Definition 3.2. *A g.a.c. will be called flat if its horizontal distribution is involutive i.e. the bracket of two horizontal vector fields is again a horizontal vector field.*

From the structure equation (3.3) it follows

Theorem 3.1. *The g.a.c.* $\tilde{\Gamma}$ *is flat if and only if* $dV=0$. Taking $dV=0$ and $a=0$ in (3.1) one obtains

$$\partial A(x)(\lambda,\mu)-\partial A(x)(\mu,\lambda)+\Gamma(x)(A(x)\mu,\lambda)-\Gamma(x)(A(x)\lambda,\mu)=0 \tag{3.4}$$

Taking again $dV=0$ in (3.1) and using (3.4) one obtains

$$R(x)(\lambda,\mu)a=0 \tag{3.5}$$

Conversely, if (3.4) and (3.5) hold, then $dV=0$, therefore we have proved the following

Theorem 3.2. *The g.a.c.* $\tilde{\Gamma}=(\Gamma, A)$ *is flat if and only if the curvature tensor* R *of* Γ *vanishes identically and* (3.4) *holds.*

When $p=\pi: TM\to M$, the conditions (3.4) is equivalent to the vanishing of the following tensor of type (1, 2) on M

$${}^{a}T(X,Y)=\nabla_X A(Y)-\nabla_Y A(X)-A[X,Y], \tag{3.6}$$

where X, Y are vector fields on M, which will be called the torsion tensor of g.a.c. $\tilde{\Gamma}$. The Theorem 3.2 has the following

Corollary 3.2. *A g.a.c.* $\tilde{\Gamma}=(\Gamma, A)$ *on the manifold* M *is flat if and only if* $R=0$ *and* ${}^{a}T=0$.

When $A=I$ (the Kronecker tensor), ${}^{a}T$ becomes the well-known torsion tensor of an a affine connection, therefore we have

Corollary 3.3 *An affine connection on a Banach manifold* M *is flat if and only if its curvature tensor and torsion tensor are both identically zero.*

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ON THE VRANCEANU'S NONHOLONOMIC CONNECTION

BY

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The first author of this Note has studied in [2], Schouten and Vranceanu's nonholonomic connections in the context of the theory of the almost product structures. In [3] the authors indicated a construction of Vranceanu's nonholonomic connection by means of a special connection, called V-connection, which is strongly connected to the almost product structures. An interesting study on some partial connections compatible to a split of the tangent bundles in pairs of complementary subbundles was made by R. Caddeo in [1].

Let M be an n-dimensional differentiable manifold. We denote with $\mathcal{F}(M)$ the ring of differentiable functions of class C^∞, and $\mathcal{X}(M)$ the Lie algebra of the vector fields on M which is a $\mathcal{F}(M)$-module. In the following, all geometrical objects appearing in the discussion will be supposed to be of differentiability class C^∞. *A field of endomorphisms J on the tangent bundle TM is called an almost product structure if*

$$J^2=I, \tag{1}$$

where I is the identity automorphism on TM.

It is known that J defines a split of the tangent bundle TM in two subbundles (or distributions) D and D', such that we have a Whitney sum $TM=D\oplus D'$.

If we denote with V and V' the projection operators corresponding to the two subbundles D and D' respectively, we have

$$\begin{gathered} J=V-V',\ V+V'=I, \\ V^2=V,\ V'^2=V',\ VV'=V'V=0. \end{gathered} \tag{2}$$

The two complementary subbundles can be characterized as follows,

$$D=\{X\in TM : VX=X\},\ D'=\{X\in TM : V'X=X\}.$$

We consider $\mathcal{F}(M)$-modules $\mathcal{S}(D)$ and $\mathcal{S}(D')$ of the sections in the bundles D and D'. If a linear connection ∇^0 on M is given, we can introduce the following two operators :

$$\tilde{\nabla}:\mathcal{X}(M)\times\mathcal{S}(D)\to\mathcal{S}(D)\ ;\ (X,Y)\to\tilde{\nabla}_X Y=V\nabla^0_X Y, \tag{3}$$

(3') $$\tilde{\nabla} : \mathcal{X}(M)\times\mathcal{S}(D')\to\mathcal{S}(D')\ ;\ (X, Y)\to\tilde{\nabla}'_X Y = V'\nabla^0_X Y.$$

Let ∇ and ∇' be the restrictions of $\tilde{\nabla}$ and $\tilde{\nabla}'$ at $\mathcal{S}(D)\times\mathcal{S}(D)$ and $\mathcal{S}(D')\times\times\mathcal{S}(D')$, respectively. Then S c h o u t e n's connection associated to ∇^0 is the connection ∇^1 on M, given by the formula

(4) $$\nabla^1_X Y = \tilde{\nabla}_X VY + \tilde{\nabla}'_X V'Y, \quad X, Y\in\mathcal{X}(M),$$

while V r a n c e a n u's connection associated to ∇^0 is the connection ∇^2 given by:

(5) $$\nabla^2_X Y = \nabla_{VX} VY + \nabla'_{V'X} V'Y + V[V'X, V'Y] + V'[VX, VY],\ X, Y\in\mathcal{X}(M).$$

Let $\mathcal{C}(M)$ be the space of the linear connections on M. The formulas (4) and (5) define the operators $\tilde{p}:\mathcal{C}(M)\to\mathcal{C}(M)$, $\nabla^0\to\tilde{p}(\nabla^0)=\nabla^1$ and $p:\mathcal{C}(M)\to\to\mathcal{C}(M)$, $\nabla^0\to p(\nabla^0)=\nabla^2$, having the following properties

$$\tilde{p}^2=\tilde{p},\ p^2=p,\ \tilde{p}p=p\tilde{p}=\tilde{p}.$$

As $\tilde{p}$ and p are projectors, it results that a linear connection ∇^0 on M belongs to the image of $\tilde{p}$ (or to the image of p), if and only if ∇^0 is invariant with respect to $\tilde{p}$ (or with respect to p).

The invariant connections with respect to the operator $\tilde{p}$ are characterized by the following.

Proposition 1. *A linear connection ∇^1 in M is invariant with respect to $\tilde{p}$ if and only if there exists two connections $\tilde{\nabla}$ and $\tilde{\nabla}'$ on the vector bundles D and D' respectively, so that (4) is satisfied; $\tilde{\nabla}$ and $\tilde{\nabla}'$ are uniquely determined by the connection ∇^1*

Remark. The connection ∇^1 is the sum of the connections $\tilde{\nabla}$ and $\tilde{\nabla}'$ corresponding to W h i t n e y's sum $TM=D\oplus D'$.

In order to characterize the invariant connections with respect to the operator p, we first introduce the following definition:

Definition 2. *An operator*

(6) $$\nabla:\mathcal{S}(D)\times\mathcal{S}(D)\to\mathcal{S}(D),\quad X, Y\to\nabla_X Y,$$

which is $\mathcal{F}(M)$-linear with respect to the argument X and a $\mathcal{F}(M)$-derivation with respect to Y, is called V-connection.

The notion of V'-connection is defined analogously.

We remark that in (5) the operator ∇ is a V-connection and ∇' is a V'-connection. Now we define the *torsion* of ∇ by the formula

(7) $$T(X, Y)=\nabla_X Y-\nabla_Y X-V([X, Y]),\ X, Y\in\mathcal{S}(D)$$

It is immediate that T is $\mathcal{F}(M)$-linear in both the arguments. Analogously the torsion T' is introduced for a V'-connection. The V-connection ∇ is symmetric if $T=0$; analogously for ∇'.

Proposition 3. *A linear connection ∇^2 on M is invariant with respect to p if and only if there exists a V-connection ∇ and a V'-connection ∇', so that (5) is satisfied; ∇ and ∇' are uniquely determined by the invariant connection ∇^2.*

Remark. From propositions 1 and 3, it results that S c h o u t e n's connection leads to the well-known notion of linear connection (of K o s z u l) on vector bundles, while V r a n c e a n u's connection leads to a new concept, that of V-connection. We remark that the subbundles D and D' are parallel to respectively ∇^1 and ∇^2.

If the connection ∇^0 is symmetric, V r a n c e a n u's connection ∇^2 is generally not symmetric because we have

(8) $$T^2(X, Y)=V(T^0(VX, VY))+V'(T^0(V'X, V'Y))-$$
$$-V[V'X, V'Y]-V'[VX, VY],$$

where T^0 and T^2 are the torsions associated to ∇^0 and ∇^2 respectively.

We have the following result

Proposition 4. *Let ∇^0 be a symmetric connection on M. Then Vranceanu's connection ∇^2 is symmetric if and only if D and D' are integrable subbundles.*

Now we have the formulas,

(9) $$\begin{aligned} &VT^2(X, Y)=T(X, Y), && X, Y\in\mathcal{S}(D),\\ &V'T^2(X', Y')=T'(X', Y'), && X', Y'\in\mathcal{S}(D'),\\ &T^2(X, X')=0. \end{aligned}$$

From these we obtain,

Proposition 5. *Vranceanu's connection ∇^2 is symmetric if and only if ∇ and ∇' are symmetric.*

Now we can generalize the fundamental theorem of the pseudoriemannien geometry as follows:

Theorem 6. *Let g be a pseudoriemannian metric on the subbundle D. Then there exists a V-connection ∇, uniquely determined, which is symmetric, so that g is covariant constant with respect to ∇, that is we have*

a) $$\nabla_X Y-\nabla_Y X=V(X, Y),\qquad X, Y, Z\in\mathcal{S}(D),$$

b) $$Xg(Y, Z)=g(\nabla_X Y, Z)+g(Y, \nabla_X Z).$$

Proof. The connection ∇ is given by the formula

(10) $$2g(\nabla_X Y, Z)=Xg(Y, Z)+Yg(Z, X)-Zg(X, Y)+$$
$$+g(X, V[Y, Z])+g(Z, V[X, Y])-g(Y, V[Z, X])$$

It is easily seen that ∇ given by (10) satisfies the properties of a V-connection, which is unique and satisfies *a)* and *b)*.

Now we introduce a curvature operator R for a V-connection ∇, by

(11) $$R(X, Y)\,Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z-V[V'[X, Y], Z],$$

where $X, Y, Z\in\mathcal{S}(D)$. It is immediate that R is $\mathcal{F}(M)$-linear in each argument.

Remark. In [3] the notion of V-connection and the curvature operator are introduced in an equivalent way supposing that $X\in\mathcal{X}(M)$ in (6) and

$X, Y \in \mathscr{X}(M)$ in (11) and asking for the invariance of ∇ and R with respect to V. If $J = V$ (i.e. $V' = 0$), then ∇, T, R are reduced to the usual notions of linear connection, torsion and curvature tensors.

Proposition 7. *The following identity of Bianchi type for a V-connection ∇ is satisfied :*

$$\mathscr{P} R(X, Y) Z = \mathscr{P} T(X, V[Y, Z]) + \mathscr{P} \nabla_X (T(Y, Z)) \tag{12}$$

where $\mathscr{P}$ means circular permutation for the arguments X, Y and Z. A detailed study of the V-connection will be attempted in a following paper.

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LA GÉOMÉTRIE DIFFÉRENTIELLE DES COMPLEXES DE QUADRIQUES DANS L'ESPACE EQUIAFFINE A_3

PAR

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0. Considérations générales. En continuant et élargissant nos préoccupations des Notes [4]—[10], nous allons étudier, dans l'éspace équiaffine à trois dimensions A_3, les variétés à trois paramètres, engendrées par une quadrique (réelle ou vide, irréductible ou degénérée.) Cette étude inclura donc 17 types de variétés, selon la classification des quadriques dans A_3[1]. Vu que chacune des variétés dépend de trois paramètres, nous les nommons *complexes de quadriques*, en analogie avec les complexes de droites [11].

Nous utiliserons plus fréquemment les indices $\alpha, \beta = \overline{0,3}$; $i, j, k = 1, 2, 3$; $m = 0, 2, 3$, de même que la convention d'Einstein des indices muets.

Les diverses considérations générales faites sur les complexes de quadriques dans P_3 [9], peuvent être adaptées facilement à notre étude, en remplaçant le groupe projectif G_{15} par le groupe équiaffine G_{11}, les repères $\{B_\alpha, V\}$ et $\{A_\alpha, U\}$ par $\{O ; i_k\}$, respectivement $\{M, I_j\}$ et en faisant : $\omega_\alpha^0 = 0$, $\omega_0^i = \omega^i$, avec les motivations données dans [4].

En notant par R respectivement r les rangs des formes quadratiques $H = a_{ij} x^i x^j + 2b_i x^i x^0 + c(x^0)^2$, $h = a_{ij} x^i x^j$, nous désignons ces complexes par $V(3 ; (R, r) A_3)$ ou, plus brièvèment $V(R ; r)$, vu que le nombre 3 des paramètres principaux de la variété et l'espace A_3 sont les mêmes pour toutes les variétés ici considérées.

D'une manière analogue à celle de [9], pour chaque couple $(R ; r)$ fixé, nous étudions une seule variété que nous noterons $V_1(R ; r)$, l'étude d'autres vatiétés $V_p(R, r)$, $p = 2, 3, \ldots$ étant réservé pour une autre Note. Pour chaque complexe, nous allons déterminer le système complet d'invariants et le repère Frenet, à l'aide de la méthode du repère mobile de Cartan ; de même nous allons donner des interprétations géométriques aux éléments du repère Frenet.

1. Complexes de quadriques à centre unique ou à droite de centres ou à plan de centres. Par définition, nous considérons comme repères d'ordre zéro, les repères $R = \{M, I_k\}$, par rapport auxquels la quadrique générique a l'équation normale

$$Q = \varepsilon_i (x^i)^2 + \varepsilon_0 = 0, \tag{1.1}$$

où $\varepsilon_1 = 1$ et ε_m ont les valeurs données par le tableau