$$G_7=(\{S_7\},\{a,b\},P_7,S_7),\quad \text{where } P_7:\begin{cases}(1)\ S_7\to aS_7\,;\\(2)\ S_7\to b\,;\end{cases}$$

$$G_8=(\{S_8\},\{x,y\},P_8,S_8),\quad \text{where } P_8:\begin{cases}(a)\ S_8\to xS_8\\(b)\ S_8\to y\,;\end{cases}$$

$$G_9=(\{S_9\},\{1,2\},P_9,S_9),\quad \text{where } P_9:\begin{cases}(x)\ S_9\to 11S_9\\(y)\ S_9\to 2.\end{cases}$$

it is easy to prove that

$$L=CL^3(G_7,G_8,G_9\,;\,ab)=\{a^{2n}b/n\geqslant 0\}\,.$$

Then $\tilde{L}=\{ba^{2n}/n\geqslant 0\}$ is the language obtained by the mirror image operation from L. This language is not an n-cyclic interactive language. We suppose that there exist an initial word w_0 and an n-cyclic interactive system $(G_1,G_2...,G_n)$, $n\geqslant 3$, such that

$$L=CL^n(G_1,G_2\dots G_n\,;\,w_0).$$

The words ba and baa are in $CL^n(G_1,G_2...G_n,w_0)$, therefore they are the control words for the derivations in the grammar G_2.

If the word ba is a control word for the derivation in the grammar G_2, it results that the word $g_{G_2}(ba)$ has not nonterminal symbols. But the word baa is also a control word for the derivation in the grammar G_2, and this leads to a contradiction since after applying the productions labeled with b and a one obtains a word which has not nonterminal symbols; therefore we can not apply a second time the production labeled with a.

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ON A DISCRETE OPTIMIZATION PROBLEM

BY

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In this paper we deal with a discrete optimization problem, which has been formulated as a mathematical model for some scheduling problems, [1]. A set of optimal solutions for such a problem has been characterized by means of a continued fraction associated with the problem, [2], or by means of some binary vectors, [3]. By using an explicit form of the optimal solutions, in this paper we characterize these solutions and, for some particular cases, a simple method for finding them is given.

1. Let us consider the integer programming problem, $P(m,s,b)$:

$$\Sigma x_j=x+x_2...+x_m \ \text{min}.$$

subject to

$$\text{(1.1)}\qquad \sum_{j=0}^{s-1} x_{i\oplus j}\geqslant b,\ i=1,\dots,m;\qquad x_i\in Z_+,\ i=1,\dots,m,$$

where m,s,b are positive integers and $\oplus$ denotes the addition mod.m, when the residuals mod.m are considered $1,\dots,m$.

If $\lceil a\rceil$, $a\in R$, denotes the smallest integer greater than a, in [2] it is shown that the minimum of the objective function of $P(m,s,b)$ is $\lceil mb/s\rceil$ when $s\leqslant m$ and in [3] it is shown that this result is true for all s.

It is clear that for every feasible solution, x, of the problem $P(m,s,b)$ there exist the non-negative integers δ_i, $i=1,\dots,m$, such that $(x\,;\,\delta)$ is a solution of the system

$$\text{(1.2)}\qquad \begin{aligned}&\text{a)}\quad \sum_{j=0}^{s-1} x_{i\oplus j}=b+\delta_i,\ i=1,\dots,m,\\ &\text{b)}\quad x_i,\ \delta_i\in Z_+;\ i=1,\dots,m,\end{aligned}$$

and conversely, if $(x\,;\,\delta)$ is a solution of the system (1.2), then x is a feasible solution of the problem $P(m,s,b)$. By adding the equalities (1.2a) one obtains

$$\text{(1.3)}\qquad s\Sigma x_j=mb+\Sigma\delta_j,$$

and supposing

$$\text{(1.4)}\qquad mb=sq+r,\ 1\leqslant r\leqslant s,$$

it follows that for every solution $(x\,;\,\delta)$ of (1.2) there exists a non-negative integer t such that

$$\Sigma\delta_j = st+s-r. \tag{1.5}$$

We note that, if r is defined by (1.4) then $\lceil mb/s\rceil=(mb+s-r)/s$. By using the formulas (1.3)—(1.5), it results that for every feasible solution, x, of the problem $P(m,s,b)$, there exists a non-negative integer t such that

$$\Sigma x_j = \lceil mb/s\rceil + t \tag{1.6}$$

and $\Sigma x_j=\lceil mb/s\rceil$, iff $\Sigma\delta_j=s-r$, where r is defined by (1.4) and $(x\,;\,\delta)$ is the corresponding solution of the system (1.2). Since for an optimal solution of $P(m,s,b)$ we have $\Sigma x_j=\lceil mb/s\rceil$, because of (1.5) and (1.6), it follows that the problem of finding the optimal solution of $P(m,s,b)$ can be reduced to that of finding the solutions of the system

$$\begin{aligned} &\text{a)}\quad \sum_{j=0}^{s-1} x_{i+j} = b+\delta_i,\ i=1,\dots,m, \qquad \text{b)}\quad \Sigma\delta_j = s-r, \\ &\text{c)}\quad x_i,\ \delta_i \in Z_+,\ i=1,\dots,m, \end{aligned} \tag{1.7}$$

where r is defined by (1.4).

Theorem 1. *Consider the problem $P(m,s,b)$, where $s=mq+k$, $1\leqslant k\leqslant m-1$ and the problem $P(m,k,\hat b)$ where $\hat b$ is defined by*

$$\hat b = b - q\lceil mb/s\rceil. \tag{1.8}$$

If $\hat x$ is an optimal solution of $P(m,k,\hat b)$ and y is a partition of $\lceil mb/s\rceil$ — $\lceil m\hat b/k\rceil$ into m non-negative integers, then $u=\hat x+y$ is an optimal solution of the problem $P(m,s,b)$.

Proof. It is easy to prove that $\lceil mb/s\rceil-\lceil m\hat b/k\rceil=\lfloor (s-r)/k\rfloor$, where $\lfloor a\rfloor$, $a\in R$, denotes the greatest integer less then, or equal to a and r is defined by (1.4). Consider $\hat x=(\hat x_i)$ an optimal solution of $P(m,k,\hat b)$ and $y=(y_i)$ a partition of $\lfloor (s-r)/k\rfloor\geqslant 0$ in to m non-negative integers. Since $\Sigma y_i=\lceil mb/s\rceil-\lceil m\hat b/k\rceil$ and $\Sigma\hat x_i=\lceil m\hat b/k\rceil$, it is clear that

$$\Sigma\hat x_i+\Sigma y_i=\lceil mb/s\rceil. \tag{1.9}$$

Because of (1.8) and $y_i\geqslant 0$ one can prove without difficulties that $\hat x+y$ is a feasible solution of $P(m,s,b)$ and therefore, by (1.9), $\hat x+y$ is an optimal solution of $P(m,s,b)$ and theorem is proved.

Theorem 2. *Consider the problem $P(m,k,b)$ such that $1\leqslant k\leqslant m-1$, $b\leqslant k$ and m, k relative prime numbers. If $M=\{1+tk\,;\ t=0,1,\dots,mb/k-1\}$ then an optimal solution of $P(m,k,b)$ is given by*

$$x_i=\begin{cases}1 & ,\ i\in M,\\ 0 & ,\ i\notin M.\end{cases} \tag{1.10}$$

Proof: Consider a mb-dimensional vector $y=(y_i)$ defined by

$$y_i=\begin{cases}1 & ,\ i\in\{1+tk\,;\ 1+tk\leqslant mb\},\\ 0 & ,\ i\notin\{1+tk\,;\ 1+tk\leqslant mb\}.\end{cases} \tag{1.11}$$

It is easy to prove that the vector $x=(x_i)$ defined by (1.10) has the form

$$x=\sum_{h=0}^{b-1} z^h, \tag{1.12}$$

where the vectors $z^h=(z_i^h)$, $h=0,1,\dots,b-1$, are defined by

$$z_i^h = y_{mh+i},\ h=0,1,\dots,b-1\,;\ i=1,\dots,m. \tag{1.13}$$

On the other hand it is obvious that z^0 is an optimal solution of $P(m,k,1)$. Now suppose that $\tilde x$ is an optimal solution of $P(m,k,\tilde b)$, where $\tilde b<b$ and $\tilde x$ is defined by

$$\tilde x=\sum_{h=0}^{\tilde b-1} z^h. \tag{1.14}$$

Because of (1.13) it is easy to prove that $\tilde x+z^{\tilde b}$ is an optimal solution of $P(m,k,\tilde b+1)$. Hence we can state that the vector $x=(x_i)$, defined by (1.12) is an optimal solution of $P(m,k,b)$ and the theorem is proved.

Remark 1. We remark that the vector x defined by (1.12) is obtained by adding b m-dimensional binary vectors z^h, $h=0,1,\dots,b-1$, some of them being optimal solution of $P(m,k,1)$. It is easy to prove that when z^p, $0\leqslant p\leqslant b-1$, is an optimal solution of $P(m,k,1)$, then we can replace in (1.12) z^p by any cyclic transformation of z^p and one obtains also an optimal solution of $P(m,k,b)$.

Remark 2. Consider the problem $P(m,k,b)$ such that $b=qk+r$, $1\leqslant r\leqslant m-1$ and suppose that x is an optimal solution of $P(m,k,r)$. Then $x+y$ is an optimal solution of $P(m,k,b)$, where y is a m-dimensional vector the components of which are all equal to q, so that theorems 1 and 2 and this remark can serve to prove that the optimal value of the objective function in $P(m,s,b)$ is $\lceil mb/s\rceil$ for all s. On the other hand, Remark 1 provides a method for finding a class of optimal solutions for $P(m,s,b)$.

Example 1.1. Consider the problem $P(7,11,6)$. By theorem 1, if x is an optimal solution of $P(7,4,2)$, then x is an optimal solution of $P(7,11,6)$ because in this case $\lceil mb/s\rceil=\lceil m\hat b/k\rceil$. Taking into account theorem 2 it follows that $x=z^0+z^1$ is an optimal solution of $P(7,4,2)$, where $z^0=(1,0,0,0,1,0,0)$, $z^1=(0,1,0,0,0,1,0)$ and by using Remark 1 one obtains the following optimal solutions of $P(7,4,2)$: $(1,1,0,0,1,1,0)$, $(1,0,1,0,1,0,1)$, $(2,0,0,1,1,0,0)$, $(1,0,1,0,1,1,0)$, $(2,0,0,0,2,0,0)$ and their cyclic transformations, which are optimal solutions also for $P(7,11,6)$.

Lemma 1. *Suppose in problem $P(m,s,b)$, $s=mq+k$, $1\leqslant k\leqslant m-1$ and consider $\hat b=b-q\lceil mb/s\rceil$. Then x is an optimal solution of $P(m,s,b)$ iff there exists $\delta=(\delta_i)$ such that $(x\,;\,\delta)$ is a solution of the system*

(1.15) $\quad$ a) $\sum_{j=0}^{k-1} \Sigma x_{i\oplus j}=\hat{b}+\delta_i \quad i=1,\dots,m \qquad$ b) $\Sigma\delta_i=s-r,$

$$\text{c) } x_i,\ \delta_i\in Z_+ \quad i=1,\dots,m,$$

where r is defined by (1.4).

Proof: We have shown that x is an optimal solution of $P(m, s, b)$ iff $(x\,;\,\delta)$ is a solution of the system (1.7). It is obvious that (1.7a, b) implies $\Sigma x_i=\lceil mb/s\rceil$; hence the system (1.7) is equivalent to the system (1.15) and the lemma is proved.

2. In this section we shall prove two lemmas, which will be useful in the next section.

Lemma 2. *Consider the integers m, k, $1\leqslant k\leqslant m-1$, and let d be the greatest common divisor of m and k, denoted by $d=(m, k)$. Then, if $i\in\{1,\dots, m\}$, there exists a unique $j\in\{1,\dots, d\}$ such that the equation*

(2.1) $$i\oplus yk=m-d+j$$

has solutions $y\in Z$, where $\oplus$ denotes the addition mod. m, *when the residuals* mod. m *are considered* $1,\dots, m$.

Proof: Suppose $i\equiv h_i$ (mod. m), $1\leqslant h_i\leqslant d$. It is well known that the equation (2.1) has solutions $y\in Z$ iff d divides $j-h_i$ and because $1\leqslant h_i\leqslant d$, it follows that (2.1) has solutions $y\in Z$ for $1\leqslant j\leqslant d$ iff $j=h_i$ and the lemma is proved.

Remark 3. Let m, k be two integers such that $1\leqslant k\leqslant m-1$, $d=(m, k)$, $k=\alpha d$, $m=\beta d$. If we define M_i, $i=1,\dots, d$, by

(2.2) $$M_i=\{i\oplus ky,\ 0\leqslant y\leqslant\beta-1\} \quad i=1,\dots, d,$$

then one can easily prove that

(2.3)
a) $M_i\cap M_j=\varnothing$, for all $i, j, i\neq j$, $\quad$ b) $|M_i|=\beta$, $i=1,\dots, d$,
c) If $i\in\{1,\dots, m\}$, $i\equiv h_i$ (mod d), $1\leqslant h_i\leqslant d$, *then* i, $m-d+h_i\in M_{h_i}$

Lemma 3. *Suppose* $m, k\in Z$, $1\leqslant k\leqslant m-1$, $d=(m, k)$, $k=\alpha d$, $m=\beta d$. *If t_1 and γ are defined by*

(2.4)
$$t_1=\min\{y\,;\ 0\leqslant y\leqslant\beta-1,\ 1\oplus yk=m-d+1\},$$
$$\gamma=|\{1+td\,;\ 0\leqslant t\leqslant\alpha-1\}\cap\{1\oplus yk\,;\ 0\leqslant y<t_1\}|$$

then we have

(2.5) $$\gamma m=t_1k+d$$

Proof: Because of (2.3c) it follows:

(2.6) $$\{1\oplus tk\,;\ 0\leqslant t\leqslant\beta-1\}\supset\{1+td\,;\ 0\leqslant t\leqslant\alpha-1\}.$$

Now suppose $p_i\in Z$, $i=1,\dots,\gamma$, such that

(2.7)
$$0=p_1<p_2<\dots<p_\gamma<t_1,$$
$$1\oplus p_ik\in\{1+td\,;\ 0\leqslant t\leqslant\alpha-1\},\quad i=1,\dots,\gamma.$$

where t_1 and γ are defined by (2.4). Since $1\leqslant k\leqslant m-1$ and t_1, γ are defined by (2.4), it is clear that

(2.8) $$1+p_ik=1\oplus p_ik+(i-1)\,m,\quad i=1,\dots,\gamma.$$

Because $1\oplus t_1k=m-d+1$ and $k=\alpha d$, it follows

(2.9) $$1+(t_1+1)\,k=1+(\alpha-1)\,d+\gamma m.$$

which shows that (2.5) is true, and the lemma is proved.

In the next section we will use some notations which we give here.

Notation 1. *Suppose $m, k\in Z$, $1\leqslant k\leqslant m-1$ and $d=(m, k)$. Then for all i, $i=1,\dots, m$, $i\equiv h_i$* (mod. d), *$1\leqslant h_i\leqslant d$, we shall denote by t_i, $i=1,\dots, m$, the numbers defined by*

(2.10) $$t_i=\min\{y\,;\ 0\leqslant y\leqslant\beta-1,\ i\oplus yk=m-d+h_i\},\ i=1,\dots, m.$$

We note that when $1\leqslant i\leqslant d$, then $t_i=t_1$ and $t_{i+p-h_i}=t_i$ for all p, $p=h_i,\dots, d$.

Notation 2. *Consider m real numbers a_i, $i=1,\dots, m$, and $k\in Z$, $1\leqslant k\leqslant m-1$. If $i, j\in\{1,\dots, m\}$ such that $\{t\,;\ t\geqslant 1, i\oplus tk=j\}\neq\varnothing$ then $a_{(i,j)}$ denotes the number defined by*

(2.11) $$a_{(i,\,j)}=\sum_{t=0}^{T}\Sigma a_{i\oplus tk},$$

where $T=\min\{t-1,\ t\geqslant 1,\ i\oplus tk=j\}$.

We remark that $\{t\,;\ t\geqslant 1, i\oplus tk=j\}\neq\varnothing$ iff $i\equiv j$ (mod. d), where $d=(m, k)$ is the greatest common divisor of m and k.

3. Now we return to the problem $P(m, s, b)$ defined in the first section. Suppose $s=mq+k$, $0\leqslant k\leqslant m-1$; if $k=0$, it is clear that x is a feasible solution of $P(m, s, b)$ iff

(3.1) $$q\Sigma x_j\geqslant b$$

and therefore every optimal solution of $P(m, s, b)$ is a partition of $\lceil b/q\rceil$ into m non-negative integers and conversely. Hence, we can suppose $s=mq+k$, $1\leqslant k\leqslant m-1$. In section 1 we have shown that in order to find the optimal solutionss of the problem $P(m, s, b)$, it is sufficient to find the solution of the system (1.15). If we define f_i, $i=1,\dots, m$, by

(3.2) $$f_i=\sum_{j=0}^{k-1} x_{i\oplus j},\ i=1,\dots, m,$$

then obviously the system (1.15) becomes

(3.3) $\quad$ a) $f_i=\hat{b}+\delta_i \quad i=1,\dots, m$, b) $\Sigma\delta_i=s-r,$

$$\text{c) } x_i,\ \delta_i\in Z_+,\ i=1,\dots, m.$$

Lemma 4. *Consider* $d=(m, k)$ $i\equiv h_i$ (mod. d), $1\leqslant h_i\leqslant d$, $i=1,\dots, m$. *Then the system* (3.3a) *has the same solutions* $(x\,;\,\delta)$ *as the system*

(3.4) $\quad f_{[i,i^*)}-f_{[i+1,i^*\oplus 1)}=\delta_{[i,i^*)}-\delta_{[i+1,i^*\oplus 1)},\ i=1,\dots,m-1,\ f_m=\hat{b}+\delta_m$

where $i^*=m-d+h_i,\ i=1,\dots,m.$

Proof: Taking into account (2.11), it is easy to prove that

$$a_{[i,i^*)}-a_{[i\oplus 1,i^*\oplus 1)}=a_i-a_{i\oplus 1},\ \text{if}\ t_i=1,\quad a_{[i,i^*)}-a_{[i\oplus 1,i^*\oplus 1)}-a_{[i\oplus k,i\oplus k^*)}+ \\ +a_{[(i\oplus k)\oplus 1,(i\oplus k)^*\oplus 1)}=a_i-a_{i+1}\ ,\text{if}\ t_i>1. \tag{3.5}$$

Hence the system (3.4) is equivalent to the system

$$f_i-f_{i+1}=\delta_i-\delta_{i+1},\ i=1,\dots,m-1,\ f_m=\hat{b}+\delta_m \tag{3.6}$$

and obviously this system is equivalent to the system (3.3a). The theorem is proved.

Lemma 5. *Suppose* $1\leqslant k\leqslant m-1,\ d=(m,k),\ i\in\{1,\dots,m-1\},\ i\equiv h_i$ (mod. d), $1\leqslant h_i\leqslant d$. *Then*

$$\begin{aligned} &\text{i) if } 1\leqslant i\leqslant m-d \textit{ we have } f_{[i,i^*)}-f_{[i+1,i^*\oplus 1)}=x_i-x_{i^*}\\ &\text{ii) if } m-d+1\leqslant i\leqslant m-1 \textit{ we have}\end{aligned} \tag{3.7}$$

$$f_{[i,i^*)}-f_{[i+1,i^*\oplus 1)}=0, \tag{3.8}$$

where $i^*=m-d+h_i\quad i=1,\dots,m-1.$

Proof. By using Notations 1, 2 and because of (3.2) we can write

$$f_{[i,i^*)}=\sum_{j=0}^{t_ik-1}x_{i\oplus j},\ i=1,\dots,m-1,\quad f_{[i+1,i^*\oplus 1)}=\sum_{j=1}^{t_ik}x_{i\oplus j},\ i=1,\dots,m-1, \tag{3.9}$$

where t_i, $1\leqslant i\leqslant m-1$, is defined by (2.10).
Since $i\oplus t_ik=m-d+h_i$ it follows that

$$f_{[i,i^*)}-f_{[i+1,i^*\oplus 1)}=x_i-x_{m-d+h_i},\ i=1,\dots,m-1. \tag{3.10}$$

It is clear that when $m-d+1\leqslant i\leqslant m-1$ then $i=i^*=m-d+h_i$ and therefore the formulas (3.10) prove the lemma. By lemmas 4, 5 and the fact that the system (1.15) is equivalent to the system (3.3), it follows that the system (1.15) is equivalent to the system

$$\begin{aligned} &\text{a)}\quad x_i-x_{i^*}=\delta_{[i,i^*)}-\delta_{[i+1,i^*\oplus 1)},\ i=1,\dots,m-d,\\ &\text{b)}\quad 0=\delta_{[i,i^*)}-\delta_{[i+1,i^*\oplus 1)}\ ,i=m-d+1,\dots,m-1,\\ &\text{c)}\quad x_m+\sum_{j=1}^{k-1}x_j=\hat{b}+\delta_m,\quad \text{d)}\quad \Sigma\delta_i=s-r,\quad \text{e)}\quad x_i,\ \delta_i\in Z_+,\ i=1,\dots,m,\end{aligned} \tag{3.11}$$

where $d=(m,k)$, $i^*=m-d+h_i$, $i=1,\dots,m-1$ and h_i, $i=1,\dots,m-1$, satisfy $1\leqslant h_i\leqslant d$, $i\equiv h_i$ (mod. d), $i=1,\dots,m-1$.

Taking into account Remark 3, it follows that (3.11 b) is equivalent to

$$\delta_{[i,i)}=\delta_{[i+1,i+1)}\ ,i=1,\dots,d-1. \tag{3.11b$'$}$$

Hence (3.11 b, d) can be replaced by

$$\delta_{[i,i)}=(s-r)/d\quad i=1,\dots,d. \tag{3.12}$$

From the above results it follows that we can state the next theorem

Theorem 3. *The solutions of the system (1.15) are the same as the solutions of the system*

$$\begin{aligned} &\text{a)}\quad x_i-x_{i^*}=\delta_{[i,i)}-\delta_{[i+1,i^*\oplus 1)}\ ,i=1,\dots,m-d,\\ &\text{b)}\quad x_m+\sum_{j=1}^{k-1}x_j=\hat{b}+\delta_m,\quad \text{c)}\quad \delta_{[i,i)}=(s-r)/d,\ i=1,\dots,d,\\ &\text{d)}\quad x_i,\ \delta_i\in Z_+\ ,i=1,\dots,m,\end{aligned} \tag{3.13}$$

where d *and* i^*, $i=1,\dots,m-1$, *are defined as above.*

Lemma 6. *For every solution* $(x\,;\,\delta)$ *of the system* (3.13) *we have*

$$\sum_{j=1}^{k-1}x_j-\alpha\sum_{m-d+1}^{m-1}x_j-(\alpha-1)\,x_m=\alpha\delta_{[1,m-d+1)}-(\gamma-1)\,\delta_{[1,1]}-\delta_{[k,m)} \tag{3.14}$$

where $d=(m,k)$ $k=\alpha d$, $m=\beta d$ *and* γ *is defined by* (2.4).

Proof: Suppose that $(x\,;\,\delta)$ is a solution of the system (3.13). By adding the first $k-1$ equalities of (3.13 a), one obtains

$$\sum_{j=1}^{k-1}x_j-(\alpha-1)\sum_{m-d+1}^{m}x_j-\sum_{m-d+1}^{m-1}x_j=\sum_{i=1}^{k-1}\left[\delta_{[i,i^*)}-\delta_{[i+1,i^*\oplus 1)}\right]. \tag{3.15}$$

Since γ is defined by (2.4), it results that γ amongst the numbers $1+yd$, $y=0,1,\dots,\alpha-1$, preceed $m-d+1=1\oplus t_1k$ in the sequence

$$1,\ 1\oplus k,\dots,\ 1\oplus(\beta-1)\,k \tag{3.16}$$

and the others, $\alpha-\gamma$ succeed $m-d+1$ in the same sequence. On the other hand, if $1+yk$, $0\leqslant y\leqslant\alpha-1$, is before $m-d+1$ in the sequence (3.16) then

$$\delta_{[1+yd,1)}=\delta_{[1+yd,m-d+1)}+\delta_{[m-d+1,1)} \tag{3.17}$$

and when $1+yd$, $1\leqslant y\leqslant\alpha-1$, is after $m-d+1$ in the sequence (3.16) then

$$\delta_{[1+yd,1)}=\delta_{[1+yd,m-d+1)}+\delta_{[1,m-d+1)}. \tag{3.17$'$}$$

Taking into account (3.17), (3.17′) and Remark 3 it follows

$$\sum_{i=1}^{k-1}\left[\delta_{[i,i^*)}-\delta_{[i+1,i^*\oplus 1)}\right]=\alpha\delta_{[1,m-d+1)}-(\gamma-1)\delta_{[1,1)}-\delta_{[k,m)}. \tag{3.18}$$

Hence by (3.15), (3.18) it results (3.14) and the lemma is proved.

Lemma 7. The system (3.13) is equivalent to the system

$$\text{a)}\quad x_i - x_{i^*} = \delta_{[i,i^*)} - \delta_{[i+1,i^*\oplus 1)}\ , i=1,\dots,m-d,$$

$$(3.19)\quad \text{b)}\quad \sum_{m-d+1}^{m} x_j = [d\hat{b} - (\alpha-\gamma)(s-r)]/k + \delta_{[m-d+1,1)}$$

$$\text{c)}\quad \delta_{[i,i)} = (s-r)/d\ , i=1,\dots,d, \qquad \text{d)}\quad x_i,\ \delta_i \in Z_+\ , i=1,\dots,m.$$

Proof: In order to prove this lemma it is enough to prove that in (3.13) we can replace (3.13 b) by (3.19 b). It is clear that $\delta_{[k,m)} + \delta_m = \delta_{[k,k)}$ and because of (3.13 c) it follows that $\delta_{[k,k)} = \delta_{[1,1)}$. By lemma 6 we can replace (3.13 b) by

$$(3.20)\qquad \alpha \sum_{m-d+1}^{m} x_j = \hat{b} - \alpha\delta_{[1,m+d+1)} + \gamma\delta_{[1,1)}.$$

Since $\delta_{[1,1)} = (s-r)/d$ it is easy to prove that

$$(3.21)\qquad \hat{b} + \gamma\delta_{[1,1)} = [d\hat{b} + \gamma(s-r)]\ \alpha/k.$$

Because of (3.20), (3.21) and of the fact that $\delta_{[1,m-d+1)} = \delta_{[1,1)} - \delta_{[m-d+1,1)}$, it follows that we can replace (3.13 b) by (3.19 b) and the lemma is proved.

Remark that $[d\hat{b} + \gamma(s-r)]/k \in Z$, since $\gamma m - t_1 k = d$ and then

$$(3.22)\qquad [d\hat{b} + \gamma(s-r)]/k = \gamma\lceil mb/s\rceil - t_1 b + q t_1 \lfloor mb/s\rfloor.$$

Consequently it results that the right hand side terms of the system (3.19 a, b) are all integers, when $\delta_i \in Z$, $i=1,\dots,m$. In the following, $\bar{b}$ denots the number defined by

$$(3.23)\qquad \bar{b} = [d\hat{b} - (\alpha-\gamma)(s-r)]/k,$$

where $\hat{b}$ is defined by (1.8).

Theorem 4. *Consider the system (1.15) in which $s = mq + k$, $1 \leqslant k \leqslant m-1$, $d = (m, k)$, $k = \alpha d$, $m = \beta d$ and γ, t_1 the numbers defined by (2.4). Then*

i) *If $d \geqslant 2$ the system (1.15) is equivalent to the system*

$$\text{a)}\quad x_i + \sum_{m-d+2}^{m} x_j = \bar{b} + \delta_{[m-d+1,1)} + \delta_{[i,i^*)} - \delta_{(i+1,i^*\oplus 1)},\ i \in M_1$$

$$(3.24)\quad \text{b)}\quad x_i - x_{i^*} = \delta_{[i,i^*)} - \delta_{[i+1,i^*\oplus 1)}\ , i \notin M_1\ , i \leqslant m-d$$

$$\text{c)}\quad \delta_{[i,i)} = (s-r)/d\ , i=1,\dots,d \qquad \text{d)}\quad x_i,\ \delta_i \in Z_+,\ i=1,\dots,m$$

where M_1 is defined by (2.2) and $i^* = m-d+h_i$, $i=1,\dots,m$ are defined as above.

ii) *If $d=1$ then the system (1.15) is equivalent to the system*

$$(3.25)\quad \text{a)}\quad x_i = \bar{b} + \delta_{[i,i\oplus 1)},\ i=1,\dots,m, \qquad \text{b)}\quad \Sigma\delta_i = s-r,$$

$$\text{c)}\quad x_i,\ \delta_i \in Z_+\ , i=1,\dots,m.$$

Proof: i) Consider $(m, k) = d \geqslant 2$. By lemma 7, the system (1.15) is equivalent to the system (3.19). Since the system (3.24) is obtained, from (3.19) by adding the equation (3.19 b) to the equations $1+yd$, $y=0, 1,\dots \beta-1$, the first part of the theorem is proved.

ii) If $d=1$ the system (3.19) becomes

$$(3.19')\quad \text{a)}\quad x_i - x_m = \delta_{[i,m)} - \delta_{[i+1,1)}\ , i=1,\dots,m-1 \qquad \text{b)}\quad x_m = \bar{b} + \delta_{[m,1)}$$

$$\text{c)}\quad \Sigma\delta_i = s-r \qquad \text{d)}\quad x_i,\ \delta_i \in Z_+,\ i=1,\dots,m.$$

By adding the equation (3.19′ b) to each equation (3.19′ a) it results that, in (3.19′), we can replace (3.19′ a) by

$$(3.26)\qquad x_i = \bar{b} + \delta_{[m,1)} + \delta_{[i,m)} - \delta_{[i+1,1)}\ , i=1,\dots,m-1.$$

Since, in this case $(i+1)\oplus t_1 k = i$, $i=1,\dots,m-1$, it is easy to prove

$$(3.27)\qquad \delta_{[m,1)} + \delta_{[i,m)} - \delta_{[i+1,1)} = \delta_{[i,i+1)}\ , i=1,\dots,m-1.$$

Hence (3.19′) is equivalent to (3.25) and the theorem is completely proved.

The next corollaries are direct consequences of the Theorem 4.

Corollary 1. *Suppose $(m, s) = d \geqslant 2$. If $x = (x_i)$ is an optimal solution of $P(m, s, b)$ then $\bar{x} = (\bar{x}_i)$, defined by*

$$\bar{x}_i = x_i - \sum_{m-d+2}^{m} x_j \text{ for all } i \in M_1,$$

$$(3.28)\qquad \bar{x}_i = x_i + x_{m-d+h_i}\ , \text{for all } i \notin M_1,\ i \leqslant m-d,$$

$$\bar{x}_i = 0,\ i = m-d+1,\dots,m,$$

is also an optimal solution of $P(m, s, b)$.

Remark that this corollary shows that for finding all the optimal solutions of $P(m, s, b)$ it is sufficient to find those optimal solutions with the last $d-1$ components equal to zero.

The following corollary has as direct consequences Theorems 7 and 8 of the [3].

Corollary 2. *Consider the problem $P(m, s, b)$ such that $s-r=0$, where r is defined by (1.4). If $\bar{b}$ is defined by (3.23) then*

i) *when $(m, s) = 1$, the problem $P(m, s, b)$ has only one optimal solution $x = (x_i)$, where $x_i = \bar{b}$, $i =\ ,\dots, m$.*

ii) *when $(m, s) = d \geqslant 2$, all the optimal solutions of the problem $P(m, s, b)$ are given by,*

$$(3.29)\qquad x_i = \bar{b} - \sum_{m-d+2}^{m} x_i,\ \text{for all } i \in M_1,$$

$$x_i = x_{m-d+h_i},\ \text{for all } i \notin M_1,\ i \leqslant m-d, i \equiv h_i (\mathrm{mod}\cdot d),\ 1 \leqslant h_i \leqslant d,$$

where x_i, $i = m-d+2,\dots,m$ are non-negative integers such that $\sum_{m-d+2}^{m} x \leqslant \bar{b}$.

Corollary 3. *If the problem $P(m, s, b)$ is such that $(m, s)=1$ and $\bar{b}\geq 0$ then all the optimal solutions of $P(m, s, b)$ are given by*

$$x_i=\bar{b}+\delta_{[i,i+1]},\quad i=1,\dots, m, \tag{3.30}$$

where $\delta=(\delta_i)$ is a partition of $s-r$ into m non-negative integers. Moreover the number of the optimal solutions of $P(m, s, b)$ is equal to the number of the partitions of $s-r$ into m non-negative integers. We remark that if m and s are given, then for b sufficiently large the inequality $\bar{b}\geq 0$ holds.

Theorem 5. *The problem $P(m, s, b)$ has a unique optimal solution iff m and s are relatively prime numbers and $mb/s\in Z$.*

Proof: By using Corollary 2 it is clear that when $(m, s)=1$ and $mb/s \in Z$ then $P(m, s, b)$ has a unique optimal solution. Conversely, suppose that $P(m, s, b)$ has a unique optimal solution; by Corollary 1 it follows that $(m, s)=1$ and by Theorem 1 it follows that $P(m, k, \hat{b})$ has a unique optimal solution too. Then Theorem 1 and remark 2 show that $m\hat{b}/k\in Z$. Now it is easy to prove that $m\hat{b}/k\in Z$ iff $(s-r)/k\in Z$, where r is defined by (1.4) and by using again Theorem 1 it results that $P(m, s, b)$ has a unique optimal solution only if $s-r=0$ and the theorem is proved.

Example 3.1. Consider the problem $P(m, s, b)$ where $m=19$, $s=21$, $b=42$. By Corollary 2 it follows that this problem has a unique optimal solution $x=(x_i)$, where $x_i=2$, $i=1,\dots, 19$.

Example 3.2. Consider the problem $P(m, s, b)$ where $m=6$, $s=19$, $b=21$. In this case $\hat{b}=\bar{b}=0$ and because $(m, s)=1$, $s-r=7$, it follows that all the optimal solutions of $P(6, 19, 21)$ are of the form $x_i=\delta_i$, $i=1,\dots, 6$, where $\delta=(\delta_i)$ is any partition of 7 into 6 non-negative integers.

Example 3.3. Consider the problem $P(m, s, b)$ where $m=10$, $s=4$, $b=5$. In this case $s=k=4$, $r=2$, $s-r=2$, $d=2$ and $\bar{b}=2$. By using Theorem 4 and Corollary 1 we obtain the following optimal solutions of $P(10, 4, 5)$: (2, 1, 2, 1, 2, 0, 2, 1, 2, 0), (2, 1, 2, 1, 1, 1, 2, 1, 2, 0), (3, 0, 3, 0, 2, 0, 3, 0, 2, 0), (3, 0, 2, 1, 2, 0, 3, 0, 2, 0), (3, 0, 2, 1, 2, 0, 2, 1, 2, 0), (2, 1, 1, 2, 1, 1, 2, 1, 1, 1), (2, 1, 2, 1, 1, 1, 2, 1, 1, 1) and their cyclic transformations.

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SOME REMARKS CONCERNING THE KERNELS OF A STRONG CONNECTED DIGRAPH

BY

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Let $D=<\mathcal{N}, \mathcal{A}>$ be a *strong connected digraph* [1], with $\mathcal{N}=\{n_1, n_2,\dots, n_p\}$ the set of *nodes* and $\mathcal{A}=\{a_1, a_2,\dots, a_q\}$ the set of *arcs*. In [3] we give a *sufficient condition* for the existence of a kernel in a strong connected digraph. In [2] we suggested a *method* for finding the kernel in a digraph. Moreover, the algorithm from [2] can be considered as a simplified demonstration of Richardson's theorem [4]. In this paper we give a *new and complete* form of the theorem 6 from [3], using a simplified proof, and we make some remarks concerning the kernels of a strong connected digraph.

Let n be an arbitrary node of $\mathcal{N}$. We denote by $\mathcal{N}^+_{[n]}$ the set: $\mathcal{N}^+_{[n]}=\{n\}\cup\{m\in\mathcal{N} \mid$ there exists $<n, m>\in\mathcal{A}\}$ (see [2]). For a subset $\mathcal{X}\subset\mathcal{N}$, we consider (see [2]) the set: $\mathcal{N}^+[\mathcal{X}]=\bigcup_{n\in\mathcal{X}}\mathcal{N}^+_{[n]}\supseteq\mathcal{X}$.

Theorem 1. *Let $D=<\mathcal{N}, \mathcal{A}>$ be a strong connected digraph. If it exists a unique partition $\mathcal{P}=\{\mathcal{N}_1, \mathcal{N}_2\}$ of $\mathcal{N}$, so that every arc of $\mathcal{A}$ links a node of $\mathcal{N}_1$ with a node of $\mathcal{N}_2$, then D contains at least two kernels.*

Proof. From the hypothesis, it results that for every two nodes n and m from $\mathcal{N}_1$ we have no arc $<n, m>\in\mathcal{A}$, (i.e. for every two $n, m\in\mathcal{N}_1$ we have $<n, m>\notin\mathcal{N}$). Similarly for $\mathcal{N}_2$. Moreover, because D is strong connected, we have: $\mathcal{N}^+[\mathcal{N}_1]=\mathcal{N}^+[\mathcal{N}_2]=\mathcal{N}$.

Hence, $\mathcal{N}_1$ and $\mathcal{N}_2$ are kernels of D (see [1], [2]) (Q.E.D)

Theorem 2. *If $D=<\mathcal{N}, \mathcal{A}>$ is a digraph consisting in a single elementary even circuit (with an even number of arcs), then, D contains exactly two kernels and only two.*

Proof. Let $[\lambda_0=n_{i_1}, n_{i_2}, n_{i_3},\dots, n_{i_p}, n_{i_1}]$ be the even circuit who forms D.

We define the sets: $\mathcal{X}_1=\{n_{i_1}, n_{i_3}, n_{i_5},\dots\}$ and $\mathcal{X}_2=\{n_{i_2}, n_{i_4}, n_{i_6},\dots\}$. Evidently, the last node of $\mathcal{X}_1$ is n_{p-1}, if and only if λ_0 is an even circuit.

Hence, in view of theorem 1, it results that $\mathcal{X}_1$ and $\mathcal{X}_2$ are the single kernels of D. (Q.E.D.)

Remark. Evidently, in view of theorem 1), if D is a digraph consisting in a single odd circuit, D does not contain a kernel.

Corollary. *Let $D=<\mathcal{N}, \mathcal{A}>$ be a strong connected digraph. If D contains a pair of kernels $\mathcal{P}=\{\mathcal{X}_1, \mathcal{X}_2\}$, who forms a partition of $\mathcal{N}$ so that every*