$[q_{\alpha(\lambda)\alpha'(\mu)}]=[q_{\alpha(\lambda)}]$, that is, the condition (i) of 6.4. Now, let $h: B\to A$ be a σ-complete continuous homomorphism, with B an ANR-complete algebra. By 2.5, there exist $\lambda\in\Sigma_c(A)$ and a morphism $h_\lambda: B\to\sigma(\lambda)$, such that $[q_\lambda][h_\lambda]=[h]$.

Further on by 5.5, because λ is at most countable space, there exists $\alpha(\lambda)\in\mathcal{A}(\lambda)$ and a continuous homomorphism $h_{\alpha(\lambda)}: B\to\sigma(K_{\alpha(\lambda)})$ such that $[\sigma(p_{\alpha(\lambda)})][h_{\alpha(\lambda)}]=[h_\lambda]$. For this homomorphism we have $[q_{\alpha(\lambda)}][h_{\alpha(\lambda)}]=[q_\lambda][\sigma(p_{\alpha(\lambda)})][h_{\alpha(\lambda)})]=[q_\lambda][h_\lambda]=[h]$, thus the condition (ii) of 6.4 is satisfied. Finally, let $h_{\alpha(\lambda)}, k_{\alpha(\lambda)}: B\to\sigma(K_{\alpha(\lambda)})$ be two complete continuous homomorphisms with B an ANR-complete algebra, satisfying $[q_{\alpha(\lambda)}][h_{\alpha(\lambda)}]=[q_{\alpha(\lambda)}][k_{\alpha(\lambda)}]$, that is $[q_\lambda][\sigma(p_{\alpha(\lambda)})\circ h_{\alpha(\lambda)}]=[q_\lambda][\sigma(p_{\alpha(\lambda)})k_{\alpha(\lambda)}]$. By 2.5 and 3.5, it follows $[\sigma(p_{\alpha(\lambda)})][h_{\alpha(\lambda)}]=[\sigma(p_{\alpha(\lambda)})][k_{\alpha(\lambda)}]$. By 5.5, there exists $\alpha'(\lambda)\in\mathcal{A}(\lambda)$, with $\alpha(\lambda)\prec\alpha'(\lambda)$ such that $[\sigma(p_{\alpha(\lambda)\alpha'(\lambda)}))][h_{\alpha(\lambda)}]=[\sigma(p_{\alpha(\lambda)\alpha'(\lambda)})][k_{\alpha(\lambda)}]$, which implies $[q_{\alpha(\lambda)\alpha'(\lambda)}][h_{\alpha(\lambda)}]=[q_{\alpha(\lambda)\alpha'(\lambda)}][k_{\alpha(\lambda)}]$, and we have $(\lambda, \alpha(\lambda))<(\lambda, \alpha'(\lambda))$, thus the condition (iii) of 6.4. is verified. This completes the proof of the theorem.

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GENERALIZED AFFINE CONNECTIONS ON BANACH MANIFOLDS

BY

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The theory of nonlinear connections in the category of Banach vector bundles has been developed by J. Vilms ([7], [8]). A class of nonlinear connections, called *homogeneous connections*, is of great importance in the theory of Finsler connections ([3], [5]).

The purpose of this paper is the study of another class of nonlinear connections, called *generalized affine connections* (g.a.c., for short). The term agrees with the one used in [4, p. 127]. In the first section some new results regarding the nonlinear connections are given. The second section contains the definition of g.a.c. and some of their properties (associated linear connections, geodesics and others). The flat g.a.c. are studied in the third section.

1. Nonlinear connections. Let M be a paracompact manifold of class C^∞ (smooth), modeled by the Banach space $\mathbf{M}$ and let $p: E\to M$ be a smooth vector bundle of fiber type a Banach space $\mathbf{E}$. Denote by $p^{-1}TM\to E$ the pull-back by p of the tangent bundle $\pi: TM\to M$ and by $p\,!=(Tp, \tau)$, where Tp is the tangent map to p and $\tau: TE\to E$ is the tangent bundle to the manifold E. The map $Tp: TE\to TM$ gives to TE a second (different) structure of vector bundle.

A smooth nonlinear connection is a smooth splitting of the following exact sequence

$$0\to VE\xrightarrow{i} TE\xrightarrow{p!} p^{-1}TM\to 0 \tag{1.1}$$

of vector bundles over E. Here $VE:=\ker(p\,!)=\ker(Tp)$ denotes the vertical subbundle of TE and i is the inclusion map.

The vertical subbundle $VE\to E$ is canonically isomorphic to $p^{-1}E\to E$ (the pull-back of E by p). Hence, there exists a canonical morphism (over P) $r: VE\to E$ of vector bundles, isomorphic on the fibres.

A splitting of the exact sequence (1.1), i.e. a nonlinear connection is given by a smooth morphism $V: TE\to VE$, such that $V\circ i=\mathrm{id}\mid VE$, or equivalently, a smooth morphism $W: p^{-1}TM\to TE$ such that $p\,!\circ W=\mathrm{id}\mid p^{-1}TM$. Moreover, we have $i\circ V+W\circ p\,!=\mathrm{id}\mid TE$. This implies $TE=VE+HE$, where $HE=\ker V=\operatorname{im} W$. Obviously, HE is isomorphic to

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$p^{-1}TM$ as vector bundles. The morphism (over p) $K: = r\circ V: TE\to E$ is called the connection map and $v=i\circ V$, $h=W\circ p\,!$ are called vertical and horizontal projections, respectively. The morphism $J=i\circ p\,!$ of TE satisfies $J^2=0$ since $p\,!\circ i=0$, therefore J defines an almost tangent structure on E. Obviously, $JV(E)=0$ and $\operatorname{Im} J=VE$. The morphism $\gamma=2h-I$, where I is the identity on TE, satisfies

$$J\circ\gamma=J,\ \gamma\circ J=-J. \tag{1.2}$$

Conversely, a morphism γ satisfying (1.2) determines a unique splitting of the exact sequence (1.1), i.e a nonlinear connection on $p: E\to M$. Indeed, let W' be any right splitting map of the sequence (1.1) (W' exists if M admits smooth partitions of unity). We put $W=hW'$, where $2h=I+\gamma$. The morphism W does not depend on W' and it is easy to check, using (1.2), that $p\,!\circ W=\mathrm{id}_{p^{-1}TM}$. Therefore, we have the following definition of the nonlinear connections, equivalently to that previously given.

Definition 1.1. *A nonlinear connection on $p: E\to M$ is a smooth morphism γ of TE (over $\mathrm{id}\mid_E$) satisfying* (1.2).

The Definition 1.1 generalizes a definition of nonlinear connections on finite dimensional manifolds given by J. Grifone [3]. As in finite dimensional case (see[3]), one can proves the following.

Theorem 1.1. *A smooth morphism γ of TE is a nonlinear connection on $p: E\to M$ if and only if it defines an almost product structure on E ($\gamma\circ\gamma=I$) such that for every $u\in E$, the eigenspace of γ_u (the restriction of γ to $p^{-1}(u)$) which corresponds to the eigenvalue -1 be V_uE.*

2. Generalized affine connections. Let $\mathbf{F}$ be a Banach space. The map$\to$: $\mathbf{F}\times\mathbf{F}\to\mathbf{F}$ given by $(u,v)\to\overrightarrow{uv}=v-u$ defines the so-called canonical affine structure on $\mathbf{F}$. Every vector bundle can be considered as an affine bundle if one considers its fibers with the canonical affine structure.

Let $\mathbf{F}'$ be another Banach space. A map $t: \mathbf{F}\to\mathbf{F}'$ is said to be affine if $t(u)=T(u)+t(0)$ for every $u\in\mathbf{F}$, where $T: \mathbf{F}\to\mathbf{F}'$ is a linear map. If we regard $\mathbf{F}$ and $\mathbf{F}'$ as affine spaces, the map t is affine if and only if it is an affine morphism.

Given two vector bundles $E\to M$ and $E'\to M'$, a map $h: E\to E'$ which preserves the fibers is said to be affine if it is smooth and its restrictions to fibers are affine. Of course, h can be considered as a morphism in the category of affine bundles.

Definition 2.1. *A nonlinear connection on $p: E\to M$ will be called generalized affine connection (briefly g.a.c.) if its connection map, denoted above by K, is an affine map with respect to the structure of vector bundle of TE given by $Tp: TE\to TM$.*

An examination of the local situation will be suitable to leed us to the essentiel properties of g.a.c. Let (U,φ) be a local chart on M. We identify U with $\varphi(U)$ and, restricting U if necessary, suppose that there exists a bundle chart $U\times\mathbf{E}\cong E\mid_U$. Then the tangent map gives a local chart $U\times\mathbf{E}\times\mathbf{M}\times\mathbf{E}\cong TE\mid_U$ and the sequence (1.1) restricted to U becomes

$$0\to U\times\mathbf{E}\times 0\times\mathbf{E}\xrightarrow{i} U\times\mathbf{E}\times\mathbf{M}\times\mathbf{E}\xrightarrow{p!} U\times\mathbf{E}\times\mathbf{M}\to 0, \tag{2.1}$$

where $p\,!(x,a,\lambda,b)=(x,a,\lambda)$, $x\in U$, $\lambda\in\mathbf{M}$, $a,b\in\mathbf{E}$.

The map T_p is locally given by $T_p(x,a,\lambda,b)=(x,\lambda)$, therefore the fibers of bundle $T_p: TE\to TM$ are isomorphic to $x\times\mathbf{E}\times\lambda\times\mathbf{E}\cong\mathbf{E}^2$. J. Vilms has proved (see [7]) the following.

Lemma. *A morphism (over p) $K: TE\to E$ is the connection map of a nonlinear connection on $p: E\to M$, if and only if it is locally given by*

$$K(x,a,\lambda,b)=(x,b+\omega(x,a)\lambda),\ x\in U,\ \lambda\in\mathbf{M},\ a,b\in\mathbf{E}, \tag{2.2}$$

where $\omega: U\times\mathbf{E}\to L(\mathbf{M},\mathbf{E})$ is smooth. For the above the nonlinear connection we shall prove.

Lemma 2.1. *A morphism (over p) $K: TE\to E$ is the connection map of a g.a.c. if and only if it is locally given by*

$$K(x,a,\lambda,b)=(x,b+\Gamma(x)(a,\lambda)+A(x)\lambda), \tag{2.3}$$

where $\Gamma: U\to L^2(\mathbf{E},\mathbf{M};\mathbf{E})$ and $A: U\to L(\mathbf{M},\mathbf{E})$ are smooth maps.

Proof. Let K be the connection map of a g.a.c. . By Definition 2.1, the map $(x,a,\lambda,b)\to(x,b+\omega(x,a)\lambda)$ must be affine on Tp-fibers. Consequently, the map $(a,b)\to b+\omega(x,a)\lambda$ of $\mathbf{E}\times\lambda\times\mathbf{E}\to\mathbf{E}$ must be affine with respect to both the variables. Being linear, hence affine with respect to b, it remains to be affine with respect to a. This happens if and only if there exists a smooth map $\tilde\omega: U\to L(\mathbf{E},L(\mathbf{M},\mathbf{E}))$ such that $\omega(x,a)=\tilde\omega(x)(a)+\omega(x,0)$. We put $A(x)=\omega(x,0)$. Since $L(\mathbf{E},L(\mathbf{M},\mathbf{E}))\cong L^2(\mathbf{E},\mathbf{M};\mathbf{E})$, $\tilde\omega$ determines a unique smooth map $\Gamma: U\to L^2(\mathbf{E},\mathbf{M};\mathbf{E})$ such that $\tilde\omega(x)(a)\lambda=\Gamma(x)(a,\lambda)$. Therefore, $\omega(x,a)\lambda=\Gamma(x)(a,\lambda)+A(x)\lambda$ and (2.3) follows from (2.2).

The maps Γ and A will be called local components of the g.a.c..

Remarks. If the connection map of a nonlinear connection is linear on Tp-fibers, the connection becomes a linear connection. A g.a.c. is linear if and only if A vanishes on U.

When $\omega(x,a)$ is 1-homogeneous with respect to a, or equivalently, K is 1-homogeneous on Tp-fibers, the nonlinear connection is called homogeneous connection. The class of homogeneous connections is used in the theory of Finsler connections (see [3], [5]). In the definition of an homogeneous connections one needs a greater generality, namely the smoothness of it must be assumed only on $E-0$, otherwise it becomes linear. Our considerations from the first section remain true in such a generality (with the appropriate modifications), but it is not necessary for the theory of g.a.c.

Let (U,φ) and (V,ψ) two local charts on M, such that $U\cap V\neq\varnothing$. We put $f=\psi\circ\varphi^{-1}$. If $\Phi: p^{-1}(U)\to U\times\mathbf{E}$ and $\Psi: p^{-1}(V)\to V\times\mathbf{E}$ are bundle local charts, we denote by $B: U\cap V\to L(\mathbf{E},\mathbf{E})$ the map $x\to B(x)=\Psi\circ\Phi^{-1}(x,-)$. In this notations the change of bundle local charts on E can be writen as $(x,a)\to(f(x),\ B(x)a)$, $a\in\mathbf{E}$ and the change of bundle local charts on $Tp: TE\to TM$ induced by it, is given by $(x,a,\lambda,b)\to(f(x)),\ B(x)a,\ \partial f(x),\ \partial B(x)(\lambda)a+B(x)b)$, $x,\lambda\in\mathbf{M}$, $a,b\in\mathbf{E}$, where ∂ means Fréchet differentiation. Let us denote by $\bar\Gamma$ and $\bar A$ the local components of g.a.c. with respect to the local chart $(\bar V,\psi)$. Using (2.3) and the expressions of changes of bundle local charts given above, we find the following transformation rule for the local components of a g.a.c.

$$\Gamma(f(x))\,(B(x)\,a,\ \partial f(x)\,\lambda)+A\,(f(x))\,\partial f(x)=B(x)\,\Gamma(x)\,(a,\lambda)+ \tag{2.4}$$
$$+B(x)\,A(x)\,\lambda-\partial B(x)\,(\lambda)\,a,\ x\in U\cap V,\ a,b\in \mathbf{E},\ \lambda\in \mathbf{M}.$$

For $a=0$, the relation (2.4) becomes

$$A(f(x))\,\partial f(x)\,\lambda=B(x)\,A(x)\,\lambda,\quad x\in U\cap V,\ \lambda\in \mathbf{M}, \tag{2.5}$$

which, used in (2.4), leeds to

$$\overline{\Gamma}(f(x))\,(B(x)\,a,\ \partial f(x)\,\lambda)=B(x)\,\Gamma(x)\,(a,\lambda)-\partial B(x)\ (\lambda)\,a. \tag{2.6}$$

The relation (2.5) shows that A is the local part of a section, denoted also by A, of the vector bundle $L\,(TM,E)\to M$ (of fiber $L(T_xM,\ E_x)$, $x\in M$). The relation (2.6) is just the transformation rule of the local connector of a linear connection on E. Therefore, Γ defines a linear connection on E, which will be denoted also by Γ. Conversely, a section A of the vector bundle $L\,(TM,E)\to M$ and a linear connection Γ on E determine a unique g.a.c. via their local parts. So, we have proved

Theorem 2.1. *Let $p:E\to M$ be a Banach vector bundle. There exists a one-to-one correspondence between the set of g.a.c. on $p:E\to M$ and the pairs (Γ, A), where Γ is a linear connection on $p:E\to M$ and A is a section of $L(TM,E)\to M$.*

Let us denote by $\mathcal{X}_E(M)$ the set of smooth sections of $p:E\to M$ and let us put $\mathcal{X}(M)=\mathcal{X}_{TM}(M)$. Now, let us regard $p:E\to M$ as an affine bundle. Its fiber in $x\in M$ will be denoted by ${}^aE_x$ and x, identified with zero of E_x will be called the *contact point* of M with ${}^aE_x$. A map $P:M\to E$ given by $x\to P_x\in {}^aE_x$ is by definition of class C^∞, if the map $a:M\to E$ defined by $x\to a_x=\overrightarrow{xP_x}\in E_x$ is of class C^∞.

Such a map P of class C^∞ will be called *a point field*. We denote by $\mathcal{P}_E(M)$ the set of point fields of class C^∞ and we put $\mathcal{P}(M)=\mathcal{P}_{TM}(M)$.

By Theorem 2.1 a g.a.c. is well determined by the pair (Γ, A). But a linear connection defines a covariant differentiation i.e. a map $\nabla:\mathcal{X}(M)\times\mathcal{X}_E(M)\to\mathcal{X}_E(M)$ with the following properties:

$$\nabla(X,\alpha a)=\alpha\nabla(X,a)+X(\alpha)\cdot a, \tag{2.7}$$

$$\nabla(X,a+b)=\nabla(X,a)+\nabla(X,b), \tag{2.8}$$

$$\nabla(\alpha X+\beta Y,a)=\alpha\nabla(X,a)+\beta\nabla(Y,a)\quad X,Y\in\mathcal{X}(M),\ a,b\in\mathcal{X}_E(M), \tag{2.9}$$

and $\alpha,\beta\in\mathcal{F}(M)$ — the module of real functions defined on M.

Using ∇ and A we shall define an analogon of ∇ for a g.a.c., namely $D:\mathcal{P}(M)\times\mathcal{P}_E(M)\to\mathcal{P}_E(M)$ given by

$$\overrightarrow{QD(P,Q)}=\nabla(X,a)+A(X),\quad \text{where}\quad X=\overrightarrow{xP}\quad\text{and}\quad a=\overrightarrow{xQ}. \tag{2.10}$$

Theorem 2.2. *The map D associated to a g.a.c. as above, has the following properties:*

$$D(P,\alpha Q+\beta R)=\alpha D(P,Q)+\beta D(P,R)+X(\alpha)\,P+X(\beta)\,Q \tag{2.11}$$

$$D(\alpha P+\beta P',\,Q)=\alpha D(P,Q)+\beta D(P',Q), \tag{2.12}$$

$$D(x,Q)=Q,\ \text{for}\ \alpha,\ \beta\in\mathcal{F}(M),\ \alpha+\beta=1,\ P\in\mathcal{P}(M),\ Q,R\in\mathcal{P}_E(M) \tag{2.13}$$

and $X=\overrightarrow{xP}$ where x is the contact point field.

Proof. To prove (2.11) we remark that it is equivalent to

$$(*)\ \overrightarrow{xD(P,\alpha Q+\beta R)}=\alpha\overrightarrow{xD(P,Q)}+\beta\overrightarrow{xD(P,R)}+X(\alpha)\,\overrightarrow{xP}+X(\beta)\,\overrightarrow{xQ},\ \text{or}$$
$$\overrightarrow{(\alpha Q+\beta R)\,D(P,\alpha Q+\beta R)}=\alpha\overrightarrow{QD(P,Q)}+\beta\overrightarrow{RD(P,R)}+X(\alpha)\,\overrightarrow{xP}+X(\beta)\,\overrightarrow{xQ}.$$

Now we can use (2.10) to obtain

$$(**)\quad \nabla(X,\alpha a+\beta b)+A(X)=\alpha\nabla(X,a)+\alpha A(X)+\beta\nabla(X,b)+\beta A(X)+X(\alpha)\,a+$$
$$+X(\beta)\,b,\ \text{where}\ a=\overrightarrow{xQ},\ b=\overrightarrow{xR}.$$

Since $\alpha+\beta=1$, $(^{**})$ is true by virtue of (2.7) and (2.8). The proof of (2.11) follows the pattern of the previous proof. The property (2.13) is equivalent to $\overrightarrow{xD(x,Q)}=\overrightarrow{xQ}$, or $\overrightarrow{QD(x,Q)}=0$. Using again (2.10) we obtain $\nabla(0,a)+A(0)=0$, which is obviously true. Conversely, given a map $D:\mathcal{P}(M)\times\mathcal{P}_E(M)\to\mathcal{P}_E(M)$ which satisfies (2.11)—(2.13) we can derive from it a covariant differentiation ∇ and a section of $L(TM,E)\to M$, as follows:

$$\nabla(X,a)=\overrightarrow{QD(P,Q)}-\overrightarrow{xD(P,x)},\quad A(X)=\overrightarrow{xD(P,x)}, \tag{2.14}$$

where

$$X=\overrightarrow{xP},\quad a=\overrightarrow{xQ}.$$

But the covariant differentiation ∇ does not define Γ, such that in our framework the map D does not determine a g.a.c. This happens when the dimension of M, as well as of E is finite (see [2]). It is also easy to prove, using (2.14), the following

Theorem 2.3. *A generalized affine connections on M is affine if and only if $D(P,x)=P$ for every $P\in\mathcal{P}(M)$.*

We obtain a g.a.c. on M when $E=TM$. Every section of $L(TM,\ TM)\to M$ is a tensor field of type (1,1). Therefore, we have:

Corollary 2.1. *Let M be a Banach manifold. There exists a one-to-one correspondence between the set of g.a.c. on M and the set of pairs consisting of a linear connection on M and a tensor field of type (1,1) on M.*

The g.a.c. ${}^a\Gamma$ which corresponds to (Γ, I), where I is the tensor of Kronecker, will be called affine connection.

Let $\tilde\Gamma$ be a g.a.c. on the Banach manifold M and let be $K:TTM\to TM$ its connection map. The map D defined above, induces a covariant differentiation $\tilde\nabla:\mathcal{X}(M)\times\mathcal{X}(M)\to\mathcal{X}(M)$, which can be expressed as $\tilde\nabla(X,Y)=K\circ TY(X)$, where TY is the tangent map to $Y:M\to TM$. Indeed, the local parts of $\tilde\nabla$ and D are given by the right part of the equality

$$\tilde\nabla_XY\,|_\varphi=\partial Y_\varphi(X_\varphi)+\Gamma_\varphi(Y_\varphi,X_\varphi)+A(X_\varphi)\,, \tag{2.15}$$

where X_φ, Y_φ are the local parts of X and Y, respectively and Γ_φ and A are the local components of g.a.c.

Let $c:(a,b)\subset R\to M$ be a smooth curve on M and let $Tc:(a,b)\times R\to\to TM$ be its tangent map. We denote by $\dot c$ the vector field on $c(a,b)\subset M$ given by $c(t)\to\dot c(t)$, where $t\in(a,b)$ and $\dot c(t)=Tc(t,1)$. In a local chart (U,φ) with $U\cap c(a,b)\neq\emptyset$, the $\dot c(t)$ is given by

$$\dot c(t)=(c(t),\ \partial c_\varphi(t),(1))\ , \tag{2.16}$$

where $c_\varphi=\varphi\circ c$.

A curve c will be called a geodesic of the g.a.c. $\tilde\Gamma$ if $\tilde\nabla_{\dot c}\dot c=0$. The local component c_φ of a geodesic of $\tilde\Gamma$ satisfies the following differential equation

$$\partial^2c_\varphi(t)+\Gamma_\varphi(\partial c_\varphi(t),\ \partial c_\varphi(t))+A(\partial c_\varphi(t))=0. \tag{2.17}$$

From the theory of differential equations, it follows the local existence and the uniqueness of a geodesic with the initial conditions $c_\varphi(t_0)=c_0\in M$ and $\partial c_\varphi(t_0)\,(1)=u_0\in\mathbf{M}$.

In the following, we shall prove that the well-known relationship between geodesics and sprays holds within the general context. A vector field S on TM, smooth on $TM-0$, is said to be a spray on M if $T\pi\circ S=\mathrm{id}|_{TM}$. Let C be the canonical vector field on TM defined locally by $C(x,a)=\\=(x,a,0,a)$, $x\in U$, $a\in\mathbf{M}$.

Lemma 2.2. *A vector field S on TM, smooth on TM-0, is a spray on M if and only if $J\circ S=C$, where J is the natural almost tangent structure on TM.*

Proof. A vector field S on TM can be writen locally as follows

$$S(x,a)=(x,a,S_1(x,a),\ S_\varphi(x,a)),\ x\in U,\ a\in\mathbf{M}.$$

The condition $T\pi\circ S=\mathrm{id}\,|_{TM}$ implies $S_1(x,a)=a$, therefore $S(x,a)=\\=(x,a,a,S_\varphi(x,a))$, where S_φ is smooth on $U\subset\mathbf{M}$-0. It follows easily $J\circ S=C$, because $J(x,a,b,c)=(x,a,0,b)$, $x\in U$, $a,b,c\in\mathbf{M}$.

Conversely, given S as above, the condition $J\circ S=C$ implies $S_1(x,a)=a$, hence $T\pi\circ S=\mathrm{id}\,|_{TM}$.

Lemma 2.3. *Let K be the connection map of a nonlinear connection on M. There exists a unique spray on M such that $K\circ S=0$, called geodesic spray.*

Proof. Locally, every spray S can be writen as follows:

$$S(x,a)=(x,a,a,S_\varphi(x,a))\ .$$

We obtain the geodesics spray if we take $S(x,a)=-\omega(x,a)\,a$, $x\in U$, $a\in\mathbf{M}$.

The geodesics of a nonlinear connection are the solutions of the following differential equation.

$$\partial^2c_\varphi(t)+\omega(c_\varphi(t),\ \partial c_\varphi(t))\,\partial c_\varphi(t)=0. \tag{2.18}$$

Theorem 2.4. *A curve $c:(a,b)\to M$ is a geodesic of a nonlinear connection N if and only if there exists an integral curve $\tilde c:(a,b)\to TM$ of the geodesic spray S of N, such that $\pi\circ\tilde c=c$.*

Proof. The curve $\tilde c$ on TM is a integral curve of S if $\dot{\tilde c}=S$, therefore in a local chart (U,φ) we have $\partial c_\varphi(t)=S_\varphi(\pi\circ\tilde c_\varphi(t),\ \tilde c_\varphi(t))$. Differentiating $\pi\circ\tilde c=c$, we obtain $\tilde c_\varphi(t)=\partial c_\varphi(t)$, therefore $\partial^2c_\varphi(t)=S_\varphi(c_\varphi(t)),\ \partial c_\varphi(t))=-\omega(c_\varphi(t),\ \partial c_\varphi(t))\,\partial c_\varphi(t)$, or $\partial^2c_\varphi(t)+\omega(c_\varphi(t),\ \partial c_\varphi(t))\,\partial c_\varphi(t)=0$, i.e. c is a geodesic of N. Conversely, if c is a geodesic of N, then the curve $\tilde c$ on TM given by $\tilde c(t)=\dot c(t)$ is a integral curve of the geodesic spray of N and $\pi\circ\tilde c=c$.

Remarks. As a corollary of the Theorem 2.4 one obtains again the local existence and uniqueness of a geodesic with given initial conditions. The geodesic spray of a g.a.c. is locally given by "$S_\varphi(x,a)=(x,a,a,-\Gamma(x)(a,a)-A(x)\,a)$. Using (2.18) one obtains again the equation (2.17) for the geodesics of a g.a.c. Let us suppose that M has finite dimension. Then a curve c can be write as follows: $x^i=x^i(t)$, $t\in(a,b)$, $i=1,2,\ldots,m=\dim M$ and the equation (2.17) becomes

$$\frac{d^2x^i}{dt^2}+\Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}+A^i_j\frac{dx^j}{dt}=0, \tag{2.19}$$

where Γ^i_{jk} are the Christoffel symbols of the linear connection associated to the g.a.c. and A^i_j are the components of a tensor of type (1.1) on M. The solutions of the equation (2.19), called holomorphically planar curves have been used to give some geometrical meanings in the geometry of complex manifolds [6].

When M is the space-time manifold of the general theory of relativity, the solutions of (2.19) are the trajectories of a charged particle moving in an electromagnetic field [1].

3. **Flat generalized affine connections.** Let $V:TE\to VE$ be the splitting map which defines a g.a.c. on $p:E\to M$. The map V can be viewed as a 1-form VE valued. On the other hand, the linear connection Γ defined by $\tilde\Gamma$, induces a linear connection Γ_v on $VE\to E$.

Definition 3.1. *The exterior differential dV of the 1-form VE-valued V, accounted using the linear connection Γ_v on $VE\to E$ will be called the curvature form of $\tilde\Gamma$.*

The local component of V, denoted also by $V:U\times\mathbf{E}\to L(\mathbf{M},\mathbf{E},\mathbf{E})$ is $V(x,a)\,(\lambda,b)=b+\Gamma(x)\,(a,\lambda)+A(x)\,\lambda$, $x\in U$, $\lambda\in M$, $a,b\in\mathbf{E}$. After a calculus rather long but not difficult, one obtains the following local expression for the curvature of form $\tilde\Gamma$:

$$\begin{aligned} dV(x,a)\,((\lambda,b),\ (\mu,c))&=R(x)\,(\lambda,\mu)+\partial A(x)\,(\lambda,\mu)-\\ &-\partial A(x)\,(\mu,\lambda)+\Gamma(x)\,(A(x)\,\mu,\lambda)-\Gamma(x)\,(A(x)\,\lambda\ \ \mu)\ , \end{aligned} \tag{3.1}$$

$x\in U$, $\lambda,\mu\in\mathbf{M}$, $a,b,c\in\mathbf{E}$, where $R(x)$ is the local component of the curvature tensor of Γ.

From (3.1) it follows that dV vanishes when it is applied to a vertical vector field ($\lambda=0$ or $\mu=0$), therefore dV is an horizontal 2-form i.e.

$$dV(\mathbf{A},\mathbf{B})=dV(h\mathbf{A}\ h\mathbf{B}) \tag{3.2}$$

holds for every vector fields $\mathbf{A},\mathbf{B}$ on E. Using $dV(\mathbf{A},\mathbf{B})={}^V\nabla_{\mathbf{A}}V\mathbf{B}-{}^V\nabla_{\mathbf{B}}V\mathbf{A}-\\-V[\mathbf{A},\mathbf{B}]$, where ${}^V\nabla$ is the covariant differentiation associated to Γ_v and (3.2), one obtains.

(3.3) $dV(\mathbf{A},\mathbf{B})=-V[h\mathbf{A},h\mathbf{B}]$ (the structure equation of $\bar\Gamma$).

Definition 3.2. *A g.a.c. will be called flat if its horizontal distribution is involutive i.e. the bracket of two horizontal vector fields is again a horizontal vector field.*

From the structure equation (3.3) it follows

Theorem 3.1. *The g.a.c. $\tilde{\Gamma}$ is flat if and only if $dV=0$.* Taking $dV=0$ and $a=0$ in (3.1) one obtains

$$\partial A(x)(\lambda, \mu)-\partial A(x)(\mu, \lambda)+\Gamma(x)(A(x)\mu, \lambda)-\Gamma(x)(A(x)\lambda, \mu)=0 \tag{3.4}$$

Taking again $dV=0$ in (3.1) and using (3.4) one obtains

$$R(x)(\lambda, \mu)a=0 \tag{3.5}$$

Conversely, if (3.4) and (3.5) hold, then $dV=0$, therefore we have proved the following

Theorem 3.2. *The g.a.c. $\tilde{\Gamma}=(\Gamma, A)$ is flat if and only if the curvature tensor R of Γ vanishes identically and (3.4) holds.*

When $p=\pi: TM\to M$, the conditions (3.4) is equivalent to the vanishing of the following tensor of type (1, 2) on M

$$^aT(X, Y)=\nabla_X A(Y)-\nabla_Y A(X)-A[X, Y], \tag{3.6}$$

where X, Y are vector fields on M, which will be called the torsion tensor of g.a.c. $\tilde{\Gamma}$. The Theorem 3.2 has the following

Corollary 3.2. *A g.a.c. $\tilde{\Gamma}=(\Gamma, A)$ on the manifold M is flat if and only if $R=0$ and $^aT=0$.*

When $A=I$ (the Kronecker tensor), ^aT becomes the well-known torsion tensor of an a affine connection, therefore we have

Corollary 3.3 *An affine connection on a Banach manifold M is flat if and only if its curvature tensor and torsion tensor are both identically zero.*

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ON THE VRANCEANU'S NONHOLONOMIC CONNECTION

BY

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The first author of this Note has studied in [2], Schouten and Vranceanu's nonholonomic connections in the context of the theory of the almost product structures. In [3] the authors indicated a construction of Vranceanu's nonholonomic connection by means of a special connection, called V-connection, which is strongly connected to the almost product structures. An interesting study on some partial connections compatible to a split of the tangent bundles in pairs of complementary subbundles was made by R. Caddeo in [1].

Let M be an n-dimensional differentiable manifold. We denote with $\mathcal{F}(M)$ the ring of differentiable functions of class C^∞, and $\mathcal{X}(M)$ the Lie algebra of the vector fields on M which is a $\mathcal{F}(M)$-module. In the following, all geometrical objects appearing in the discussion will be supposed to be of differentiability class C^∞. *A field of endomorphisms J on the tangent bundle TM is called an almost product structure if*

$$J^2=I, \tag{1}$$

where I is the identity automorphism on TM.

It is known that J defines a split of the tangent bundle TM in two subbundles (or distributions) D and D', such that we have a Whitney sum $TM=D\oplus D'$.

If we denote with V and V' the projection operators corresponding to the two subbundles D and D' respectively, we have

$$\begin{gathered} J=V-V',\ V+V'=I, \\ V^2=V,\ V'^2=V',\ VV'=V'V=0. \end{gathered} \tag{2}$$

The two complementary subbundles can be characterized as follows,

$$D=\{X\in TM : VX=X\},\ D'=\{X\in TM : V'X=X\}.$$

We consider $\mathcal{F}(M)$-modules $\mathcal{S}(D)$ and $\mathcal{S}(D')$ of the sections in the bundles D and D'. If a linear connection ∇^0 on M is given, we can introduce the following two operators :

$$\tilde{\nabla} : \mathcal{X}(M)\times\mathcal{S}(D)\to\mathcal{S}(D) ;\ (X, Y)\to\tilde{\nabla}_X Y=V\nabla^0_X Y, \tag{3}$$