geq 0,\ x \in \Omega, \\
& u(t, x) = 0, \quad t \geq 0, \quad x \in \Gamma, \\
& u(0, x) = u_0(x), \quad x \in \Omega.
\end{aligned}
$$

This problem has recently been considered by the author [1], under the key assumption that $f(u)$ is continuous and has the same sign as u. In the present paper we only assume that

$$
\text{(F—1)} \qquad f \in L^{\infty}_{\mathrm{loc}}(R),
$$

$$
\text{(F—2)} \qquad \operatorname*{ess\,sup}_{-\infty < t < -s_0} f(t) \leq \operatorname*{ess\,inf}_{s_0 < t < +\infty} f(t).
$$

for some $s_0 \geq 0$.

Our approach rests heavily on the technique of [1]. It also involves some arguments previously used by Kiffe [4] and Rauch [5] in the study of discontinuous scalar Volterra integral equations and discontinuous elliptic equations, respectively.

2. Suppose that $f : R \to R$ satisfies (F—1). Then, for any $\varepsilon > 0$ and $s \in R$, one can define, as in [5]:

$$
\underline{f}_\varepsilon(s) = \operatorname*{ess\,inf}_{|t-s|<\varepsilon} f(t), \qquad \overline{f}_\varepsilon(s) = \operatorname*{ess\,sup}_{|t-s|<\varepsilon} f(t),
$$

and

$$
\underline{f}(s) = \lim_{\varepsilon \to 0+} \underline{f}_\varepsilon(s), \qquad \overline{f}(s) = \lim_{\varepsilon \to 0+} \overline{f}_\varepsilon(s).
$$

We are now in position to precise what we mean by a solution of (P).

Definition. *A function $u(t,x)$, $t\geq 0$; $x\in\Omega$ is said to be a solution to the problem (P) (where f is locally bounded), if*

(1) $$u\in C([0,\infty);L^2(\Omega))\cap L^\infty_{loc}([0,\infty);H^1_0(\Omega)),$$

(2) $$u_t\in L^\infty_{loc}([0,\infty);L^2(\Omega))\cap L^1_{loc}([0,\infty);H^{-1}(\Omega)+L^1(\Omega)),$$

(3) $$u_{tt}\in L^1_{loc}([0,\infty);H^{-1}(\Omega)+L^1(\Omega)),$$

(4) *there exists* $w\in L^1_{loc}([0,\infty);L^1(\Omega))$ *such that*
$$\underline{f}(u(t,x))\leq w(t,x)\leq\bar{f}(u(t,x)),\ a.e.\ on\ (0,\infty)\times\Omega,$$

(5) $$u_t(t,x)+\int_0^t a(t-s)\,(-\Delta u(s,x)+w(s,x))\,ds=h(t,x),t\geq 0,\ x\in\Omega,$$

(6) $$u(t,x)=0,\ on\ [0,\infty)\times\Gamma,$$

(7) $$u(0,x)=u_0(x),\ x\in\Omega.$$

To state our basic result we need the following assumptions:

(8) $$a\in W^{2,1}_{loc}([0,\infty);R);\ a(0)>0,$$

(9) $$h\in W^{1,1}_{loc}([0,\infty);L^2(\Omega)),$$

(10) $$u_0\in H^1_0(\Omega)\cap L^\infty(\Omega).$$

Theorem. *Let the conditions* (8)—(10), (F—1) *and* (F—2) *be satisfied Then problem* (P) *has at least one solution.*

Remark. As $a=1$ satisfies (8), the theorem also applies to the nonlinear wave equation
$$u_{tt}-\Delta u+f(u)=h_t,\ \text{on}\ (0,\infty)\times\Omega,$$
with appropriate initial and boundary conditions. For a related result see [5].

3. To prove the theorem, we employ a simple approximation scheme. In what follows we assume without loss of generality that

(11) $$\operatorname*{ess\,sup}_{-\infty<t<-s_0} f(t)\leq 0\leq\operatorname*{ess\,inf}_{s_0<t<+\infty} f(t),$$

Let $\rho(t)$ be a C^∞ function with support contained in $(-1,1)$ satisfying $\rho(t)\geq 0$ and $\int_{-\infty}^{\infty}\rho(t)dt=1$. Set $\rho_n(t)=n\rho(nt)$, $n=1,2,\ldots$, and define

(12) $$f_n(t)=\int_{-\infty}^{\infty}\rho_n(t-s)f(s)ds.$$

Clearly, f_n is continuous for each n. Moreover, by (F—1) and (11) there exist positive numbers S_0 and μ so that for all n

(13) $$\text{(i)}\quad tf_n(t)\geq 0,\ \text{if}\ |t|>S_0,$$
$$\text{(ii)}\quad |f_n(t)|\leq\mu,\ \text{if}\ |t|\leq S_0.$$

Then, in view of [6, Lemma 2.2] one can find a sequence of continuous functions $f_{n,k}:R\to R$, $k=1,2,\ldots$, such that

(14) $$\text{(i)}\ |f_{n,k}(x)-f_{n,k}(y)|\leq c_{n,k}|x-y|,\ \forall x,y\in R,$$
for some $c_{n,k}>0$.

(ii) $f_{n,k}$ tends to $f_n (k\to\infty)$, uniformly on bounded sets,

(iii) $xf_{n,k}(x)\geq 0$ if $|x|>S_0$.

Using the functions $f_{n,k}$ we will approximate (P) by a sequence of regularized problems. As in [1], we set
$$V=H^1_0(\Omega),\ H=L^2(\Omega),\ V'=H^{-1}(\Omega).$$
Obviously,
$$V\subset H\subset V'$$
with dense and compact injections. The norms in V and H will be denoted by $\|\cdot\|$ and $|\cdot|$, respectively. We also use $(,)$ to indicate the scalar product in H.

Introduce the linear operators $A:V\to V'$ and $A_H:H\to H$ by
$$Au=-\Delta u,\ u\in V;\ A_Hu=Au,\ u\in D(A_H)=\{u\in V;\ Au\in H\}.$$
One has
$$A_H=\partial\varphi_H(\partial=\text{subdifferential}),\ \varphi_H(u)=\begin{cases}\frac{1}{2}\|u\|^2, & u\in V\\ +\infty, & u\in H\setminus V.\end{cases}$$
Also define the Yosida approximation A_k of A_H by
$$A_k=k(I-J_k),\ J_k=\left(I+\frac{1}{k}A_H\right)^{-1},$$
where, as usually I denotes the identity on H. Recall that A_k is Lipschitz and (see [2])

(15) $$A_k=\partial\varphi_k,\ \varphi_k(u)=\varphi_H(J_ku)+\frac{1}{2k}|A_ku|^2=\inf_{v\in H}\left\{\frac{k}{2}|u-v|^2+\varphi_H(v)\right\},\ u\in H.$$

Finally, let
$$(F_{n,k}u)(x)=f_{n,k}(u(x)),\ u\in H,\ x\in\Omega,$$
and remark (cf. (14) (i)) $F_{n,k}$ is everywhere defined and Lipschitz on H.

Consider now the following approximating problem (in H)

$$\text{(16)}\quad \begin{aligned}&\text{(i)}\ u'_{n,k}(t)+\int_0^t a(t-s)\,(A_k+F_{n,k})\,u_{n,k}(s)\,ds=h(t),\ t\geq 0,\ ('=d/dt)\\&\text{(ii)}\ u_{n,k}(0)=u_0.\end{aligned}$$

Since $A_k+F_{n,k}$ is Lipschitz, (16) has a unique solution $u_{n,k}\in W^{2,2}_{\text{loc}}([0,\infty);H)$.

4. In this section we establish a number of energy estimates on the solutions $u_{n,k}$. The method is similar to that of [1] and we only sketch it.

First define $G_{n,k}: H\to R$ by

$$\text{(17)}\qquad G_{n,k}(u)=\int_\Omega g_{n,k}(u(x))\,dx,\ u\in H;\ g_{n,k}(t)=\int_0^t f_{n,k}(s)\,ds,\ t\in R.$$

It is easily seen that $u\to G_{n,k}\,u$ is Fréchet differentiable on H and

$$\text{(18)}\qquad (u'_{n,k}(t),\,F_{n,k}u_{n,k}(t))=\frac{d}{dt}G_{n,k}(u_{n,k}(t)),\ \text{a.e. on } (0,\infty).$$

Suppose next that n is fixed. From (10), (13), (14) (ii), (iii) and (17), we then deduce the existence of finite constants $C>0$ and $\gamma\in R$, independent of n, such that

$$\text{(19)}\qquad |G_{n,k}(u_0)|\leq C$$

and

$$\text{(20)}\qquad G_{n,k}(v)\geq\gamma,$$

for all $k>N(n)$ and all $v\in H$. Here $N(n)$ denotes a positive integer depending on n.

Let $T\in(0,\infty)$ be arbitrary. Multiply (16) (i) by $v_{n,k}(t)=(A_k+F_{n,k})\,u_{n,k}(t)$ and integrate the result over $(0,t)$, $0\leq t\leq T$. Employing (15), (16) (ii), (18) and [2, Lemma 3.3] yields

$$\text{(21)}\qquad \begin{aligned}&\varphi_k(u_{n,k}(t))-\varphi_k(u_0)+G_{n,k}(u_{n,k}(t))-G_{n,k}(u_0)+\\&+\int_0^t (a*v_{n,k}(s),\,v_{n,k}(s))\,ds=\int_0^t (h(s),\,v_{n,k}(s))\,ds,\ 0\leq t\leq T,\end{aligned}$$

where $*$ designates the convolution.

Combining (8)—(10), (15) and (19)—(21), we arrive at (recall n is temporarily fixed)

$$\text{(22)}\qquad \int_0^t (a*v_{n,k}(s),\,v_{n,k}(s))\,ds\leq C_1+C_2\max_{0\leq s\leq t}\left|\int_0^s v_{n,k}(\tau)\,d\tau\right|,\ t\in[0,T],$$

for sufficiently large k. From this point onward we denote by $C_1, C_2,\ldots$, etc. various finite positive constants independent of n.

Calling on (8) and [3, Proposition (a)], (22) implies

$$\text{(23)}\qquad \begin{aligned}&\text{(i)}\quad \left|\int_0^t v_{n,k}(s)\,ds\right|\leq C_3;\\&\text{(ii)}\quad \left|\int_0^t (a*v_{n,k}(s),\,v_{n,k}(s))\,ds\right|\leq C_4.\end{aligned}\qquad ,\ 0\leq t\leq T$$

From (16) (i) and (23) (i), it now follows

$$\text{(24)}\qquad |u'_{n,k}(t)|\leq C_5,\ t\in[0,T].$$

Consequently

$$\text{(25)}\qquad |J_k u_{n,k}(t)|\leq C_5,\ 0\leq t\leq T.$$

By (19)—(21) and (23) we have

$$\text{(26)}\qquad \varphi_k(u_{n,k}(t))\leq C_6,\ t\in[0,T],$$

and (see (15) and the definition of φ_n)

$$\text{(27)}\qquad \|J_k u_{n,k}(t)\|\leq C_7,\ t\in[0,T].$$

Remark. The above estimates hold for large k. In other words there is $N(n)$ such that if $k>N(n)$ then (23)—(27) are satisfied.

We finally differentiate (16) (i) in H, multiply the resultant equation by $u_{n,k}(t)$ and integrate over $(0,T)$. After some calculations involving (8)—(10) and (23) (i), (24) we get

$$\int_0^T (F_{n,k}\,u_{n,k}(t),\,u_{n,k}(t))\,dt\leq C_8,$$

or equivalently

$$\text{(28)}\qquad \int_Q f_{n,k}(u_{n,k}(t,x))\cdot u_{n,k}(t,x)\,dt\,dx\leq C_8,\ Q=(0,T)\times\Omega.$$

Invoking (14) (ii), (iii), (28) yields (for sufficiently large k)

$$\text{(29)}\qquad \int_Q |f_{n,k}(u_{n,k})\cdot u_{n,k}|\,dt\,dx\leq C_9.$$

5. On account of (24)—(27) and (29) passage to the limit in (16) now procedes easily. In what follows we use various subsequences, for which we do not bother changing notation.

We first let $k\to\infty$ with n fixed. From (25), (27) and the

Ascoli theorem it follows that

(30) (i) $J_k u_{n,k} \to u_n (k \to \infty)$, strongly in $C([0, T]; H)$,

(ii) $J_k u_{n,k} \to u_n$, weakly-star in $L^\infty(0, T; V)$.

In view of (15), (26) and (30) (i), one also has

(31) $u_{n,k} \to u_n$, strongly in $C([0, T]; H)$.

This and (24) lead to

(32) $u'_{n,k} \to u'_n$, weakly-star in $L^\infty(0, T; H)$.

By (31) we may assume that

(33) $u_{n,k}(t, x) \to u_n(t, x)$, a.e. on Q.

Calling on (14) (ii), this gives

(34) $f_{n,k}(u_{n,k}) \to f_n(u_n)$, a.e. on Q.

Taking into account (12), (14) (ii), (29) and (34) we can apply Theorem 1.1 of [6], thereby obtaining

(35) $F_{n,k}(u_{n,k}) \to f_n(u_n)$, strongly in $L^1(Q) = L^1(0, T; L^1(\Omega))$.

Finally remark that (30) (ii) implies

(36) $A_k u_{n,k} \to A u_n$, weakly-star in $L^\infty(0, T; V')$.

Letting $k \to \infty$ in (16) then gives (use (23) (i), (31), (32), (35), (36))

(37) (i) $u'_n(t) + \int_0^t a(t-s)\,(Au_n(s) + f_n(u_n(s)))\,ds = h(t)$, $t \in [0, T]$,

(ii) $u_n(0) = u_0$.

The next step is to pass to the limit in (37) as $n \to \infty$. Note that (24), (27) in conjunction with (30) (ii) and (32) lead to

(38) $\{u_n;\ n = 1, 2, \ldots\}$ is bounded in $L^\infty(0; T; V)$

$\{u'_n;\ n = 1, 2, \ldots\}$ is bounded in $L^\infty(0, T; H)$.

From (38) it follows

(39) (i) $u_n \to u$, strongly in $C([0, T]; H)$,

(ii) $u_n \to u$, weakly-star in $L^\infty(0, T; V)$;

(iii) $u'_n \to u_t = u'$, weakly-star in $L^\infty(0, T; H)$.

By (39) (ii) one has

(40) $Au_n \to Au$, weakly star in $L^\infty(0, T; V')$.

It remains to establish the convergence of $f_n(u_n)$. Using (29), (33), (34) and Fatou's lemma we get

(41) $$\int_Q |u_n f_n(u_n)|\,dt\,dx \leqslant C_0.$$

Combining (12), (41) and the simple estimate

$$|f_n(s)| \leqslant m^{-1} |sf_n(s)| + \sup_{|s| \leqq m} |f_n(s)|, \quad \forall m > 0,$$

one can readily show that (cf. [5]) for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ so that if $\omega \subset Q$ with meas $(\omega) < \delta$, then $\int_\omega |f_n(u_n)| < \varepsilon$ $\forall n$. This implies (by the Dunford-Pettis criterion) that

(42) $f_n(u_n) \to w$, weakly in $L^1(Q)$.

From (37), (39), (40), (42) and the arbitrariness of T, assertions (1)—(3) and (5)—(7) of our theorem now follow easily. To complete the proof we need only verify (4). To this end we use an argument of [5].

By (39) (i) we may assume that $u_n \to u$, a.e. on Q.

Fix $\eta > 0$. Invoking the theorems of Luzin and Egorov we can choose a subset $q \subset Q$ such that meas $(q) < \eta$, $u \in L^\infty(Q \setminus q)$ and $u_n \to u$, uniformly on $Q \setminus q$. Therefore, for any $\varepsilon > 0$ there is $n_0 > 2/\varepsilon$ such that for $n > n_0$

$$|u_n(\xi) - u(\xi)| < \frac{\varepsilon}{2}, \quad \forall \xi \in Q \setminus q.$$

Then, if $|u_n(\xi) - s| < \dfrac{1}{n}$ $(n > n_0,\ \xi \in Q \setminus q)$ we have $|u(\xi) - s| < \varepsilon$ and hence (cf. (12))

(43) $\underline{f_\varepsilon}(u(\xi)) \leqslant f_n(u_n(\xi)) \leqslant \overline{f_\varepsilon}(u(\xi))$, $\xi \in Q \setminus q$, $n > n_0$

Let $h \in L^\infty(Q \setminus q)$ with $h \geqslant 0$, a.e. Multiply (43) by $h(\xi)$ and integrate over $Q \setminus q$. Then, let $n \to \infty$ to obtain (see (42)).

(44) $$\int_{Q\setminus q} \underline{f_\varepsilon}(u(\xi))\,h(\xi)\,d\xi \leqslant \int_{Q\setminus q} w(\xi)\,h(\xi)\,d\xi \leqslant \int_{Q\setminus q} \overline{f_\varepsilon}(u(\xi))\,h(\xi)\,d\xi.$$

The dominated convergence theorem allows us to take $\varepsilon \to 0$ in (44). As $h \geqslant 0$ was arbitrary, we conclude that

(45) $\underline{f}(u(\xi)) \leqslant w(\xi) \leqslant \overline{f}(u(\xi))$, a.e. on $Q \setminus q$

Recall that meas $(q) < \eta$, with η as small as we like. Then (45) clearly implies (4) and the proof is closed.

Remark. The following properties of $-\Delta$ have played an essential role in the proof: $-\Delta$ is cyclically maximal monotone, coercive and weakly continuous from V to V'.

REFERENCES

1. Aizicovici, S. — *On a semilinear Volterra integrodifferential equation*, Israel J. Math., to appear.
2. Brézis, H. — *Opérateurs Maximaux Monotones et Semigroupes de Contractions dans les Espaces de Hilbert*, Math. Studies 5, North Holland, Amsterdam, 1973.
3. Crandall, M.G., Londen, S.O. and Nohel, J.A. — *An abstract nonlinear Volterra integrodifferential equation*, J. Math. Anal. Appl. 64 (1978), 701 — 735.
4. Kiffe, T. — *A discontinuous Volterra integral equation*, J. Integral Equations, 1 (1979), 193 — 200.
5. Rauch, J. — *Discontinous semilinear differential equations and multiple valued maps*, Proc. Amer. Math. Soc. 64 (1977), 277 — 282.
6. Strauss, W.A. — *On weak solutions of semi-linear hyperbolic equations*, An. Acad. Brasil, Ci. 42 (1970), 645 — 651.

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ON SOME FUNDAMENTAL n-th ORDER FINITE DIFFERENCE INEQUALITIES

BY

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1. Introduction. In the theory of finite difference equations, discrete analogues of Gronwall's inequality [1] and its variants are of fundamental importance, since the bounds provided by these inequalities are adequate in many applications (see, [3]—[14]). However, some other recent developments in the theory of numerical solutions of n-th order differential and integrodifferential equations have become of compelling interest in the study of such models without reducing them to a system of n equations of the first order. In such situations the bounds provided by the Gronwall type discrete inequalities are inadequate in applications. In this paper we present some new n-th order finite difference inequalities which can be used in accelerating the growth of literature on various aspects of n-th order finite difference equations of the more general type.

2. Main Results. Before giving the main results in this section, we first introduce a few of the basic notions involved in our subsequent discussion. Let N be a set of points $k_0+r(r=0, 1, 2,...)$, where $k_0 \geq 0$ is a given integer. The expression $\sum_{s=k_0}^{k-1} f(s)$ represents a solution of the linear difference equation $\Delta x(k)=f(k)$ for all $k \in N$ under the initial condition $x(k_0)=0$, where Δ is the operator defined by $\Delta x(n)=x(n+1)-x(n)$. The operators $\overset{2}{\Delta} x(k), ..., \overset{n}{\Delta} x(k)$ are defined in the usual way. It is supposed that $\sum_{s=k_0}^{k_0-1} f(s)=0$. The expression $\prod_{s=0}^{k-1} g(s)$ represents a solution of the linear difference equation $x(k+1)=g(k)\,x(k)$ for all $k \in N$ under the initial condition $x(k_0)=1$. It is supposed that $\prod_{s=k_0}^{k_0-1} g(s)=1$.

A new and useful n-th order finite difference inequality is embodied in the following theorem.

Theorem 1. *Let $x(k)$, $\Delta x(k), ..., \overset{n}{\Delta} x(k)$, $a(k)$, $b(k)$, and $c(k)$ be real-valued nonnegative functions defined on N, for which the inequality*