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A K. MORITA TYPE THEOREM FOR CLOSURE ALGEBRAS

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0. **Introduction.** In 1975 Kiiti Morita showed, [1], that the notion of *shape map of topological spaces* can be described using also the ANR-system approch. To each topological space he associates a $\mathcal{H}$-CW-system, that is a homotopical inverse system of CW-complexes (or of ANR-spaces, [8]). In 1977, M. R. Bunjatov defined [3] the basic concepts of the homotopy theory of continuous homomorphisms of closure algebras.

In this note we define the ANR-algebras, which are special C-algebras that abstract and generalize the ANR-spaces and then we prove a K. Morita type theorem for σ-complete closure algebras (Theorem 6.5). This theorem is used in [9] to give an introduction in the shape theory of closure algebras.

The needed notations and the results on closure algebras, are those of the paper [5] of Roman Sikorski.

1. **ANR-algebras.** Denote by σ-$\mathcal{CB}$ the category of the σ-complete closure algebras and the σ-complete continuous homomorphisms and let $\mathcal{CB}$ be the subcategory of σ-$\mathcal{CB}$ of the complete closure algebras with the complete continuous homomorphisms.

1.1. Definition. *A Boolean ideal J of a closure algebra A is called a closure ideal if for every $x \in J$ its closure $\bar{x}$ is also in J.*

2.1. Definition. *A morphism $h : B \to A$ in the category $\mathcal{CB}$ is called a homeomorphism in preimage, if there exist a closure complete ideal J of the closure algebra B and a continuous complete map $k : A \to J$ (i.e.k preserves all suprema and all infima and $\overline{k(a)} \leqslant k(\bar{a})$, for every $a \in A$) such that :*

(i) *$h(J) = A$,* (ii) *$k \circ (h/J) = 1_J$, $(h/J) \circ k = 1_A$,* (iii) *$c = \vee J$ is a closed element of the closure algebra B. An ideal J which satisfies the above conditions is denoted by $h_c^{-1}(A)$.*

Let X be an arbitrary topological space. Denote $\sigma(X)$ the closure algebra of all subsets of X.

3.1. Proposition. *Let $f : X \to Y$ be a continuous map of topological spaces and let $h = \sigma(f) : \sigma(Y) \to \sigma(X)$ be the continuous homomorphism of closure algebras induced by f. Then, f is a homeomorphism of X on a closed subset of Y iff h is a homeomorphism in preimage.*

Proof. Let $f : X \to Y_0 \subseteq Y$ be a homeomorphism, with Y_0 a closed subset of Y. Then $J = \sigma(Y_0)$ is a closure complete ideal of the field $\sigma(Y)$ and together with the map $k=\sigma(f^{-1}):\sigma(X)\to\sigma(Y_0)$ satisfy 2.1 with respect to $h=\sigma(f)$.

Conversely, if $h=\sigma(f)$ is a homeomorphism in preimage, then, the set $c=\bigcup h_c^{-1}(\sigma(X))$ of the space Y is closed and the map $g:c\to X$, given by $g(y)=h(\{y\})=f^{-1}(y)$ is well-defined and continuous. Moreover, g is the inverse of the map $f:X\to c$, such that, this last map is a homeomorphism.

4.1. Let A be a closure algebra and let $a\in A$. Denote by A_a the principal ideal generated by a, that is $A_a=\{x\in A/x\leqslant a\}=\{x\in A/x\wedge a=x\}$.

5.1. **Definition.** *If a, $b\in A$, then, a map $f:A_a\to A_b$, with $f(a)=b$ is called continuous if, for every element $x\in A_a$ we have $b\wedge\overline{f(x)}\leqslant\overline{f(x\wedge a)}$.* One can prove easily the following lemma:

6.1. **Lemma.** *Let A be a closure algebra and let c be a closed element in A and o an open neighborhood of c, i.e. an open element in A such that $c<o$. Let $j:A_0\to A_c$ be the map defined by $j(x)=x\wedge c$. Then, the following assertions hold:*

(i) *A_c is a closure complete ideal of A,*
(ii) *A_0 is a complete ideal of A and if $y\in A_0$, then* Int $y\in A_0$,
(iii) *j is a continuous and complete map.*

7.1. **Definition.** *Let A be a closure algebra and let c be a closed element of A. We say that c is an absolute neighborhood retract in A, if there exist an open neighborhood o of c and a complete homomorphism $r':A_c\to A_0$, such that:*

(i) $r'(c)=o$,
(ii) *r' is continuous,*
(iii) $jor'=1_{Ac}$.

If $o=1$ (i.e. the unity element of A), then c is called an absolute retract of A.

8.1. **Proposition.** *A closed subset X_0 of a topological space X is an absolute (neighborhood) retract in X iff X_0 is an absolute (neighborhood) retract in the closure algebra $\sigma(X)$.*

Proof. Let X_0 be a closed subset of X and let O be an open neighborhood of X_0 in X, such that X_0 is a retract of O by the retraction $r:O\to X_0$. This retraction induces the continuous homomorphism $r'=\sigma(r):\sigma(X_0)\to\to\sigma(O)$. For X_0, $O\in\sigma(X)$, we have $\sigma(X_0)=A_{X_0}$ and $\sigma(O)=A_O$ and are verified the conditions of 7.1.

Conversely, the conditions of 7.1 involve that the map r' defines a continuous map $r:O\to X_0$ such that $r/X_0=1_{X_0}$ (if $y\in O$ then $y\in O\wedge\overline{r'(X_0)}\subseteq\subseteq r'(\overline{X}_0\cap X_0)=r'(X_0)$).

9.1. We recall from [5] that a closure algebra A is called a C-algebra if A admits a C-basis, i.e. a countable sequence $\{r_n\}$ of open elements of A such that every open element $g\in A$ is the sum of all elements r_n such that $\overline{r}_n<g$. The notion of C-algebra is a generalization of that of separable metrizable space in the sense that a topological space X is separable metrizable space iff the field $\sigma(X)$ is a C-algebra.

10.1. **Definition.** *A complete C-algebra (C-field) A is called an ANR-algebra (ANR-field) if for every complete C-algebra (C-field) B and every homeomorphism in preimage, $h:B\to A$, the closed element $c=h_c^{-1}(A)$ is an absolute neighborhood retract in B. If A is only σ-complete C-algebra, we say σ-ANR-algebra (σ-ANR-field).*

11.1. **Theorem.** *A topological space X is an ANR-space for the class of separable metrizable spaces iff $\sigma(X)$ is an ANR-field.*

Proof. Suppose that $\sigma(X)$ is an ANR-field. It follows that $\sigma(X)$ is a C-algebra and therefore [5, Theorem 10.1], X is a metrizable separable space. Then, let Y be another metrizable separable space and let $f:X\to Y$ be a homeomorphism on a closed subset $f(X)$ of Y. By 3.1 the induced map $\sigma(f):\sigma(Y)\to\sigma(X)$ is a continuous homomorphism in preimage. Since $\sigma(X)$ is an ANR-field and because $\sigma(Y)$ is a C-field, it follows [5, Theorem 10.1] that $c=\vee h_c^{-1}(\sigma(X))$ is an absolute neighborhood retract in $\sigma(Y)$. By 8.1 this implies that $f(X)$ is a closed subset of Y and it is an absolute neighborhood retract in Y.

Conversely, suppose that X is an ANR for the class of separable metrizable spaces. It follows that $\sigma(X)$ is a C-field. Now, let B be an arbitrary complete C-field and let $h:B\to\sigma(X)$ be a homeomorphism in preimage. From [5] there exist a metrizable separable space Y and a subfield B' of the field $\sigma(Y)$ such that, defining for an element $b\in B'$ its closure $\overline{b}$ as the intersection of all Borel sets of Y which contain b, one obtaines a C-field isomorphic to B. We define $f:X\to Y$ by $f(x)=k(\{x\})$, where k is the map of 2.1. Then, f is well-defined and it is continuous. Moreover, f is a homeomorphism on the closed image $f(X)$. Since X in an ANR with respect to the class of separable metrizable spaces, it follows that $f(X)$ is an absolute neighborhood retract in Y. This implies, by 8.1, that $c=\vee f(X)=\vee h_c^{-1}(\sigma(X))$ is an absolute neighborhood retract in $\sigma(Y)$. But, since $h_c^{-1}(\sigma(X))$ is an ideal of B, it follows that c is an absolute neighborhood retract in B. Thus, $\sigma(X)$ is an ANR-field.

2. The Bunjatov spectrum representation theorem and some related results

1.2. For an object A of the category $\mathcal{T}\mathcal{B}$, denote by $\Sigma(A)$ the set of all partitions of unity of the closure algebra A, with the following partial order: if λ, $\mu\in\Sigma(A)$, then $\lambda\prec\mu$ iff for every $x\in\mu$ there exists $y\in\lambda$ such that $x\leqslant y$ (that is $\lambda\prec\mu$ iff μ is a refinement of λ). Since the sets λ and μ are disjoint, the relation $\prec$ implies a map $p_\lambda^\mu:\mu\to\lambda$.

Let $O(A)$ be the set of open elements of the closure algebra A and let $\lambda\in\Sigma(A)$. We consider the class $O_\lambda^A=\{G\subset\lambda/\vee G\in o(A)\}$. This is the class of the open sets of some topology of λ. If λ, $\mu\in\Sigma(A)$ and if $\lambda\prec\mu$, then, by considering on μ the topology O_μ^A and on λ the topology O_λ^A, the map p_λ^μ, defined above, is continuous. In fact, let be $G\in O_\lambda^A$, that is $G=\{y\in\lambda/\vee y\in\in O(A)\}$. We have $(p_\lambda^\mu)^{-1}(G)=\{x\in\mu/\exists\ y\in G,\ x\leqslant y\}$. But, $\vee(p_\lambda^\mu)^{-1}(G)=\vee G$, since the inequality $\vee(p_\lambda^\mu)^{-1}(G)\leqslant\vee G$ is obvious and, if $y\in G$, then $y=\bigvee_{x\in\mu}(\vee x)\wedge y=\bigvee_{x\in(p_\lambda^\mu)^{-1}(y)}x$. Hence $\vee(p_\lambda^\mu)^{-1}(G)\in O(A)$ and $(p_\lambda^\mu)^{-1}(G)\subset\mu$, such that $(p_\lambda^\mu)^{-1}(G)\in O_\mu^A$. Moreover, the system $\{(\lambda,O_\lambda^A),p_\lambda^\mu\}_{\lambda,\mu\in\Sigma(A)}$ is an inverse spectrum of topological spaces.

2.2. A topological space X is the inverse limit of the inverse spectrum of topological spaces $\{(\lambda,O_\lambda^{\sigma(X)}),p_\lambda^\mu\}_{\lambda,\mu\in\Sigma(\sigma(X))}$. For a disjoint covering λ of the space X, we have $\lambda\prec X$ and the projection $p_\lambda^X=p_\lambda:X\to\lambda$ is defined by $p_\lambda(x) = \lambda_x$ iff $x \in \lambda_x$. By applying the functor σ to the inverse spectrum $\{(\lambda, O_\lambda^{\sigma(X)}), p_\lambda^\mu\}$, we obtain a direct spectrum of closure algebras $\{\sigma(\lambda), q_\mu^\lambda = \sigma(p_\lambda^\mu)\}$ and the maps $q_\lambda = \sigma(p_\lambda) : \sigma(\lambda) \to \sigma(X)$ are defined by $q_\lambda(N) = \bigcup_{j \in J} N_j$, for $N = \{N_j\}_{j \in J} \in \sigma(\lambda)$.

From 2.2 we obtain:

3.2. The field $\sigma(X)$ is the direct limit of the direct spectrum $\{\sigma(\lambda), q_\mu^\lambda\}_{\lambda,\mu \in \Sigma(\sigma(X))}$ of closure algebras. Proposition 3.2 can be generalized for an arbitrary closure algebra, namely:

4.2. **Bunjatov's Theorem,** [4, Theorem 1]. *A closure algebra A is the direct limit of the direct spectrum $\{\sigma(\lambda), q_\mu^\lambda = \sigma(p_\lambda^\mu)\}_{\lambda,\mu \in \Sigma(A)}$, in the category $\mathcal{C}\mathcal{B}$.*

5.2. If A is only a σ-complete closure algebra, we denote by $\Sigma_c(A)$ the set of all at most countable partitions of unity of A. For a $\lambda \in \Sigma_c(A)$ we can consider O_λ^A and this is also a topology on λ. All the other results of this section remain true when we replace a complete closure algebra with a σ-complete closure algebra and the set $\Sigma(A)$ with the set $\Sigma_c(A)$. But the proofs in this case are more complicated. Thus, we can state:

6.2. **Theorem.** *Let A be a σ-complete closure algebra and let $\Sigma_c(A)$ be the set of all at most countable partitions of unity of A. Then, A is the direct limit of direct spectrum $\{\sigma(\lambda), q_\mu^\lambda = \sigma(p_\lambda^\mu)\}_{\lambda,\mu \in \Sigma_c(A)}$, in the category σ-$\mathcal{C}\mathcal{B}$.*

Proof. We define the maps $q_\lambda : \sigma(\lambda) \to A$ by $q_\lambda(N) = \vee N$, for $N \in \sigma(\lambda)$, $\lambda \in \Sigma_c(A)$. We prove that q_λ is a σ-complete continuous homomorphism of closure algebras. If $N_1, N_2 \in \sigma(\lambda)$, then $q_\lambda(N_1 \cup N_2) = \vee(N_1 \cup N_2) = (\vee N_1) \vee (\vee N_2) = q_\lambda(N_1) \vee q_\lambda(N_2)$. If $N \in \sigma(\lambda)$, we have $q_\lambda(\complement_\lambda N) = \vee \complement_\lambda N$, but $(\vee N) \vee (\vee \complement_\lambda N) = \vee \lambda = 1$, and therefore $\vee \complement_\lambda N = (\vee N)'$ (if $x \in A$, then x' is the complement of x), that is $q_\lambda(\complement_\lambda N) = (q_\lambda(N))'$. It is easy to see that q_λ is σ-complete. Then, if $G \in O_\lambda^A$, $q_\lambda(G) = \vee G$ is an open element in A, and, by [5, Theorem 5.2], it follows that q_λ is continuous.

Now we have the commutative diagram

$$\begin{tikzcd}
\sigma(\lambda) \arrow[r, "q_\lambda"] \arrow[d, "q_\mu^\lambda"'] & A \\
\sigma(\mu) \arrow[ur, "q_\mu"'] &
\end{tikzcd}$$

for every $\lambda, \mu \in \Sigma_c(A)$, with $\lambda \prec \mu$. In fact, if $N \in \sigma(\lambda)$, then $q_\mu(q_\mu^\lambda(N)) = \vee q_\mu^\lambda(N) = \vee (p_\lambda^\mu)^{-1}(N) = \vee N$, as we saw at the beginning of this section. Therefore $q_\mu(q_\mu^\lambda(N)) = q_\lambda(N)$.

Now, let B be an arbitrary σ- complete closure algebra and let φ_λ: $\sigma(\lambda) \to B$ be a σ- complete continuous homomorphism, for every $\lambda \in \Sigma_c(A)$, such that the diagram

$$\begin{tikzcd}
\sigma(\lambda) \arrow[r, "\varphi_\lambda"] \arrow[d, "q_\mu^\lambda"'] & A \\
\sigma(\mu) \arrow[ur, "\varphi_\mu"'] &
\end{tikzcd}$$

for $\lambda, \mu \in \Sigma_c(A)$ $\lambda \prec \mu$ be commutative.

For an element $a \in A$, let λ_a be the partition of unity $\lambda_a = \{a, a'\}$, where a' is the complement of a. Obviously $\lambda_a \in \Sigma_c(A)$ and let be $\varphi_a = \varphi_{\lambda_a}$ Then, we define $\varphi : A \to B$ by $\varphi(a) = \varphi_a(a)$. First we prove that the diagram

$$\begin{tikzcd}
& \sigma(\lambda) \arrow[dl, "\varphi_\lambda"'] \arrow[dr, "q_\lambda"] & \\
B & & A \arrow[ll, "\varphi"']
\end{tikzcd}$$

are commutative, for every $\lambda \in \Sigma_c(A)$. For this, let $N \in \sigma(\lambda)$. Then, we have $\{\vee N, (\vee N)'\} \prec \lambda$ since if $x \in \lambda$ and $x \in N$, $x \leqslant \vee N$, while if $x \notin N$, then $x \leqslant (\vee N)'$ because $x = (x \wedge (\vee N)) \vee (x \wedge (\vee N)')$. Thus we have the commutative diagram:

$$\begin{tikzcd}
& \sigma(\lambda) \arrow[dl, "\varphi_\lambda"'] \arrow[dr, "q_\lambda"] & \\
B & & A \\
& \sigma(\lambda_a) \arrow[ul, "\varphi_a"] \arrow[ur, "q_a"'] \arrow[uu, "q_a^\lambda"'] &
\end{tikzcd}$$

where $a = \vee N$ and $q_a^\lambda = q_{\lambda_a}^\lambda$. Hence we obtain $\varphi q_\lambda(N) = \varphi(\vee N) = \varphi(a) = \varphi_a(a) = \varphi_\lambda q_a^\lambda(a) = \varphi_\lambda q_a^\lambda(\vee N) = \varphi_\lambda(N)$.

Now, we prove that φ is a σ-complete continuous homomorphism. If $a, b \in A$, then $\lambda_{a \vee b} = \{a \vee b, a' \wedge b'\}$ and $\varphi(a \vee b) = \varphi_{a \vee b}(a \vee b)$. If we set $\lambda_{ab} = \lambda_a \wedge \lambda_b \wedge \lambda_{a \vee b}$, then we have $\lambda_a, \lambda_b, \lambda_{a \vee b} \prec \lambda_{ab}$ and therefore $\varphi(a) \vee \varphi(b) = \varphi_a(a) \vee \varphi_b(b) = \varphi_{\lambda_{ab}} q_{ab}^a(a) \vee \varphi_{\lambda b}^b q_{ab}^b(b) = \varphi_{\lambda_{ab}}(q_{ab}^a(a) \vee q_{ab}^b(b)) = \varphi_{\lambda_{ab}}(q_{ab}^{a \vee b}(a \vee b)) = \varphi_{a \vee b}(a \vee b) = \varphi(a \vee b)$. Analogously, it follows that $\varphi(\bigvee_{(\alpha)} a_\alpha) = \bigvee_{(\alpha)} \varphi(a_\alpha)$, for (α) a set at most countable, by using the following partition of unity $(\bigvee_{(\alpha)} \lambda_{a_\alpha}) \wedge (\lambda_{\bigvee_{(\alpha)} a_\alpha})$, which belongs to $\Sigma_c(A)$. Further on, if $a \in A$, then $\lambda_{a'} = \{a', a\} = \lambda_a$ and therefore $\varphi(a') = \varphi_a'(a') = \varphi_a(a') = (\varphi_a(a))' = \varphi(a)'$. Hence, φ is a σ-homomorphism. It is also continuous since if $a \in A$ is an open element, then $\varphi(a) = \varphi_a(a)$ is an open element in B, because φ_a is continuous and the element a is open in λ_a and therefore in $\sigma(\lambda_a)$. The uniqueness of the homomorphism φ follows because if ψ is another homomorphism with the same properties as φ, and if $a \in A$, then $\psi q_{\lambda_a}(a) = \varphi_{\lambda_a}(a)$, that is $\psi(a) = \varphi_a(a) = \varphi(a)$. This completes the proof of the theorem.

The following theorem has an analogous proof as 6.2.

7.2. **Theorem.** *Let A be a σ-complete C-algebra with a C-basis $\{r_n\}$. Let $\Sigma_X(A)$ be the set of all partitions of unity of A of the form $(\wedge \lambda_{r_k}) \wedge \lambda_{\vee r_k}$ for $\{r_k\}$ a subset of $\{r_n\}$. Then $A = \lim\limits_{\overrightarrow{\lambda \in \Sigma_X(A)}} \sigma(\lambda)$, in the category σ-$\mathcal{TB}$.*

8.2. *Remark.* In the case of B u n j a t o v's Theorem, 4.2, the universal property of A with respect to the spectrum $\{\sigma(\lambda),\ q^\lambda_\mu = \sigma(p^\mu_\lambda)\}_{\lambda,\mu \in \Sigma(A)}$ admits a different proof than that of 6.2. Precisely, in this case there exists λ_0 the finest partition of unity of A and let $j: A \to \sigma(\lambda_0)$ be the map defined by $j(a) = \{x \in \lambda_0 / x \wedge a \neq 0\}$. It is easy to see that $j(\vee a_\alpha) = \bigcup j(a_\alpha)$. Then, $j(a') = j(a)'$; in fact, if for $x \in \lambda_0$ we have $x \wedge a' \neq 0$, it follows $x \wedge a = 0$, otherwise if it would exist $x_0 \in \lambda_0$ such that $x_0 \wedge a \neq 0$ and $x_0 \wedge a' \neq 0$, then the set $\{x / x \in \lambda_0,\ x \neq x_0\} \cup \{x_0 \wedge a,\ x_0 \wedge a'\}$ involube a partition of unity of A and so a refinement of λ_0; thus $j(a') = \{x \in \lambda_0 / x \wedge a' \neq 0\} = \{x \in \lambda_0 / x \wedge a = 0\} = j(a)'$. Hence, j is a complete monomorphism. It is also continuous, because, if a is an open element in A, then, by the equality $\vee j(a) = a$, it follows that $j(a)$ is an open element in λ_0 and therefore in $\sigma(\lambda_0)$. Now, if $\varphi_\lambda : \sigma(\lambda) \to B$, are completely by continuous homomorphisms, such that $\varphi_\mu \circ q^\lambda_\mu = \varphi_\lambda$, for λ, $\mu \in \Sigma(A)$, then we can define $\varphi : A \to B$ by $\varphi = \varphi_{\lambda_0} \circ j$[1]).

9.2. *Remark.* In the case of B u n j a t o v's Theorem, 4.2, it also results that $q_{\lambda_0} : \sigma(\lambda_0) \to A$ is a complet continuous epimorphism.

We give an application to 8.2, namely:

10.2. **Theorem.** *If A is an ANR-algebra and λ_0 is the finest partition of unity of A, then $\sigma(\lambda_0)$ is an ANR-field.*

Proof. Let $\{r_n\}$ be a C-basis in A. If $j: A \to \sigma(\lambda_0)$ is the continuous monomorphism of 8.2, then, we can consider the countable set $\{R_n = j(r_n)\}$ in $\sigma(\lambda_0)$ which is a C-basis. Then, if $h: B \to \sigma(\lambda_0)$ is a homeomorphism in preimage, $q_{\lambda_0} \circ h : B \to A$ is a homeomorphism in preimage and $\vee (q_{\lambda_0} \circ h)^{-1}_c(A) = \vee h^{-1}_c(\sigma(\lambda_0))$.

3. **Homotopic homomorphisms in the category $\mathcal{TB}$.** 1.3. If $h: A \to B$ is a morphism in the category $\mathcal{TB}$ and if $\lambda \in \Sigma(A)$, then $h(\lambda) \in \Sigma(B)$ and moreover the map $h_\lambda = (h/\lambda)^{-1}: h(\lambda) \to \lambda$ is well defined. If we consider on $h(\lambda)$ and on λ the topologies $O^B_{h(\lambda)}$ and O^A_λ respectivelly, then, the map h_λ is continuous with respect to these topologies. In fact, if $G \in O^A_\lambda$, then $h^{-1}_\lambda(G) = h(G)$. We have $\vee h^{-1}_\lambda(G) = \vee h(G) = h(\vee G)$. But, since $\vee G$ is an open element in A and h is a continuous homomorphism, it follows [5, Theorem 5.2], that $h(\vee G)$ is an open element in B, that is $h^{-1}_\lambda(G) \in O^B_{h(\lambda)}$.

Now, we recall from M. R. B u n j a t o v, [3]:

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[1]) I have no information on the proof given by M. R. Bunjatov for 4.2.

2.3. **Definition.** [3. Definition 1]. *Two morphisms $h_0, h_1 : A \to B$, in the category $\mathcal{TB}$ are homotopic if for any $\lambda \in \Sigma(A)$, there exists $\mu \in \Sigma(B)$ satisfying the conditions:*

(i) $\nu_0 = h_0(\lambda) \prec \mu$ and $\nu_1 = h_1(\lambda) \prec \mu$,

(ii) $[h_{0\lambda} \circ p^\mu_{\nu_0}] = [h_{1\lambda} \circ p^\mu_{\nu_1}]$, *where $[f]$ denotes the homotopy class of the continuous map f of topological spaces.*

The homotopy relation of morphisms, in the category $\mathcal{TB}$, is an equivalence relation and it is compatible with the composition of morphisms. If the morphisms h_0, h_1, in the category $\mathcal{TB}$, are homotopic, one denotes $[h_0] = [h_1]$ or $h_0 \simeq h_1$.

3.3. **Proposition.** *Let $f_0, f_1 : X \to Y$ be two continuous maps of topological spaces and let $h_i = \sigma(f_i) : \sigma(Y) \to \sigma(X)$, $i = 1, 2$, be the induced homomorphisms of closure algegras. Then, the maps f_0, f_1 are homotopic iff the continuous homomorphisms h_0, h_1 are homotopic.*

Proof. First, we suppose that $[h_0] = [h_1]$ from 2.3. We take for the closure algebra $\sigma(Y)$, $\lambda \in \Sigma(\sigma Y))$, with $\lambda = \{\{y\} / y \in Y\}$. Then, the set λ can be identified with Y and the topology $O^{\sigma(Y)}_\lambda$ with the topology of Y, such that we have $(\lambda, O^{\sigma(Y)}_\lambda) \equiv Y$, a topological isomorphism. Then, $h_0(\lambda) = f_0^{-1}(\lambda) = \{\{x\} / x \in X\}$ and $h_1(\lambda) = f_1^{-1}(\lambda) = \{\{x\} / x \in X\}$. By the condition (i) of 2.3, it follows necessarily $\mu = h_0(\lambda) = h_1(\lambda)$. Moreover $(\mu, O^{\sigma(x)}_\mu) \equiv X$. Using the notations of 2.3 for our case, we have $h_{0\lambda} \circ p^\mu_{\upsilon_0} = h_{0\lambda} = f_0$ and $h_{1\lambda} \circ p^\mu_{\upsilon_1} = h_{1\lambda} = f_1$. These equalities and the condition (ii) of 2.3 imply $[f_0] = [f_1]$.

Conversely, suppose $[f_0] = [f_1]$. Let $\lambda \in \Sigma(\sigma(Y))$, with $\lambda = \{M_j\}_{j \in J}$. Then, $\nu_0 = h_0(\lambda) = \{f_0^{-1}(M_j)\}_{j \in J}$ and $\nu_1 = h_1(\lambda) = \{f_1^{-1}(M_j)\}_{j \in J}$. Let $\mu = \{N_{jj'}\}_{(j,j') \in J \times J}$, where $N_{jj'} = f_0^{-1}(M_j) \cap f_1^{-1}(M_{j'})$. We have $\mu \in \Sigma(\sigma(X))$ and the condition (i) of 2.3 is obviously satisfied. We consider the diagram

$$\begin{tikzcd}
\mu \arrow[r, "p^\mu_{\nu_i}"] & \nu_i \arrow[r, "h_{i\lambda}"] & \lambda \\
 & X \arrow[ul, "w"] \arrow[u, "u_i"] \arrow[r, "f_i"] & Y \arrow[u, "v"']
\end{tikzcd}$$

$i = 0, 1$, where: for $x \in X$, $u_i(x) = f_i^{-1}(M_j)$ iff $x \in f_i^{-1}(M_j)$; for $y \in Y$, $v(y) = M_j$ iff $y \in M_j$ and, for $x \in X$, $w(x) = N_{jj'}$ iff $x \in N_{jj'}$. The above diagram is commutative and all morphisms of this are continuous. We have the following equalities: $h_{i\lambda} \circ p^\mu_{\nu_i} \circ w = v f_i$. Let $H: X \times I \to Y$ be a homotopy of the maps $f_0, f_1 : X \to Y$. Then, $v \circ H$ is a homotopy of the maps $v f_0, v f_1 : X \to \lambda$, that is a homotopy between the continuous maps $h_{0\lambda} \circ p^\mu_{\upsilon_0} \circ w$ and $h_{1\lambda} \circ p_{\upsilon_1} \circ w$. Now, we observe that, if $N_{jj'} \in \mu$, then the restrictions $h_{0\lambda} \circ p^\mu_{\upsilon_0} \circ w / N_{jj'}$ and $h_{1\lambda} \circ p^\mu_{\upsilon_1} \circ w / N_{jj'}$ are constant maps. In fact, if $x, x' \in N_{jj'}$, then $w(x) = w(x') = N_{jj'}$ and therefore $(h_{i\lambda} \circ p^\mu_{\upsilon_i} \circ w)(x) = (h_{i\lambda} \circ p^\mu_{\upsilon_i} \circ w)(x')$, $i = 1, 2$. Then, we can suppose (if necessary, we take instead of $X_{jj'}$ a finer partition, for example $\{\{x\}/x\in X\}$) that, for any $X_{jj'}\in\mu$, we have $vH(x,t)=vH(x',t)$, $\forall x,x'\in X_{jj'}$, $\forall t\in I$. In this way, we define a continuous map $H':\mu\times I\to\lambda$ by $H'(X_{jj'},t)=vH(x,t)$, for a point $x\in X_{jj'}$. The map H' is a homotopy between the maps $h_{0\lambda}\circ p^{\mu}_{\mathcal{V}_0}$ and $h_{1\lambda}\circ p^{\mu}_{\mathcal{V}_1}$. This completes the proof of the proposition.

4.3. **Theorem.** *Let X be a topological space having the homotopy type of an ANR-space, for the class of separable metrizable spaces. Then, the field $\sigma(X)$ has the homotopy type of an ANR-field.*

Proof. Let X be homotopy equivalent with the space Y which is an ANR, for the class of separable metrizable spaces. Then, by 3.3 the fields $\sigma(X)$ and $\sigma(Y)$ are homotopy equivalent. But, by 11.1, $\sigma(Y)$ is an ANR-field.

5.3. If $h_0,h_1:A\to B$ are two morphisms in the category $\sigma\mathcal{FB}$, we say that these morphisms are σ-homotopic (or Σ_c-homotopic) if the conditions of 2.3 are verified when we replace $\Sigma(A)$ by $\Sigma_c(A)$ and $\Sigma(B)$ by $\Sigma_c(B)$.

6.3. *Remark.* Two complete homomorphisms which are homotopic, obviously are σ-homotopic.

4. **$\mathcal{HB}$-systems.** 1.4. For a subcategory $\mathcal{C}$ of the category Top, we denote $\mathcal{HC}$ the corresponding subcategory to $\mathcal{C}$ in the homotopy category of topological spaces. Also, for a subcategory $\mathcal{C}'$ of the category $\mathcal{FB}$ (σ-$\mathcal{FB}$) denote by $\mathcal{HC}'$ the corresponding subcategory in the homotopy category of complete closure algebras (respectively of σ-complete closure algebras).

2.4. **Definition.** a) *A directed projective system $\mathbf{X}=\{X_j,[p_{jj'}],J\}$ in a subcategory $\mathcal{HC}$ of the category $\mathcal{H}$ Top is called a $\mathcal{H}$-topological system.*

b) *A directed inductive system $\mathbf{A}=\{A_j,[q_{jj'}],J\}$ in a subcategory $\mathcal{HC}'$ of the category $\mathcal{HFB}$ (or $\mathcal{H}\sigma-\mathcal{CB}$) is called a $\mathcal{HC}'$-Boolean system.*

Using some previous results, we obtain:

3.4. **Proposition.** *If $\mathbf{X}=\{X_j,[p_{jj'}],J\}$ is a $\mathcal{H}$ Top-topological system, then $\sigma(\mathbf{X})=\{\sigma(X_j),[\sigma(p_{jj'})],J\}$ is well defined and is a $\mathcal{HFB}$-Boolean system.*

4.4. **Definition.** K. Morita, [1]. *Let $\mathcal{HC}$ be a subcategory of the category $\mathcal{H}$ Top and let $\mathbf{X}=\{X_j,[p_{jj'}],J\}$ be a $\mathcal{H}$ Top-topological system. We say that $\mathbf{X}$ is $\mathcal{HC}$-associated with a topological space X, if there are continuous maps $p_j:X\to X_j$, for $j\in J$ such that the following conditions are satisfied:*

(i) $[p_{jj'}][p_{j'}]=[p_j]$, *if* $j<j'$.

(ii) *For any continuous map $f:X\to Q$ with Q an object in $\mathcal{HC}$, there exist $j\in J$ and a continuous map $f_j:X_j\to Q$ such that $[f]=[f_j][p_j]$.*

(iii) *For $j\in J$ and for two continuous maps $f_j,g_j:X_j\to Q$, with Q an object in $\mathcal{HC}$ such that $[f_j][p_j]=[g_j][p_j]$, there exists $j'\in J$ with $j<j'$ such that $[f_j][p_{jj'}]=[g_j][p_{jj'}]$.*

5.4. By K. Morita's Theorem [1, Theorem 1.3] and [2, Theorem 4.3], for any topological space X, there exists an $\mathcal{H}$ ANR-topological system $\mathcal{H}ANR$ associated with X.

6.4. **Definition.** *Let $\mathcal{HC}'$ be a subcategory of the category $\mathcal{HFB}$ and $\mathbf{A}=\{A_j,[q_{jj'}],J\}$ be a $\mathcal{HFB}$-Boolean system. We say that $\mathbf{A}$ is $\mathcal{HC}'$-associated with a complete closure algebra A, if there are complete continuous homomorphisms $q_j:A_j\to A$, for $j\in J$, such that the following conditions are verified:*

(i) $[q_{j'}][q_{jj'}]=[q_j]$, *if* $j<j'$.

(ii) *For any complete continuous homomorphism $h:B\to A$, with B an object in $\mathcal{HC}'$, there exist $j\in J$ and a complete continuous homomorphism $h_j:B\to A_j$, such that $[q_j][h_j]=[h]$.*

(iii) *For $j\in J$ and for two complete continuous homomorphisms h_j, $k_j:B\to A_j$, with B an object in $\mathcal{HC}'$, such that $[q_j][h_j]=[q_j][k_j]$, there exists $j'\in J$, with $j<j'$ such that $[q_{jj'}][h_j]=[q_{jj'}][k_j]$.*

7.4. If we replace in Definition 6.4 the completness with σ-completness and the homotopy by σ-homotopy, we obtain a corresponding definition for a homotopical system associated with a σ-complete closure algebra.

5. **The main result.** 1.5. Denote by $\mathcal{F}$ the category of σ-complete closure fields and by σ-C the category of σ-complete C-algebras.

2.5. **Theorem.** *Let A be a σ-complete closure algebra. Then, the $\mathcal{H}\sigma$-$\mathcal{F}$-Boolean system $\{\sigma(\lambda),[q^{\lambda}_{\mu}],\Sigma_c(A)\}$ is $\mathcal{H}\sigma$-C-associated with the closure algebra A.*

Proof. At first, by 6.2, we remark that the considered system is a $\mathcal{H}\sigma\mathcal{F}$ Boolean system. Then, by the same Theorem 6.2, there are σ-complete continuous homomorphisms $q_\lambda:\sigma(\lambda)\to A$, such that $q_\mu\circ q^{\lambda}_{\mu}=q_\lambda$, for every $\lambda,\mu\in\Sigma_c(A)$, when $\lambda\prec\mu$. Now, let $h:B\to A$ be a σ-complete continuous homomorphism, with B a σ C-algebra. Let $\{r_n\}$ be a C-basis for B. Then, if $\lambda'\in\Sigma_N(B)$, $h(\lambda')\in\Sigma_c(A)$ and, moreover, there exist $\lambda_0=\bigwedge_{\lambda'\in\Sigma_N(B)}h(\lambda')$ and $\lambda_0\in\Sigma_c(A)$. Thus, we can consider the following homomorphism $\sigma(\lambda')\xrightarrow{h}\sigma(h(\lambda'))\xrightarrow{q^{h(\lambda')}_{\lambda_0}}\sigma(\lambda_0)$, for every $\lambda'\in\Sigma_N(B)$ and $\lambda'\prec\mu'$ implies the commutativity of the diagram

$$\begin{tikzcd}
\sigma(\lambda') \arrow[rr, "q^{h(\lambda')}_{\lambda_0}\circ h"] \arrow[d, "q^{\lambda'}_{\mu'}"'] & & \sigma(\lambda_0) \\
\sigma(\mu') \arrow[urr, "q^{h(\mu')}_{\lambda_0}\circ h"'] & &
\end{tikzcd}$$

By 7.2, there exists a σ-complete continuous homomorphism $h_{\lambda_0}:B\to\sigma(\lambda_0)$, such that $h_{\lambda_0}\circ q'_{\lambda'}=q^{h(\lambda')}_{\lambda_0}\circ h$, $\forall\lambda'\in\Sigma_N(B)$. Then, we obtain $q_{\lambda_0}\circ h_{\lambda_0}\circ q'_{\lambda'}=q_{\lambda_0}\circ q^{h(\lambda')}_{\lambda_0}\circ h=q_{h(\lambda')}\circ h=h\circ q'_{\lambda'}$, $\forall\lambda'\in\Sigma_N(B)$. This implies $q_{\lambda_0}\circ h_{\lambda_0}=h$, thus the condition (ii) of 6.4. is verified.

Now, let h_λ, $k_\lambda:B\to\sigma(\lambda)$, $\lambda\in\Sigma_c(A)$, be two σ-homomorphisms such that $[q_\lambda\circ h_\lambda]=[q_\lambda\circ k_\lambda]$ (here the corresponding morphisms are σ-homotopic).

By 2.3 and 5.3, the homotopy equivalence $q_\lambda \circ h_\lambda \simeq q_\lambda \circ k_\lambda$ means that for any $\lambda' \in \Sigma_c(B)$ there exists $\mu' \in \Sigma_c(A)$ such that $[(h_\lambda)_{\lambda'} \circ (q_\lambda)_{h_\lambda(\lambda')} \circ p^{\mu'}_{q_\lambda}(h_\lambda(\lambda'))] = [(k_\lambda)_{\lambda'} \circ (q_\lambda)_{k_\lambda(\lambda')}\, p^{\mu'}_{q_\lambda(k_\lambda(\lambda'))}]$.

It is easy to see that there exists a partition of unity $\mu'' \in \Sigma_c(\sigma(\lambda))$ such that $h_\lambda(\lambda') \prec \mu''$, $k_\lambda(\lambda') \prec \mu''$ and $q_\lambda(\mu'') = \mu'$(if is necessary one replace μ' by a finer partition). Then, we have the commutative diagram :

$$\begin{tikzcd}
\mu' \arrow[r, "p^{\mu'}_{q_\lambda(h_\lambda(\lambda'))}"] & q_\lambda(h_\lambda(\lambda')) \arrow[r, "(q_\lambda)_{h_\lambda(\lambda')}"] & h_\lambda(\lambda') \\
\mu'' \arrow[u, "(q_{\mu''})_{\mu'}"] \arrow[urr, "p^{\mu''}_{h_\lambda(\lambda')}"'] & &
\end{tikzcd}$$

and an analogous diagram for k_λ.

We obtain $[(h_\lambda)_{\lambda'} \circ (q_\lambda)_{h_\lambda(\lambda')}\, p^{\mu'}_{q_\lambda(h_\lambda(\lambda'))}\, (q_{\mu''})_{\mu'}] = [(k_\lambda)_{\lambda'} \circ (q_\lambda)_{k_\lambda} \circ p^{\mu'}_{q_\lambda}(k_\lambda(\lambda'))(q_{\mu''\mu'}]$, which implies $[(h_\lambda)_{\lambda'} \circ p^{\mu''}_{h_\lambda}(\lambda') = (k_\lambda)_{\lambda'} \circ p^{\mu''}_{k_\lambda(\lambda')}]$, i. e. $[h_\lambda] = [k_\lambda]$, and obviously then it follows $[q_\lambda \circ h_\lambda] = [q_\lambda \circ k_\lambda]$, i.e. the condition (iii) of 6.4.

3.5. *Remark*. For the Boolean system of 2.5 the condition (iii) of 6.4 is satisfied in the stronger form : $[q_\lambda \circ h_\lambda] = [q_\lambda \circ k_\lambda]$ implies $[h_\lambda] = [k_\lambda]$.

4.5. Denote by $sANR$ the category of the separable ANR-spaces, by ANR-$\mathcal{B}$ the category of the ANR-algebras and by ANR-$\mathcal{F}$ the category, of the ANR-fields.

5.5 **Theorem.** *Let X be a topological space at most countable. Then, there exists a $\mathcal{H}$-ANR.$\mathcal{F}$-Boolean system which is $\mathcal{H}$-ANR-$\mathcal{B}$ associated with the closure field $\sigma(X)$.*

Proof. By K. Morita's Theorem [1, Theorem 1.3], there exists a $\mathcal{H}$-ANR-topological system which is $\mathcal{H}$-ANR—associated with the space X. This system $\{K_\alpha, [p_{\alpha\alpha'}], \mathcal{A}\}$ is obtained by considering the set $\{\mathcal{U}_\alpha / \alpha \in \mathcal{A}\}$ of all locally finite normal open coverings of X, taking for K_α the nerve of $\mathcal{U}_\alpha$ and $p_{\alpha\alpha'}$, : $K_{\alpha'} \to K_\alpha$ being the induced map by the refinement relation $\mathcal{U}_\alpha \prec \mathcal{U}_{\alpha'}$. If X is at most countable, then every K_α has a set at most countable of simplexes. But, by this, the polyhedron K_α is a separable space [7, Theorem 1.10.11, p. 48]. Thus, if X is at most countable, the system $\{K_\alpha, [p_{\alpha\alpha'}], \mathcal{A}\}$ is a $\mathcal{H}$-$sANR$-system. By 3.4 we can consider the $\mathcal{H}\mathcal{F}$-Boolean system $\{\sigma(K_\alpha), [\sigma(p_{\alpha\alpha'})], \mathcal{A}\}$ and, by 4.3, this is a $\mathcal{H}$-ANR-$\mathcal{F}$-Boolean system.

Now, if $p_\alpha : X \to K_\alpha$ satisfy the condition (i) of 4.4, then, by 3,3, $\sigma(p_\alpha) : \sigma(K_\alpha) \to \sigma(X)$ satisfy the condition (i) of 6.4. Let $h : B \to \sigma(X)$ be a continuous homomorphism, with B an ANR-complete algebra. Let λ_0 be the finest partition of unity of B and let $q_{\lambda_0} : \sigma(\lambda_0) \to B$ be the continuous epimorphism of 9.2. We consider the continuous homomorphism $h \circ q_{\lambda_0}$: $\sigma(\lambda_0) \to \sigma(X)$. By the result due to R. Sikorski [6, Theorem 1.1 and 5, Theorem 5.1], there exists a continuous map $f_{\lambda_0} : X \to \lambda_0$, inducind $h \circ q_{\lambda_0}$,

that is $\sigma(f_{\lambda_0}) = h \circ q_{\lambda_0}$. By 10.2 and 11.1, λ_0 is an ANR-space for the class of separable metrizable spaces. By condition (ii) of 4.4 and by 5.4, there exist $\alpha_0 \in \mathcal{A}$ and a continuous map $f_{\alpha_0} : K_{\alpha_0} \to \lambda_0$. such that $[f_{\lambda_0}] = [f_{\alpha_0}]\,[p_{\alpha_0}]$. By 3.3. we obtain $[\sigma(f_{\lambda_0})] = [\sigma(p_{\alpha_0})]\,[\sigma(f_{\alpha_0})]$, that is $[h \circ q_{\lambda_0}] = [\sigma(p_{\alpha_0})]\,[\sigma(f_{\alpha_0})]$. We consider the continuous homomorphism $B \xrightarrow{j} \sigma(\lambda_0) \xrightarrow{\sigma(f_{\alpha_0})} \sigma(K_{\alpha_0})$, where j is defined in 8.2. If we denote $h_{\alpha_0} = (f_{\alpha_0}) \circ j$, then $[\sigma(p_\alpha) \circ \sigma(f_{\alpha_0}) \circ j] = [h \circ q_{\lambda_0} \circ j] = [h]$, because $q_{\lambda_0} \circ j = 1_B$. Thus, the condition (ii) of 6.4. is satisfied. Finally, let $h_\alpha, k_\alpha : B \to \sigma(K_\alpha)$, be two continuous homomorphism satisfying $[\sigma(p_\alpha)]\,[h_\alpha] = [\sigma(p_\alpha)]\,[k_\alpha]$. We consider $h_\alpha \circ q_{\lambda_0}, k_\alpha \circ q_{\lambda_0} : \sigma(\lambda_0) \to \sigma(K_\alpha)$ and let $f_\alpha, g_\alpha : K_\alpha \to \lambda_0$ such that $\sigma(f_\alpha) = h_\alpha \circ q_{\lambda_0}$ and $\sigma(g_\alpha) = k_\alpha \circ q_{\lambda_0}$. Let us consider the following maps $f_\alpha \circ p_\alpha, g_\alpha \circ p_\alpha : X \to \lambda_0$. For these maps, we have $[\sigma(f_\alpha \circ p_\alpha)] = [\sigma(p_\alpha)\sigma(f_\alpha)] = [\sigma(p_\alpha)][h_\alpha \circ q_{\lambda_0}] = [\sigma(p_\alpha)][h_\alpha][q_{\lambda_0}] = [\sigma(p_\alpha)][k_\alpha][q_{\lambda_0}] = [\sigma(g_\alpha \circ p_\alpha)]$ and by 3.3, it results $[f_\alpha \circ p_\alpha] = [g_\alpha \circ p_\alpha]$. By the condition (iii) of 4.4, there exists $\alpha' \in \mathcal{A}$, with $\alpha < \alpha'$ (that is $\mathcal{U}_\alpha \prec \mathcal{U}_{\alpha'}$) such that $[f_\alpha][p_{\alpha\alpha'}] = [g_\alpha][p_{\alpha\alpha'}]$. Using again 3.3, we can write $[\sigma(p_{\alpha\alpha'})][\sigma(f_\alpha)] = [\sigma(p_{\alpha\alpha'})][\sigma(g_\alpha)]$, which implies $[\sigma(p_{\alpha\alpha'})][h_\alpha][q_{\lambda_0}] = [\sigma(p_{\alpha\alpha'})][k_\alpha][q_{\lambda_0}]$. By composing at right with $[j_{\lambda_0}]$, we obtain $[\sigma(p_{\alpha\alpha'})][h_\alpha] = [\sigma(p_{\alpha\alpha'})][k_\alpha]$, which accomplishes the proof of the theorem.

Now we can prove the main result :

6.5. **Theorem.** *For any σ-complete closure algebra A, there exists a $\mathcal{H}$-ANR-$\mathcal{F}$-Boolean system which is $\mathcal{H}$-ANR-$\mathcal{B}$-associated with A.*

Proof. By 2.5, the $\mathcal{H}$ σ-$\mathcal{F}$-Boolean system $\{\sigma(\lambda), [q^\lambda_\mu], \Sigma_c(A)\}$ is $\mathcal{H}\sigma$-C-associated with A. Then, because every $\lambda \in \Sigma_c(A)$ is a topological space at most countable, by 5.5, there exists, for any such λ, a $\mathcal{H}$-ANR-$\mathcal{F}$-Boolen-system $\{\sigma(K_{\alpha(\lambda)}), \sigma(p_{\alpha(\lambda),\alpha'(\lambda)})\ \mathcal{A}(\lambda)\}$ which is $\mathcal{H}$-ANR-$\mathcal{B}$-associated with the closure field $\sigma(\lambda)$. Now we consider the set $\mathcal{A} = \{(\lambda, \alpha(\lambda)) / \lambda \in \Sigma_c(A), \alpha(\lambda) \in \mathcal{A}(\lambda)\}$ (recall that by $\mathcal{A}(\lambda)$ we denote the set of all locally finite normal open coverings of the space (λ, O^A_λ)). On the set $\mathcal{A}$ we consider the following partial order : if $(\lambda, \alpha(\lambda))$ $(\mu, \alpha'(\mu)) \in \mathcal{A}$, then we set $(\lambda, \alpha(\lambda)) < (\mu, \alpha'(\mu))$ if $\lambda \prec \mu$ and $(p^\mu_\lambda)^{-1}(\alpha(\lambda)) \prec \alpha'(\mu)$, where $p^\mu_\lambda : \mu \to \lambda$ is the cotinuous map defined in the section 2.

Using 2.5 and 5.5 we can prove that $\mathcal{A}$, with this partial order, is a directed set. In fact, let $(\lambda, \alpha(\lambda))$, $(\mu, \alpha'(\mu)) \in \mathcal{A}$. We consider $\lambda \wedge \mu \in \Sigma_c(A)$, for which we have $\lambda \prec \lambda \wedge \mu$ and $\mu \prec \lambda \wedge \mu$. Then $(p^{\lambda\wedge\mu}_\lambda)^{-1}(\alpha(\lambda))$, $(p^{\lambda\wedge\mu}_\mu)^{-1}(\alpha'(\mu)) \in \mathcal{A}(\lambda \wedge \mu)$, there exists $\alpha''(\lambda \wedge \mu) \in \mathcal{A}(\lambda \wedge \mu')$, such that $(p^{\lambda\wedge\mu}_\lambda)^{-1}(\alpha(\lambda)) \prec \alpha''(\lambda \wedge \mu)$ and $(p^{\lambda\wedge\mu}_\mu)^{-1}(\alpha'(\mu)) \prec \alpha''(\lambda \wedge \mu)$. For the pair $(\lambda \wedge \mu, \alpha''(\lambda \wedge \mu))$ we have $(\lambda, \alpha(\lambda)) < (\lambda \wedge \mu, \alpha''(\lambda \wedge \mu))$ and $(\mu, \alpha'(\mu)) < (\lambda \wedge \mu, \alpha''(\lambda \wedge \mu))$. Now, we consider the following Boolean system $\{\sigma(K_{\alpha(\lambda)}), [q_{\alpha(\lambda),\alpha'(\mu)}], \mathcal{A}\}$, where $[q_{\alpha(\lambda),\alpha'(\mu)}]$ is defined, for $(\lambda, \alpha(\lambda)) < (\mu, \alpha'(\mu))$ thus : we denote $(p^\mu_\lambda)^{-1}(\alpha(\lambda)) = \alpha''(\mu)$ and, by 2.5 and 3.5, there exists an unique morphism $[f_{\alpha(\lambda),\alpha'(\mu)}] : \sigma(K_{\alpha(\lambda)}) \to \sigma(K_{\alpha''(\mu)})$ such that $[\sigma(p_{\alpha''(\mu)})][f_{\alpha(\lambda),\alpha'(\mu)}] = [\sigma(p^\mu_\lambda)][\sigma(p_{\alpha(\lambda)})]$; then we put $[q_{\alpha(\lambda),\alpha'(\mu)}] = [\sigma(p_{\alpha''(\mu),\alpha'(\mu)})][f_{\alpha(\lambda),\alpha'(\mu)}]$. By 6.2, 5.5 and 3.5, the system thus obtained is a $\mathcal{H}$-ANR-$\mathcal{F}$-Boolean system. Now we prove that this system is $\mathcal{H}$-ANR-$\mathcal{B}$-associated with A. First we define $q_{\alpha(\lambda)} : \sigma(K_{\alpha(\lambda)}) \to A$ by the composition $\sigma(K_{\alpha(\lambda)}) \xrightarrow{\sigma(p_{\alpha(\lambda)})} \sigma(\lambda) \xrightarrow{q_\lambda} A$. By some usual calculus, under 6.2 and 5.5 we obtain for $(\lambda, \alpha(\lambda)) < (\mu, \alpha'(\mu))$ the relation $[q_{\alpha'(\mu)}]$

$[q_{\alpha(\lambda)\alpha'(\mu)}]=[q_{\alpha(\lambda)}]$, that is, the condition (i) of 6.4. Now, let $h : B \to A$ be a σ-complete continuous homomorphism, with B an ANR-complete algebra. By 2.5, there exist $\lambda \in \Sigma_c(A)$ and a morphism $h_\lambda : B \to \sigma(\lambda)$, such that $[q_\lambda][h_\lambda]=[h]$.

Further on by 5.5, because λ is at most countable space, there exists $\alpha(\lambda) \in \mathcal{A}(\lambda)$ and a continuous homomorphism $h_{\alpha(\lambda)}: B \to \sigma(K_{\alpha(\lambda)})$ such that $[\sigma(p_{\alpha(\lambda)})][h_{\alpha(\lambda)}]=[h_\lambda]$. For this homomorphism we have $[q_{\alpha(\lambda)}][h_{\alpha(\lambda)}]=[q_\lambda][\sigma(p_{\alpha(\lambda)})][h_{\alpha(\lambda)})]=[q_\lambda][h_\lambda]=[h]$, thus the condition (ii) of 6.4 is satisfied. Finally, let $h_{\alpha(\lambda)}, k_{\alpha(\lambda)} : B \to \sigma(K_{\alpha(\lambda)})$ be two complete continuous homomorphisms, with B an ANR-complete algebra, satisfying $[q_{\alpha(\lambda)}][h_{\alpha(\lambda)}]=[q_{\alpha(\lambda)}][k_{\alpha(\lambda)}]$, that is $[q_\lambda][\sigma(p_{\alpha(\lambda)})\circ h_{\alpha(\lambda)}]=[q_\lambda][\sigma(p_{\alpha(\lambda)})k_{\alpha(\lambda)}]$. By 2.5 and 3.5, it follows $[\sigma(p_{\alpha(\lambda)})][h_{\alpha(\lambda)}]=[\sigma(p_{\alpha(\lambda)})][k_{\alpha(\lambda)}]$. By 5.5, there exists $\alpha'(\lambda) \in \mathcal{A}(\lambda)$, with $\alpha(\lambda) \prec \alpha'(\lambda)$ such that $[\sigma(p_{\alpha(\lambda)\alpha'(\lambda)}))][h_{\alpha(\lambda)}]=[\sigma(p_{\alpha(\lambda)\alpha'(\lambda)})][k_{\alpha(\lambda)}]$, which implies $[q_{\alpha(\lambda)\alpha'(\lambda)}][h_{\alpha(\lambda)}]=[q_{\alpha(\lambda)\alpha'(\lambda)}][k_{\alpha(\lambda)}]$, and we have $(\lambda, \alpha(\lambda))<(\lambda, \alpha'(\lambda))$, thus the condition (iii) of 6.4. is verified. This completes the proof of the theorem.

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GENERALIZED AFFINE CONNECTIONS ON BANACH MANIFOLDS

BY

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The theory of nonlinear connections in the category of Banach vector bundles has been developed by J. Vilms ([7], [8]). A class of nonlinear connections, called *homogeneous connections*, is of great importance in the theory of Finsler connections ([3], [5]).

The purpose of this paper is the study of another class of nonlinear connections, called *generalized affine connections* (g.a.c., for short). The term agrees with the one used in [4, p. 127]. In the first section some new results regarding the nonlinear connections are given. The second section contains the definition of g.a.c. and some of their properties (associated linear connections, geodesics and others). The flat g.a.c. are studied in the third section.

1. **Nonlinear connections.** Let M be a paracompact manifold of class C^∞ (smooth), modeled by the Banach space $\mathbf{M}$ and let $p : E \to M$ be a smooth vector bundle of fiber type a Banach space $\mathbf{E}$. Denote by $p^{-1}TM \to E$ the pull-back by p of the tangent bundle $\pi : TM \to M$ and by $p\,! = (Tp, \tau)$, where Tp is the tangent map to p and $\tau : TE \to E$ is the tangent bundle to the manifold E. The map $Tp : TE \to TM$ gives to TE a second (different) structure of vector bundle.

A smooth nonlinear connection is a smooth splitting of the following exact sequence

$$0 \to VE \xrightarrow{i} TE \xrightarrow{p!} p^{-1}\,TM \to 0 \tag{1.1}$$

of vector bundles over E. Here $VE := \ker\,(p\,!) = \ker\,(Tp)$ denotes the vertical subbundle of TE and i is the inclusion map.

The vertical subbundle $VE \to E$ is canonically isomorphic to $p^{-1}E \to E$ (the pull-back of E by p). Hence, there exists a canonical morphism (over P) $r : VE \to E$ of vector bundles, isomorphic on the fibres.

A splitting of the exact sequence (1.1), i.e. a nonlinear connection is given by a smooth morphism $V : TE \to VE$, such that $V \circ i = \mathrm{id} \mid VE$, or equivalently, a smooth morphism $W : p^{-1}TM \to TE$ such that $p\,! \circ W = \mathrm{id} \mid p^{-1}TM$. Moreover, we have $i \circ V + W \circ p\,! = \mathrm{id} \mid TE$. This implies $TE = VE + HE$, where $HE = \ker\ V = \mathrm{im}\,W$. Obviously, HE is isomorphic to

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