ou $\varepsilon_2 = \varepsilon_3 = 0$, (1.3) devient (1.7), respectivement (1.10) et les calculs ultérieurs de même que les résultats différent du cas des variétés V_1 (4, 3).

Voici l'explication d'avoir repris, presque chaque fois l'application de l'algorithme de la méthode du repère mobile.

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SOME PROPERTIES OF n-CYCLIC INTERACTIVE LANGUAGE

BY

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In [1] the notions of interactive system of grammars, interactive language, n-cyclic interactive system and n-cyclic interactive language, are introduced. Some properties of interactive language are also given.

In the present paper some properties of n-cyclic interactive languages are studied.

Definition 1. (see [1], also [2], 43—50). *A phrase-structure grammar is a 4-tuple $G = (V_N, V_T, P, S)$, where*

1) *V_N is a finite set of nonterminal symbols ;*
2) *V_T is a finite set of terminal symbols ;*
3) *P is a finite set of productions ;*
4) *S is a start symbol in V_N.*

Let $P_1 = \{p_1, p_2, \ldots, p_r\}$ be a set of labels for the productions in P, distinct productions in P have distinct labels.

Let

$$S = u_0 \xrightarrow{p_{j(0)}} u_1 \xrightarrow{p_{j(1)}} u_2 \xrightarrow{p_{j(2)}} \ldots \xrightarrow{p_{j(m)}} u_{m+1},\ m \geq 0$$

be a left-most derivation.

The word $c = p_{j(0)} p_{j(1)} \ldots p_{a(m)}$, is called a control word. The mapping $g_G : P_1 \to (V_N \cup V_T)^*$ is defined by $g_G(c) = u_{m+1}$.

Definition 2. *Let $G_1 = (V_{N_1}, V_{T_1}, P_1, S_1)$ and $G = (V_{N_2}, V_{T_2}, P_2, S_2)$ be two phrase-structure grammars as in definition 1 with $V_{T_1} \subseteq P_{1_2}$ and $V_{T_2} \subseteq P_{1_1}$. The interactive derivation denoted $w_1 \underset{G_1G_2}{\Longrightarrow} w_2$ is defined by*

$$5) \qquad w_1, w_2 \in L(G_1)\ ;$$

$$6) \qquad \exists\, y \in L(G_2),\ y = g_{G_2}(w_1) \text{ and } g_{G_1}(y) = w_2.$$

The relation $\underset{G_1G_2}{\Longrightarrow}$ is called an interactive derivation between G_1 and G_2. The reflexive and transitive closure of relation $\underset{G_1G_2}{\Longrightarrow}$ will be denoted by $\underset{G_1G_2}{\Longrightarrow}^*$. A pair of grammars (G_1, G_2) is called an interactive systen.

Definition 3. *The interactive language $L(G_1, G_2;\ w_0)$ is defined by*

$$L(G_1, G_2;\ w_0) = \{w / w \in L(G_1),\ w_0 \underset{G_1G_2}{\Longrightarrow}^* w\}\ ,$$

where $w_0 \in L(G_1)$ is the initial word.

Definition 4. *Let $(G_1, G_2, \ldots, G_n)$, $n \geq 3$, be phrase-structure grammars. The n-cyclic interactive derivation denoted by $w_1 \underset{G_1G_2\ldots G_n}{\Longrightarrow} w_2$, is defined by*

$$\exists\ x_1 \in L(G_1),\ w_1 = x_1\ ;$$

$$\exists\ x_2 \in L(G_2),\ g_{G_2}(x_1) = x_2\ ;$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$\exists\ x_n \in L(G_n),\ g_{G_n}(x_{n-1}) = x_n\ ;$$

$$\text{and } g_{G_n}(x_n) = w_2,\ w_2 \in L(G_1)\ ,$$

where $V_{T_i} \subseteq P_{1_{i+1}}$, $1 \leqslant i < n$, *and* $V_{T_n} \subseteq P_{1_1}$.
The reflexive and transitive closure of relation $\underset{G_1 \ldots G_n}{\Longrightarrow}$ is denoted by $\underset{G_1 \ldots G_n}{\Longrightarrow}{}^{*}$.

Definition 5. *The n-cyclic interactive language generated by an ordered n-tuple of grammars* $(G_1, G_2 \ldots G_n)$ *is defined by*

$$CL^n(G_1, G_2, \ldots G_n\ ;\ w_0) = \{w / w \in L(G_1),\ w_0 \underset{G_1 \ldots G_n}{\Longrightarrow}{}^{*} w\},$$

where w_0 *is the initial word.*

Theorem 1. *If the lanugage L contains two words of the form* w_1aw_2a *and* w_3bw_4b, *where* $w_1, w_2, w_3, w_4 \in V_{T_1}^{*}$, *and* $a, b \in V_{T_1}$, *then it can not be generated by an n-cyclic interactive system* $(G_1, G_2, \ldots G_n)$ *in which*:

a) *there exists a linear grammar* G_i, $2 \leqslant i \leqslant n$;

b) *the grammars* G_k, $2 \leqslant k < i$, *have not productions of the form* $u \to \lambda$, *nor productions which strike out terminal symbols.*

Proof. Suppose there exist an initial word w_0 and an n-cyclic interactive system $(G_1, G_2, \ldots G_n)$, which satisfy the conditions of the theorem, such that

$$L = CL^n(G_1, G_2, \ldots G_n\ ;\ w_0).$$

Then it follows that there exists an n-cyclic interactive derivation $w_1aw_2a \underset{G_1 \ldots G_n}{\Longrightarrow} w_3bw_4b$ or $w_3bw_4b \underset{G_1 \ldots G_n}{\Longrightarrow} w_1aw_2a$.

Without loss of generality it can be suppose the n-cyclic interactive derivation is of the first form.

If in the grammar G_2 the production whose label is a has the nonterminal symbols in the right-hand, then $x_2 = g_{G_2}(w_1aw_2a)$ should contain some nonterminal symbols ; this contradicts the hypothesis on the grammars G_k, $2 \leqslant k < i$, it results that $x_2 = g_{G_2}(w_1aw_2a)$ has the following form $x_2 = w_{12}a_2\ w_{22}a_2$, where $a_2 \in V_{T_2}$.

If we repeat the same reasoning, then :
i) one obtains a word which has nonterminal symbols, and this word can not the control word (therefore the contradiction) ; or, ii) after repeating (i—2) times the reasoning we obtain

$$x_{i-1} = w_{1i-1}\,a_{i-1}w_{2i-1}\,a_{i-1} \text{ with } a_{i-1} \in V_{T_{i-1}} \text{ and } x_{i-1} = g_{G_{i-1}}(x_{i-2}).$$

The word x_{i-1} is the control word of the derivation in the grammar G_i. In the grammar G_i the production labeled with a_{i-1} has the form :

7) $$(a_{i-1})\ A \to \alpha_i B\ ,$$

or

8) $$(a_{i-1})\ A \to \alpha_i,$$

where A, $B \in V_{N_i}$ and $\alpha_i \in V_{T_i}^{*}$.

If the production labeled with a_{i-1} has the form 7), then $x_i = g_{G_i}(x_{i-1})$ contains the nonterminal symbols ; therefore x_i can not be the control word for the derivation in the grammar G_{i+1}. The n-cyclic interactive derivation can not continue, hence a contradiction.

If the production labeled with a_{i-1} has the form 8), then after having applied it the first time, the word obtained will not contain the nonterminal symbols, therefore the n-cyclic interactive derivation can not continue.

It follows that the word w_3bw_4b can not be obtained from w_1aw_2a in an n-cyclic interactive derivation.

Remark 1. From the proof of the above theorem, it can be easily seen that it is sufficient for the productions labeled with $a_1, a_2, \ldots a_{i-2}$, not to have to λ in the right-hand.

If we put $w_1 = w_2 = w_3 = w_4 = \lambda$, it results :

Corollary 1. The language L which contains two words aa and bb is not an n-cyclic interactive language generated by an n-cyclic interactive system as in theorem 1.

Theorem 2. *An infinite language L which contains a word of the form ww with* $w \in V_T^{*}$ *is not an n-cyclic interactive language.*

Proof. We suppose that there exist an initial word w_0 and an n-cyclic interactive system $(G_1, G_2, \ldots G_n)$, $n \geqslant 3$, such that $L = CL^n(G_1, G_2, \ldots, G_n\ ;\ w_0)$.

Let w be of the form $w = a_1a_2 \ldots a_m$, $a_i \in V_{T_1}$, $1 \leqslant i \leqslant m$. The word ww is a control word for the derivation in the grammar G_2 ; therefore the production in G_2 labeled with a_1 has in the left-hand the start symbol, S_2, of the grammar G_2.

If applying the productions labeled with $a_1, a_2, \ldots, a_m$ one obtains a word which contains the start symbol, S_2, then applying once again these production one obtains a word which will contain the start symbol, S_2, hence a word which is not a control word for the derivation in the grammar G_3.

If applying the productions labeled with $a_1a_2, \ldots a_m$ one obtains a word which does not contain the symbol S_2, then the production labeled with a_1 can not be applied once again.

In both cases the n-cyclic interactive derivation can not be continued from ww, and this contradicts the infinity of the language L.

Corollary 2. An n-cyclic interactive language L which contains a word of the form ww is a finite language and the word ww is the final in the n-cyclic interactive derivation.

Corollary 3. An n-cyclic interactive language L on a one-letter alphabet $\{a\}$ has $|L| = 1$ or $|L| = 2$; in the second case the initial word is a.

Theorem 3. *The infinite language L which contains a word of the form awa, with* $a \in V_T$ *and* $w \in V_T^{*}$, *can not be an n-cyclic interactive language.*

Proof. We suppose that there exists an initial word w_0 and an n-cyclic interactive system $(G_1, G_2, \ldots G_n)$, $n \geqslant 3$, such that

$$L=CL^n(G_1, G_2, \dots G_n\ ;\ w_0).$$

The word awa is a control word for the derivation in the grammar G_2. Therefore the production labeled with a in the grammar G_2 has in the left-hand the start symbol S_2, of the grammar G_2.

If the production labeled with a in the grammar G_2 has in the right-hand only the terminal symbols, then the derivation in grammar G_2 can not be continued. Therefore the language L can not be infinite.

If the production labeled with a in the grammar G_2 has in the right-hand the nonterminal symbols, then after applying the last time of production labeled with a one obtains a word which contains the nonterminal symbols, hence it is not a control word for the derivations in the grammar G_3; therefore the language L can not be infinite, and so the theorem is proved.

Corollary 4. An n-cyclic interactive language L which contains a word of the form awa, $a\in V_T$ and $w\in V_T^*$, is a finite language and the word awa is the final word in the n-cyclic interactive derivation.

Corollary 5. The language L which contains two words of the form aw_1a and bw_2b ; $a, b\in V_T$ and $w_1, w_2\in V_T^*$, can not be an n-cyclic interactive language.

Theorem 4. *If an n-cyclic interactive language contains at least two words, then for any two words* $u, v\in L$ *with* $u \underset{G_1\dots G_n}{\Longrightarrow} v$ *it results that* $|v|\leqslant k\,|u|$, *where k is independent on* u *and* v.

Proof. Let L be an n-cyclic interactive language, therefore $L=CL^n(G_1G_2,\dots G_n\ ;\ u_0)$, $n\geqslant 3$.

Let k_i, $1\leqslant i\leqslant n$, be the maximum number of symbols in the right-hand of the productions of the grammar G_i.

From $u \underset{G_1\dots G_n}{\Longrightarrow} v$ it results that there exist $x_1, x_2,\dots x_n$ where $x_i\in L(G_i)$ $1\leqslant i\leqslant n$, such that $u=x_1$, $g_{G_i}(x_1)=x_2,\dots, g_{G_n}(x_{n-1})=x_n$, $g_{G_1}(x_n)=v$.

From these relations it results:

$$|v|=|g_{G_1}(x_n)|\leqslant k_1\,|x_n|=k_1\,|g_{G_n}(x_{n-1})|\leqslant k_1k_n\,|x_{n-1}|=k_1k_n\times$$
$$\times\,|g_{G_{n-1}}(x_{n-2})|\leqslant\dots\leqslant k_1k_nk_{n-1},\dots k_3|\,x_2\,|=k_1k_nk_{n-1}\dots k_3\,|g_{G_2}(x_1)|\leqslant k_1k_n\dots$$
$$\dots k_2\,|x_1|=k_1k_2\dots k_n\,|u|.$$

If we set $k=k_1k_2\dots k_n$, one obtains $|v|\leqslant k\,|u|$. Q.E.D.

Remark. This theorem establishes the dependence between the length of a word of an n-cyclic interactive language and that of the preceding one in the n-cyclic interactive derivation.

Lemma 1. *The empty set,* $\varnothing$, *is not an n-cyclic interactive language.*

Proof. Every n-cyclic interactive language contains at least a word — the initial word.

Using the properties of n-cyclic interactive languages and lemma 1, it can be proved that the family of n-cyclic interactive languages is not closed with respect to the regular operations.

Theorem 5. *The family of n-cyclic interactive languages is not closed with respect to the operations :*

1) *union ;*
2) *intersection ;*
3) *complement ;*
4) *concatenation ;*
5) *star operation ;*
6) *homomorphism ;*
7) *intersection with regular sets ;*
8) *mirror image.*

Proof. Without loss of generality we suppose $n=3$. Let G_1, G_2 and G_3 be the following grammars :

$$G_1=(\{A\}, \{a\}, P_1, A), \text{ where } P_1:\begin{cases} 1)\ A\to aA\ ; \\ 2)\ A\to a \end{cases}$$

$$G_2=(\{B\}, \{3, 4\}, P_2, B) \text{ where } P_2: (a)\ B\to 34$$

$$G_3=(\{C\}, \{1, 2\}, P_3, C), \text{ where } P_3:\begin{cases} (3)\ C\to 1C \\ (4)\ C\to 2 \end{cases}$$

and L_1, L_2 the following languages :

$$L_1=CL^3(G_1, G_2, G_3\ ;\ a)=\{a, aa\}\ ;$$
$$L_2=CL^3(G_1, G_2, G_3\ ;\ a\,a\,a)=\{a\,a\,a\}\ .$$

1) $L_1\cup L_2=\{a, aa, aaa\}$ is not a 3-cyclic interactive language (Corollary 3).
2) $L_1\cap L_2=\varnothing$ is not a 3-cyclic interactive language (Lemma 1).
3) $L=V_{T_1}^*\setminus L_1$ is not a 3-cyclic interactive language (Corollary 3).
4) $L_1\,.\,L_2=\{a^4, a^5\}$ is not a 3-cyclic interactive language (Corollary 3).
5) L_1^* is not a 3-cyclic interactive language (Corollary 3).
6) Let G_4, G_5, and G_6 be the following grammars :

$$G_4=(\{S_4\}, \{a, b, c\}, P_4, S_4), \text{ where } P_4:\begin{cases} (0)\ S_4\to aS_4 \\ (1)\ S_4\to bS_4 \\ (2)\ S_4\to bc \end{cases}$$

$$G_5=(\{S_5\}, \{x, y, z\}, P_5, S_5), \text{ where } P_5:\begin{cases} (a)\ S_5\to xS_5 \\ (b)\ S_5\to yS_5 \\ (c)\ S_5\to z \end{cases}$$

$$G_6=(\{S_6\}, \{0, 1, 2\}, P_6, S_6), \text{ where } P_6\begin{cases} (x)\ S_6\to 0S_6 \\ (y)\ S_6\to 1S_6 \\ (z)\ S_6\to 2 \end{cases}$$

and $L=CL^3(G_4, G_5, G_6\ ;\ abc)=\{ab^nc/n\geqslant 1\}$.

We define the homomorphism h by $h(a)=\alpha$, $h(b)=\beta$, $h(c)=\alpha$; then $h(L)=\{\alpha\beta^n\alpha\,|\,n\geqslant 1\}$, which is not a 3-cyclic interactive language as it results from theorem 3.

7) $\varnothing$ is a regular set, $\varnothing\cap L_1=\varnothing$ and $\varnothing$ is not an n-cyclic interactive language (from Lemma 1).

8) Let G_7, G_8 and G_9 be the following grammars :

$$G_7=(\{S_7\},\{a,b\},P_7,S_7),\quad \text{where } P_7:\begin{cases}(1)\ S_7\to aS_7;\\(2)\ S_7\to b;\end{cases}$$

$$G_8=(\{S_8\},\{x,y\},P_8,S_8),\quad \text{where } P_8:\begin{cases}(a)\ S_8\to xS_8\\(b)\ S_8\to y;\end{cases}$$

$$G_9=(\{S_9\},\{1,2\},P_9,S_9),\quad \text{where } P_9:\begin{cases}(x)\ S_9\to 11S_9\\(y)\ S_9\to 2.\end{cases}$$

it is easy to prove that

$$L=CL^3(G_7,G_8,G_9;\ ab)=\{a^{2n}b/n\geqslant 0\}.$$

Then $\tilde{L}=\{ba^{2n}/n\geqslant 0\}$ is the language obtained by the mirror image operation from L. This language is not an n-cyclic interactive language. We suppose that there exist an initial word w_0 and an n-cyclic interactive system $(G_1,G_2...,G_n)$, $n\geqslant 3$, such that

$$L=CL^n(G_1,G_2\dots G_n;\ w_0).$$

The words ba and baa are in $CL^n(G_1,G_2...G_n,w_0)$, therefore they are the control words for the derivations in the grammar G_2.

If the word ba is a control word for the derivation in the grammar G_2, it results that the word $g_{G_2}(ba)$ has not nonterminal symbols. But the word baa is also a control word for the derivation in the grammar G_2, and this leads to a contradiction since after applying the productions labeled with b and a one obtains a word which has not nonterminal symbols; therefore we can not apply a second time the production labeled with a.

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ON A DISCRETE OPTIMIZATION PROBLEM

BY

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In this paper we deal with a discrete optimization problem, which has been formulated as a mathematical model for some scheduling problems, [1]. A set of optimal solutions for such a problem has been characterized by means of a continued fraction associated with the problem, [2], or by means of some binary vectors, [3]. By using an explicit form of the optimal solutions, in this paper we characterize these solutions and, for some particular cases, a simple method for finding them is given.

1. Let us consider the integer programming problem, $P(m,s,b)$:

$$\Sigma x_j=x+x_2...+x_m \ \min.$$

subject to

$$\sum_{j=0}^{s-1} x_{i\oplus j}\geqslant b,\ i=1,\dots,m;\qquad x_i\in Z_+,\ i=1,\dots,m, \tag{1.1}$$

where m,s,b are positive integers and $\oplus$ denotes the addition mod.m, when the residuals mod.m are considered $1,\dots,m$.

If $\lceil a\rceil$, $a\in R$, denotes the smallest integer greater than a, in [2] it is shown that the minimum of the objective function of $P(m,s,b)$ is $\lceil mb/s\rceil$ when $s\leqslant m$ and in [3] it is shown that this result is true for all s.

It is clear that for every feasible solution, x, of the problem $P(m,s,b)$ there exist the non-negative integers δ_i, $i=1,\dots,m$, such that $(x;\delta)$ is a solution of the system

$$(1.2)\qquad \begin{aligned} &\text{a)}\quad \sum_{j=0}^{s-1} x_{i\oplus j}=b+\delta_i,\ i=1,\dots,m;\\ &\text{b)}\quad x_i,\ \delta_i\in Z_+;\ i=1,\dots,m;\end{aligned}$$

and conversely, if $(x;\delta)$ is a solution of the system (1.2), then x is a feasible solution of the problem $P(m,s,b)$. By adding the equalities (1.2a) one obtains

$$s\Sigma x_j=mb+\Sigma\delta_j, \tag{1.3}$$

and supposing

$$mb=sq+r,\ 1\leqslant r\leqslant s, \tag{1.4}$$