Remark. The following properties of $-\Delta$ have played an essential role in the proof: $-\Delta$ is cyclically maximal monotone, coercive and weakly continuous from V to V'.

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Institute of Mathematics
University of Iaşi
6600 Iaşi, Romania

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ON SOME FUNDAMENTAL n-th ORDER FINITE DIFFERENCE INEQUALITIES

BY

B.G. PACHPATTE

1. Introduction. In the theory of finite difference equations, discrete analogues of Gronwall's inequality [1] and its variants are of fundamental importance, since the bounds provided by these inequalities are adequate in many applications (see, [3]—[14]). However, some other recent developments in the theory of numerical solutions of n-th order differential and integrodifferential equations have become of compelling interest in the study of such models without reducing them to a system of n equations of the first order. In such situations the bounds provided by the Gronwall type discrete inequalities are inadequate in applications. In this paper we present some new n-th order finite difference inequalities which can be used in accelerating the growth of literature on various aspects of n-th order finite difference equations of the more general type.

2. Main Results. Before giving the main results in this section, we first introduce a few of the basic notions involved in our subsequent discussion. Let N be a set of points $k_0+r(r=0, 1, 2,...)$, where $k_0 \geq 0$ is a given integer. The expression $\sum_{s=k_0}^{k-1} f(s)$ represents a solution of the linear difference equation $\Delta x(k)=f(k)$ for all $k \in N$ under the initial condition $x(k_0)=0$, where Δ is the operator defined by $\Delta x(n)=x(n+1)-x(n)$. The operators $\overset{2}{\Delta} x(k), \ldots, \overset{n}{\Delta} x(k)$ are defined in the usual way. It is supposed that $\sum_{s=k_0}^{k_0-1} f(s)=0$. The expression $\prod_{s=0}^{k-1} g(s)$ represents a solution of the linear difference equation $x(k+1)=g(k)\,x(k)$ for all $k \in N$ under the initial condition $x(k_0)=1$. It is supposed that $\prod_{s=k_0}^{k_0-1} g(s)=1$.

A new and useful n-th order finite difference inequality is embodied in the following theorem.

Theorem 1. *Let $x(k)$, $\Delta x(k), \ldots, \overset{n}{\Delta} x(k)$, $a(k)$, $b(k)$, and $c(k)$ be real-valued nonnegative functions defined on N, for which the inequality*

(1)
$$\Delta^{n}x(k) \leqslant a(k) + b(k)\Big(x(k) + \Delta x(k) + \ldots + \Delta^{n-1}x(k) + \sum_{s=0}^{k-1} c(s)(x(s) + \Delta x(s) + \ldots + \Delta^{n-1}x(s))\Big),$$

holds for all $k \in N$. Then:

(2)
$$\Delta^{n}x(k) \leqslant a(k)+b(k)\Big([x(0)+\Delta x(0)+\ldots+\Delta^{n-1}x(0)]\prod_{s=0}^{k-1}[2+b(s)+c(s)] + \sum_{s=0}^{k-1} a(s) \prod_{t=s+1}^{k-1} [2+b(t)+c(t)]\Big),$$

for all $k \in N$.

Proof. Define

(3)
$$m(k) = x(k)+\Delta x(k)+\ldots+\Delta^{n-1}x(k)+\sum_{s=0}^{k-1} c(s)(x(s)+\Delta x(s)+\ldots+\Delta^{n-1}x(s)),$$
$$m(0) = x(0) + \Delta x(0) + \ldots + \Delta^{n-1}x(0);$$

then

(4) $$\Delta m(k) = \Delta x(k)+\ldots+\Delta^{n-1}x(k)+\Delta^{n}x(k)+c(k)(x(k)+\Delta x(k)+\ldots+\Delta^{n-1}x(k))$$

for all $k \in N$. Using the facts that

(5) $$\Delta x(k) + \ldots + \Delta^{n-1}x(k) \leqslant m(k),$$

and

(6) $$x(k) + \Delta x(k) + \ldots + \Delta^{n-1}x(k) \leqslant m(k),$$

from (3), and

$$\Delta^{n}x(k) \leqslant a(k)+b(k)\,m(k),$$

from (1) in (4), we see that the inequality

(7) $$m(k+1) \leqslant [2+b(k)+c(k)]m(k) + a(k)$$

is satisfied for all $k \in N$. Multiplying both sides of (7) by $\prod_{s=0}^{k}[2+b(k)+c(k)]^{-1}$ and summing up both sides from 0 to $k-1$, we obtain the estimate for $m(k)$ such that

$$m(k) \leqslant [x(0) + \Delta x(0) + \ldots + \Delta^{n-1}x(0)]\prod_{s=0}^{k-1}[2+b(s)+c(s)] + \sum_{s=0}^{k-1} a(s) \sum_{t=s+1}^{k-1} [2+b(t)+c(t)].$$

for all $k \in N$. Now substituting this bound on $m(k)$ in (1), we obtain the desired bound in (2).

We next establish a nonlinear n-th order finite difference inequality in the following theorem.

Theorem 2. *Let $x(k)$, $\Delta x(k), \ldots, \Delta^{n}x(k)$, $a(k)$, $b(k)$, and $c(k)$ be real-valued nonnegative functions defined on N; and let $H(u)$ be a positive, continuous, monotonic, nondecreasing, subadditive and submultiplicative function for $u>0$, $H(0)=0$, and suppose further that the inequality*

(8)
$$\Delta^{n}x(k) \leqslant a(k) + b(k)\, H\Big(x(k) + \Delta x(k) + \ldots + \Delta^{n-1}x(k) + \sum_{s=0}^{k-1} c(s)(x(s) + \Delta x(s)+\ldots+\Delta^{n-1}x(s))\Big),$$

holds for all $k \in N$. Then for all $k \in N_0$,

(9)
$$\Delta^{n}x(k) \leqslant a(k) + b(k)\, H\Big(p_0(k) + q_0(k)\Omega^{-1}\Big[\Omega\Big(\sum_{s=0}^{k-1} r_0(s)\, H(p_0(s))\Big) + \sum_{s=0}^{k-1} r_0(s)\, H(q_0(s))\Big]\Big),$$

where

(10)
$$p_0(k) = [x(0) + \Delta x(0) + \ldots + \Delta^{n-1}x(0)]\sum_{s=0}^{k-1}[2+c(s)] + \sum_{s=0}^{k-1} a(s) \sum_{t=s+1}^{k-1}[2+c(t)],$$

(11)
$$q_0(k) = \sum_{s=0}^{k-1}[2+c(s)],$$

(12)
$$r_0(k) = b(k)\sum_{t=0}^{k}[2+c(t)]^{-1},$$

and

(13)
$$\Omega(v) = \int_{v_0}^{v}\frac{ds}{H(s)}, \quad v \geqslant v_0 > 0,$$

Ω^{-1} is the inverse function of Ω, and

$$N_0 = \Big\{k \in N : \Omega\Big(\sum_{s=0}^{k-1} r_0(s)\, H(p_0(s))\Big) + \sum_{s=0}^{k-1} r_0(s)\, H(q_0(s)) \in \mathrm{Dom}\,(\Omega^{-1})\Big\}.$$

Proof. Define $m(k)$ as in (3); then we see that (4) is true for all $k \in N$. Using (5), (6) and the fact that $\Delta^{n}x(k) \leqslant a(k)+b(k)\, H(m(k))$, from (8) in (4) we see that the inequality $m(k+1) \leqslant [2+c(k)]\, m(k) + [a(k)+b(k)\, H(m(k))]$ is satisfied for all $k \in N$. Multiplying both sides of the above inequality by

$\prod_{s=0}^{k}[2+c(k)]^{-1}$ and summing up both sides from 0 to $k-1$, we obtain the estimate for $m(k)$ such that

$$m(k) \leqslant p_0(k)+q_0(k)\left(\sum_{s=0}^{k-1} r_0(s)\, H(m(s))\right), \tag{14}$$

for all $k\in N$, where $p_0(k)$, $q_0(k)$, and $r_0(k)$ are as defined in (10), (11), and (12) respectively. Let

$$u(k)=\sum_{s=0}^{k-1} r_0(s)\, H(m(s)), \quad u(0)=0, \tag{15}$$

then $\Delta u(k)=r_0(k)\, H(m(k))$, which in view of (14) and the subadditive and submultiplicative properties of H we see that the inequality

$$\Delta u(k) \leqslant r_0(k)\, H(p_0(k))+r_0(k)\, H(q_0(k))\, H(u(k))$$

is satisfied for all $k\in N$ and can be written as

$$u(k) \leqslant \sum_{s=0}^{k-1} r_0(s)\, H(p_0(s))+\sum_{s=0}^{k-1} r_0(s)\, H(q_0(s))\, H(u(s)). \tag{16}$$

Now fix $\beta\in N_0$; then from (16) we have

$$u(k) \leqslant \sum_{s=0}^{\beta-1} r_0(s)\, H(p_0(s))+\sum_{s=0}^{k-1} r_0(s)\, H(q_0(s))\, H(u(s)), \quad 0\leqslant k\leqslant \beta. \tag{17}$$

Define

$$v(k)=\sum_{s=0}^{\beta-1} r_0(s)H(p_0(s))+\sum_{s=0}^{k-1} r_0(s)\, H(q_0(s))\, H(u(s)), \quad v(0)=\sum_{s=0}^{\beta-1} r_0(s)H(p_0(s)), \tag{18}$$

then from (18) and (17), we obtain

$$\Delta v(k)\leqslant r_0(k)\, H(q_0(k))\, H\,(v(k)), \quad 0\leqslant k\leqslant \beta. \tag{19}$$

From (13) and (19) we have

$$\Omega(v(k+1))-\Omega(v(k))=\int_{v(k)}^{v(k+1)}\frac{ds}{H(s)} \leqslant \frac{\Delta v(k)}{H(v(k))} \leqslant r_0(k)\, H(q_0(k)). \tag{20}$$

for $0\leqslant k\leqslant\beta$. Substituting successively $k=0, 1, 2, \ldots, \beta-1$ in (20), we obtain

$$\Omega(v(\beta))-\Omega(v(0)) \leqslant \sum_{s=0}^{\beta-1} r_0(s)\, H(q_0(s)),$$

which implies

$$v(\beta) \leqslant \Omega^{-1}\left[\Omega\left(\sum_{s=0}^{\beta-1} r_0(s)\, H\, p_0(s)\right)+\sum_{s=0}^{\beta-1} r_0(s)\, H(q_0(s))\right]. \tag{21}$$

Since $\beta\in N_0$ is arbitrary, hence from (21) we have

$$v(k) \leqslant \Omega^{-1}\left[\Omega\left(\sum_{s=0}^{k-1} r_0(s)H p_0(s)\right)+\sum_{s=0}^{k-1} r_0(s)\, H(q_0(s))\right], \tag{22}$$

for all $k\in N_0$. The desired bound in (9) follows from (22), (17), (14) and (8).

A slightly different version of Theorem 2 is established in the following theorem.

Theorem 3. *Let $x(k)$, $\Delta x(k), \ldots, \overset{n}{\Delta} x(k)$, $a(k)$, $b(k)$, $c(k)$, and $H(u)$ be as in Theorem 2, and suppose further that the inequality*

$$\begin{aligned}\overset{n}{\Delta} x(k) \leqslant a(k)+b(k)\,(x(k)+\Delta x(k)+\ldots+\overset{n-1}{\Delta} x(k)+ \\ +\sum_{s=0}^{k-1} c(s)H(x(s)+\Delta x(s)+\ldots+\overset{n-1}{\Delta} x(s))),\end{aligned} \tag{23}$$

holds for all $k\in N$. Then for $k\in N_1$,

$$\begin{aligned}\overset{n}{\Delta} x(k) \leqslant a(k)+b(k)\left((p(k)+q(k)\Omega^{-1}\left[\Omega\left(\sum_{s=0}^{k-1} r(s)Hp(s)\right)+\right.\right. \\ \left.\left.+\sum_{s=0}^{k-1} r(s)\, H(q(s))\right]\right),\end{aligned} \tag{24}$$

where

$$\begin{aligned}p(k)=[x(0)+\Delta x(0)+\ldots+\overset{n-1}{\Delta} x(0)]\prod_{s=0}^{k-1}[2+b(k)]+ \\ +\sum_{s=0}^{k-1} a(s)\prod_{t=s+1}^{k-1}[2+b(t)],\end{aligned} \tag{25}$$

$$q(k)=\prod_{s=0}^{k-1}[2+b(s)], \tag{26}$$

$$r(k)=c(k)\prod_{t=0}^{k}[2+b(t)]^{-1}, \tag{27}$$

and Ω, Ω^{-1} are as defined in Theorem 2, and

$$N_1=\left\{\ k\in N:\Omega\left(\sum_{s=0}^{k-1} r(s)\, H(p(s))\right)+\sum_{s=0}^{k-1} r(s)\, H(q(s))\in \mathrm{Dom}\,(\Omega^{-1})\right\}.$$

The proof of this theorem follows by an argument similar to that in the proof of Theorem 2 with suitable modifications. We omit the details.

Our next result deals with the n-th order finite difference inequality which basically involves the comparison principle.

Theorem 4. *Let $x(k)$, $\Delta x(k), \ldots, \overset{n}{\Delta} x(k)$, $a(k)$, $b(k)$, and $c(k)$ be as in Theorem 2; let $W(k, u)$ be positive, monotonic, nondecreasing in u, $u>0$, for each $k\in N$, and suppose further that the inequality*

$$(28) \qquad \overset{n}{\Delta} x(k) \leqslant a(k) + b(k)\left(x(k)+\Delta x(k)+\ldots+\overset{n-1}{\Delta}x(k)+\sum_{s=0}^{k-1} c(s)\, W(s, \overset{n}{\Delta}x(s))\right),$$

holds for all $k \in N$, *then*

$$(29) \qquad \overset{n}{\Delta} x(k) \leqslant a(k) + b(k)\,(p(k) + q(k)\, R(k)),$$

for all $k \in N$, *where* $p(k)$ *and* $q(k)$ *are as in Theorem 3, and* $R(k)$ *is the solution of*

$$(30) \qquad \Delta R(k) = r(k)\, W(k, a(k)+b(k)[p(k)+q(k)\, R(k)]), \quad R(0) = 0,$$

existing on N, *in which* $r(k)$ *is as defined in Theorem 3.*

Proof. Define

$$(31) \qquad \begin{gathered} m(k) = x(k) + \Delta x(k) + \ldots + \overset{n-1}{\Delta}x(k) + \sum_{s=0}^{k-1} c(s)\, W(s, \overset{n}{\Delta}x(s)), \\ m(0) = x(0) + \Delta x(0) + \ldots + \overset{n-1}{\Delta}x(0); \end{gathered}$$

then

$$(32) \qquad \Delta m(k) = \Delta x(k) + \ldots + \overset{n-1}{\Delta}x(k) + \overset{n}{\Delta}x(k) + c(k) W(k, \overset{n}{\Delta}x(k)).$$

Using the facts that $\Delta x(k)+\ldots+\overset{n-1}{\Delta}x(k) \leqslant m(k)$ from (31) and $\overset{n}{\Delta}x(k) \leqslant a(k) + b(k)\, m(k)$, from (28) in (32), we see that the inequality

$$m(k+1) \leqslant [2+b(k)]\, m(k) + [a(k)+c(k) W(k, a(k)+b(k)\, m(k))]$$

is satisfied for all $k \in N$, which implies

$$(33) \qquad m(k) \leqslant p(k)+q(k)\left(\sum_{s=0}^{k-1} r(s) W(s, a(s) + b(s)\, m(s))\right),$$

where $p(k)$ and $q(k)$ are as in Theorem 3. Define

$$u(k) = \sum_{s=0}^{k-1} r(s)\, W(s, a(s) + b(s)\, m(s)), \qquad u(0) = 0,$$

then $\Delta u(k) = r(k) W(k, a(k)+b(k)\, m(k))$, which in view of (33) implies

$$(34) \qquad \Delta u(k) \leqslant r(k)\; W(k, a(k) + b(k)\, [p(k) + q(k)\, u(k)]).$$

A suitable application of Theorem 1 given in [3] to (34) and (30) yields

$$(35) \qquad u(k) \leqslant R(k), \quad k \in N,$$

where $R(k)$ is the solution of (30) such that $R(0) = u(0) = 0$. Now from (35), (33) and (28) the desired bound in (29) follows.

We note that the usefulness of the inequalities established in Theorems 1–4 becomes apparent if we consider that $x(0)$, $\Delta x(0), \ldots, \Delta^{n-1} x(0)$, $a(k)$, $b(k)$ and $c(k)$ are known and $x(k)$, $\Delta x(k), \ldots, \overset{n}{\Delta}x(k)$ are unknown functions, i.e. the inequalities established in the above theorems give us the bounds in terms of the known functions which majorizes $\overset{n}{\Delta}x(k)$ and consequently $x(k)$ for all $k \in N$.

In concluding this paper, we note that the finite difference inequalities established in this paper can be used to study the boundedness, uniqueness, continuous dependence and other problems of a class of n-th order sum-difference equation of the form

$$(36) \qquad \begin{gathered} \overset{n}{\Delta}x(k) = f(k) + F\left[k, x(k), \ \Delta x(k), \ldots, \overset{n-1}{\Delta}x(k), \ \sum_{s=0}^{k-1} G(k, s, x(s), \ \Delta x(s), \ldots \overset{n-1}{\Delta}x(s))\right], \\ \overset{r}{\Delta}x(0) = x_0^r, \quad 0 \leqslant r \leqslant n-1, \end{gathered}$$

under some suitable conditions on the functions involved in (36).

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Department of Mathematics and Statistics
Marathwada University
Aurangabad 431004
(Maharashtra) India