Corollary 3. *If the problem $P(m, s, b)$ is such that $(m, s)=1$ and $\bar{b}\geq 0$ then all the optimal solutions of $P(m, s, b)$ are given by*

$$x_i=\bar{b}+\delta_{[i,l+1]}, \quad i=1,\dots, m, \tag{3.30}$$

where $\delta=(\delta_i)$ is a partition of $s-r$ into m non-negative integers. Moreover the number of the optimal solutions of $P(m, s, b)$ is equal to the number of the partitions of $s-r$ into m non-negative integers. We remark that if m and s are given, then for b sufficiently large the inequality $\bar{b}\geq 0$ holds.

Theorem 5. *The problem $P(m, s, b)$ has a unique optimal solution iff m and s are relatively prime numbers and $mb/s\in Z$.*

Proof: By using Corollary 2 it is clear that when $(m, s)=1$ and $mb/s\in Z$ then $P(m, s, b)$ has a unique optimal solution. Conversely, suppose that $P(m, s, b)$ has a unique optimal solution; by Corollary 1 it follows that $(m, s)=1$ and by Theorem 1 it follows that $P(m, k, \hat{b})$ has a unique optimal solution too. Then Theorem 1 and remark 2 show that $m\hat{b}/k\in Z$. Now it is easy to prove that $m\hat{b}/k\in Z$ iff $(s-r)/k\in Z$, where r is defined by (1.4) and by using again Theorem 1 it results that $P(m, s, b)$ has a unique optimal solution only if $s-r=0$ and the theorem is proved.

Example 3.1. Consider the problem $P(m, s, b)$ where $m=19$, $s=21$, $b=42$. By Corollary 2 it follows that this problem has a unique optimal solution $x=(x_i)$, where $x_i=2$, $i=1,\dots, 19$.

Example 3.2. Consider the problem $P(m, s, b)$ where $m=6$, $s=19$, $b=21$. In this case $\hat{b}=\bar{b}=0$ and because $(m, s)=1$, $s-r=7$, it follows that all the optimal solutions of $P(6, 19, 21)$ are of the form $x_i=\delta_i$, $i=1,\dots, 6$, where $\delta=(\delta_i)$ is any partition of 7 into 6 non-negative integers.

Example 3.3. Consider the problem $P(m, s, b)$ where $m=10$, $s=4$, $b=5$. In this case $s=k=4$, $r=2$, $s-r=2$, $d=2$ and $\bar{b}=2$. By using Theorem 4 and Corollary 1 we obtain the following optimal solutions of $P(10, 4, 5)$: (2, 1, 2, 1, 2, 0, 2, 1, 2, 0), (2, 1, 2, 1, 1, 1, 2, 1, 2, 0), (3, 0, 3, 0, 2, 0, 3, 0, 2, 0), (3, 0, 2, 1, 2, 0, 3, 0, 2, 0), (3, 0, 2, 1, 2, 0, 2, 1, 2, 0), (2, 1, 1, 2, 1, 1, 2, 1, 1, 1), (2, 1, 2, 1, 1, 1, 2, 1, 1, 1) and their cyclic transformations.

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Received 12.1.1980

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Analele ştiinţifice ale Universităţii „Al. I. Cuza" din IAŞI
Tomul XXVI, s. I a, 1980, f. 2

SOME REMARKS CONCERNING THE KERNELS OF A STRONG CONNECTED DIGRAPH

BY

DĂNUŢ MARCU

Let $D=<\mathcal{N}, \mathcal{A}>$ be a *strong connected digraph* [1], with $\mathcal{N}=\{n_1, n_2,\dots, n_p\}$ the set of *nodes* and $\mathcal{A}=\{a_1, a_2,\dots, a_q\}$ the set of *arcs*. In [3] we give a *sufficient condition* for the existence of a kernel in a strong connected digraph. In [2] we suggested a *method* for finding the kernel in a digraph. Moreover, the algorithm from [2] can be considered as a simplified demonstration of Richardson's theorem [4]. In this paper we give a *new and complete* form of the theorem 6 from [3], using a simplified proof, and we make some remarks concerning the kernels of a strong connected digraph.

Let n be an arbitrary node of $\mathcal{N}$. We denote by $\mathcal{N}^+_{[n]}$ the set: $\mathcal{N}^+_{[n]}=\{n\}\cup\{m\in\mathcal{N} \mid \text{there exists } <n, m>\in\mathcal{A}\}$ (see [2]). For a subset $\mathcal{X}\subset\mathcal{N}$, we consider (see [2]) the set: $\mathcal{N}^+[\mathcal{X}]=\bigcup_{n\in\mathcal{X}}\mathcal{N}^+_{[n]}\supset\mathcal{X}$.

Theorem 1. *Let $D=<\mathcal{N}, \mathcal{A}>$ be a strong connected digraph. If it exists a unique partition $\mathcal{P}=\{\mathcal{N}_1, \mathcal{N}_2\}$ of $\mathcal{N}$, so that every arc of $\mathcal{A}$ links a node of $\mathcal{N}_1$ with a node of $\mathcal{N}_2$, then D contains at least two kernels.*

Proof. From the hypothesis, it results that for every two nodes n and m from $\mathcal{N}_1$ we have no arc $<n, m>\in\mathcal{A}$, (i.e. for every two $n, m\in\mathcal{N}_1$ we have $<n, m>\notin\mathcal{N}$). Similarly for $\mathcal{N}_2$. Moreover, because D is strong connected, we have: $\mathcal{N}^+[\mathcal{N}_1]=\mathcal{N}^+[\mathcal{N}_2]=\mathcal{N}$.

Hence, $\mathcal{N}_1$ and $\mathcal{N}_2$ are kernels of D (see [1], [2]) (Q.E.D)

Theorem 2. *If $D=<\mathcal{N}, \mathcal{A}>$ is a digraph consisting in a single elementary even circuit (with an even number of arcs), then, D contains exactly two kernels and only two.*

Proof. Let $[\lambda_0=n_{i_1}, n_{i_2}, n_{i_3},\dots, n_{i_p}, n_{i_1}]$ be the even circuit who forms D.

We define the sets: $\mathcal{X}_1=\{n_{i_1}, n_{i_3}, n_{i_5},\dots\}$ and $\mathcal{X}_2=\{n_{i_2}, n_{i_4}, n_{i_6},\dots\}$. Evidently, the last node of $\mathcal{X}_1$ is $n_{i_{p-1}}$, if and only if λ_0 is an even circuit.

Hence, in view of theorem 1, it results that $\mathcal{X}_1$ and $\mathcal{X}_2$ are the single kernels of D. (*Q.E.D.*)

Remark. Evidently, in view of theorem 1), if D is a digraph consisting in a single odd circuit, D does not contain a kernel.

Corollary. *Let $D=<\mathcal{N}, \mathcal{A}>$ be a strong connected digraph. If D contains a pair of kernels $\mathcal{P}=\{\mathcal{X}_1, \mathcal{X}_2\}$, who forms a partition of $\mathcal{N}$ so that every* *arc of $\mathcal{A}$ links a node of $\mathcal{K}_1$ with a node of $\mathcal{K}_2$, then $\mathcal{P}$ is unique with this property.*

Proof. The proof. is obvious, having in view theorem 1 and the definition of the kernel [2].

Theorem 3. *If $D=<\mathcal{N},\mathcal{A}>$ is a strong connected digraph and $\mathcal{K}\subset\mathcal{N}$ kernel of D, then we have $\mathcal{N}^+[\mathcal{N}\setminus\mathcal{K}]=\mathcal{N}$.*

Proof. Since D is strong connected, for every node $n_0\in\mathcal{K}$, it exists $m_0\in\mathcal{N}$, so that, $<m_0, n_0>\in\mathcal{A}$ (see [2]). Because $\mathcal{K}$ is a kernel, then $m_0\in\mathcal{N}\setminus\mathcal{K}$(see [2]). Hence, $\mathcal{N}^+[r]\ \mathcal{N}\setminus\mathcal{K}\ [r]=\mathcal{N}$(Q.E.D.)

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Received 11.11.1980

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Analele ştiinţifice ale Universităţii „Al. I. Cuza" din IAŞI
Tomul XXVI, s. I a, 1980, f. 2

RECENZII

JULIUS PAWEL SCHAUDER: *Oeuvres*, Panstwowe Wydawnictwo Naukowe, Varşovia, 1978, 488 p.

Institutul de Matematică al Academiei Poloneze de Ştiinţe reuneşte într-un singur volum, editat sub redacţia lui J. Kisynski, W. Orlicz, M. Stark, toate lucrările lui J. P. Schauder, cu excepţia a două comunicări ţinute la un congres de la Oslo. Reprezentant strălucit şi continuator al şcolii matematice create de Ştefan Banach şi Hugo Steinhaus la Lwow, Julius Pawel Schauder este cunoscut mai ales prin teorema sa de punct fix, prin introducerea şi studiul noţiunii de grad topologic al unei aplicaţii neliniare şi prin crearea teoriei bazelor în spaţii vectoriale normate. De asemenea, rezultatele cercetărilor sale în domeniul ecuaţiilor cu derivate parţiale eliptice şi hiperbolice au rămas un fond durabil în teoria ecuaţiilor. În afara lucrărilor lui Schauder, volumul conţine o scurtă notă biografică scrisă de W. Orlicz, o prezentare a operei aparţinînd lui J. Leray, precum şi Bibliografia completă a autorului. Colectivul de redacţie al volumului a adăugat şi scurte comentarii şi note de subsol la unele articole.

C. Ilioi

FRIEDRICH HERNECK: *Albert Einstein*, 5-te, ergänzte Auflage. Biographien hervorragender Naturwissenschaftler, Techniker und Mediziner, Band 14. B. G. Teubner Verlagsgesellschaft, Leipzig, 1980, 116 S; 7,50 M.

Cartea este o încercare de a prezenta o imagine fidelă, purificată de aspectele legendare şi bazată pe izvoare istorice, a vieţii şi operei lui Albert Einstein. Capitolele cărţii prezintă pe rînd: perioada copilăriei şi a studiilor universitare, anii deosebit de fructuoşi petrecuţi în Elveţia; perioada din jurul primului război mondial; perioada republicii de la Weimar; anii petrecuţi la Princeton. Autorul descrie atît opera ştiinţifică revoluţionară a lui Einstein, contribuţiile sale fundamentale care au modificat radical fizica secolului nostru, activitatea de profesor şi de popularizator al ideilor noi, cît şi activitatea lui socială, atitudinea sa umanitară, progresistă şi antirăzboinică.

Jack Weinstein

PETER SCHREIBER: *Die Mathematik und ihre Geschichte im Spiegel der Philatelie*. Mathematische Schülerbücherei, Nr. 68, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1980, 101 S; 9,80 M.

Autorul prezintă un scurt istoric al dezvoltării matematicii din cele mai vechi timpuri pînă în epoca modernă. Sînt amintite cele mai importante personalităţi ale diferitelor epoci şi ţări. Este descris materialul filatelic care poate fi încadrat în tema discutată. Cartea se încheie cu o listă (pe ţări) a timbrelor alese de autor.

Jack Weinstein

Abstracts. The 18-th Scandinavian Congress of Mathematicians. Aarhus, August 18—22, 1980. Various Publications Series No. 33. Aarhus Universitet-Matematisk Institut, 1980, 144 p.

Volumul cuprinde rezumatele comunicărilor de la congres şi reflectă activitatea curentă de cercetare matematică din ţările scandinave. Rezumatele au fost grupate pe domenii, conform clasificării curente: teoria numerelor; inele comutative; varietăţi algebrice proiective; inele asociative; grupuri algebrice; măsură, probabilitate şi teoria potenţialului; variabile complexe; aplicaţii cvasiconforme; ecuaţii diferenţiale şi operatori; analiză funcţională; algebre de operatori şi reprezentări de grupuri; diferite teme din analiză; geometrie; geometrie diferenţială şi topologie; metode numerice; teoria dispersiei; aplicaţii ale matematicii; subiecte diverse.

Jack Weinstein