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A FUZZY EXTENSION OF DE BAKKER'S SEMANTICS

BY

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§1. **Introduction. Elements of Programming Languages Semantics.** We give here a series of definitions and results as given in [1], which will be used below. References to [1] and possibly to [2] will be constantly required in order to clarify the topics under discussion. Some of the notations and concepts which will be used in this paper are contained in the following.

Notations and Definitions :

1.1. *Ivar*$=\{x, y,...\}$, *the set of integer variables.*

1.2. *Cons*$=\{..., -1, 0, 1,...\}$, *the set of integer constants.*

1.3. $V=\{..., -1, 0, 1,...\}$, *the set of integers.*

1.4. $W=\{tt, ff\}$, *the set of truth-values.*

1.5. $\Sigma_0 = I\,var \to V$, *the set of states, with* σ *as arbitrary elements.*

1.6. $\Sigma=\Sigma_0 \cup \{bs\}$, *where „bs" is the undefined or „bottom" state.*

1.7. $\mathrm{s} ::= m/x/\mathrm{s}_1+\mathrm{s}_2/.../$ *if* b *then* s_1 *else* s_2 *fi, integer expressions, where* m *is an integer constant ; the operation „+" as well as other operations if needed are defined,* b *is an assertion.*

1.8. $\mathrm{b} ::= true/false/\mathrm{s}_1=\mathrm{s}_2/.../\overline{\mathrm{b}}/\mathrm{b}_1 \Rightarrow \mathrm{b}_2/\ \exists x[\mathrm{b}]$, *assertions, boolean expressions in particular.*

1.9. $S ::= x := \mathrm{s}/S_1;\ S_2/$ *if* b *then* S_1 *else* S_2 *fi, statements.*

It is assumed that the following mappings : „+" : $V \times V \to V$, „=" : $V \times V \to W$, „—" : $W \to W$, „⇒" : $W \times W \to W$, „∧" : $W \times W \to W$ have been defined, possessing those properties that are known for the operation of addition, equality relation, negation, implication, conjunction.

For each of the notions above „meanings" are established which in fact constitute mappings defining certain rules for working with integer expressions, assertions, statements, etc. The basic interpretation as made evident in the quoted papers is that the execution of a given statement is characterized by a change of state which is defined as a mapping $\Phi : \Sigma \to \Sigma$ possessing certain properties. It is essential to organize the set of states as a complete partially ordered set.

1.10. *Let C be a partially ordered set. A chain in C is defined as a sequence of elements* $x_1, x_2,...$, *so that* $x_i \preceq x_{i+1}$ *for* $i=1, 2,...$, *where „*$\preceq$*" is the relation of partial ordering of C.*

1.11. *An element x is called* lub *(least upper bound) for the chain $<x_i>$, denoted as* lub $<x_i>$, *if*

(i) $x_i \preceq x$, *for* $i=1, 2, \ldots$,

(ii) *for every y satisfying* (i) *we have $x \preceq y$.*

1.12. *The partially ordered set C is called complete partially ordered* (cpo), *if any chain in C has* lub.

1.13. *Let C_1 and C_2 be partially ordered sets. The mapping $\Phi : C_1 \to C_2$ is called monotonous if for any x, y where $x \preceq y$ we have $\Phi(x) \preceq \Phi(y)$. We have given a similar notation to the order relations in C_1 and C_2.*

1.14. *The mapping $\Phi : C_1 \to C_2$, where C_1 and C_2 are* cpo *sets, is called continuous if for any chain $<x_i>$ in C_1 we have Φ* (lub $<x_i>$)$=$lub$<\Phi(x_i)>$.

The Tarski-Kleene theorem gives a constructive solution to the fixed-point equation $x=\Phi(x)$, for Φ a continuous mapping (in the sense of **Def. 1.14**) from C (cpo set) to C. Such results are rendered necessary for example by the fact that with while statements or recursive procedures the final state is the solution to such an equation, where Φ changes states into states or Φ is a change of state changes. The aim of this paper is to extend the results obtained in the above-mentioned works for the case of a new set of constants featuring a new ordering and new operations.

These new constants which we shall call fuzzy constants illustrate a common situation in practical work, where a given real value has a certain degree of certitude. The fuzzy constant will thus be a function with values in [0, 1] whereby a certitude degree will be attributed to each argument (real constant) in the interval [0, 1]. We shall try to see whether in executing statements containing expressions based on such constants, we may rely on the same semantic rules, and, if so, under conditions.

§2. Partial order in the set of fuzzy constants. Let us consider μ as a measure on $\mathbf{R}$ and the usual, induced topology on [0, 1]. Let F be the set of functions $F=\{r/r : \mathbf{R} \to [0, 1], r \text{ is } \mu\text{-measurable}\}$. We shall consider the sets $D^0=(0, b_0]$ and $D^1=[b_1, 1)$ with $0<b_0 \leqslant b_1<1$, and the sets :

$$L^0_e(r)=\{a/r(a) \leqslant e\} \text{ and } L^1_e(r)=\{a/1-r(a) \leqslant e\}. \tag{2.1}$$

Remarks 2.1. The sets defined in (2.1) are measurable for any e in D^0 and D^1 respectively.

2.2. We shall stress here that the sets in (2.1) can be empty. The working hypothesis will be that both sets are non-empty, since the opposite hypothesis would be non-significant.

Definition 2.1. 2.1. *$r_1 \overset{e}{\preceq} r_2$, if there exists $e^0 \in D^0$ and $e^1 \in D^1$ so that:*

$$L^0_e(r_1) \subseteq L^0_e(r_2), \quad \forall 0 \leqslant e \leqslant e^0, \text{ and} \tag{2.2}$$

$$L^1_e(r_1) \supseteq L^1_e(r_2), \quad \forall 0 \leqslant e \leqslant e^1. \tag{2.3}$$

We shall note that the relation $\overset{e}{\preceq}$ is a preordering relation. We shall now introduce the equivalence relation over F :

$$r_1 \overset{e}{=} r_2, \text{ iff } r_1 \overset{e}{\preceq} r_2 \text{ and } r_2 \overset{e}{\preceq} r_1. \tag{2.4}$$

Lemma 2.1. *$r_1 \overset{e}{=} r_2$ iff there exists $e^0 \in D^0$ and $e^1 \in D^1$ so that $r_1(a)=$ $=r_2(a)$, $\forall a \in L^0_{e^0}(r_1) \cup L^1_{e^1}(r_1)$.*

Proof. We shall consider only the case involving the neighbourhood of 0, as the other case is perfectly analogous. From $r_1 \overset{e}{=} r_2$ follows that there is an e^0 so that $L^0_e(r_1)=L^0_e(r_2)$, $\forall 0 \leqslant e \leqslant e^0$, or, equivalently :

$$r_1(a)=r_2(a), \quad \forall a \in L^0_{e^0}(r_1) \quad (=L^1_{e^1}(r_2)). \tag{2.5}$$

This is because $L^0_e(r_1) \subseteq L^0_e(r_2)$, $\forall 0 \leqslant e \leqslant e^0$, implies $r_1(a) \geqslant r_2(a)$, $\forall a \in L^0_{e^0}(r_2)$. Indeed, if $r_1(a)<r_2(a)$ for $a \in L^0_{e^0}(r_2)$, let there be an e' so that $r_1(a) \leqslant e' <$ $<r_2(a)$. It follows that $r_1(a) \leqslant e'$ implies $r_2(a)>e'$, with $0 \leqslant e' < e^0$, which is a contradiction. The direct implication is thus proved, and the reverse implication follows immediately.

Remarks 2.3. In what follows, equality will be used in two senses : *common* or *strong equality*, when $r_1(a)=r_2(a)$ for any $a \in \mathbf{R}$ and *e-equality* or *weak equality* when $r_1 \overset{e}{=} r_2$. Obviously $r_1=r_2$ implies $r_1 \overset{e}{=} r_2$, but the converse is not true.

2.4. For constants of the type $r(a) \in \{0, 1\}$, $\forall a \in \mathbf{R}$, we have $r_1=r_2$ iff $r_1 \overset{e}{=} r_2$, so that for real, nonfuzzy constants the strong and the weak equalities coincide.

Definition 2.2. *We shall call a chain in F, the sequence $<r_i>$, $i=$ $=1, 2, \ldots, r_i \in F$, for which :*

$$r_1 \overset{e}{\preceq} r_2 \overset{e}{\preceq} \ldots r_i \overset{e}{\preceq} r_{i+1} \overset{e}{\preceq} \ldots \tag{2.6}$$

Lemma 2.2. *If $<r_i>$ is a chain in F, then there exists the descendant sequences $<e^0_i>$ and $<e^1_i>$ in D^0 and D^1 respectively, for which :*

$$L^0_{e^0_1}(r_1) \subseteq L^0_{e^0_2}(r_2) \subseteq \ldots L^0_{e^0_{n-1}}(r_{n-1}) \subseteq L^0_{e^0_{n-1}}(r_n) \subseteq \ldots \tag{2.7}$$

$$L^1_{e^1_1}(r_1) \supseteq L^1_{e^1_2}(r_2) \supseteq \ldots \supseteq L^1_{e^1_{n-1}}(r_{n-1}) \supseteq L^1_{e^1_{n-1}}(r_n) \supseteq \ldots \tag{2.8}$$

$$\textit{For } a \in L^0_{e^0_j}(r_j) \textit{ we have } r_i(a) \geqslant r_{i+1}(a) \geqslant \ldots, \ \forall i \geqslant j. \tag{2.9}$$

Proof. $r_1 \overset{e}{\preceq} r_2$ iff $\exists e^0_1 \in D^0$, $L^0_e(r_1) \subseteq L^0_{e^0}(r_2)$, $\forall 0 \leqslant e \leqslant e^0_1$. $r_2 \overset{e}{\preceq} r_3$ iff $\exists e^0_2 \in D^0$ so that $L^0_e(r_2) \subseteq L^0_e(r_3)$, $\forall 0 \leqslant e \leqslant e^0_2$. By taking $e^0_2 := \min\{e^0_1, e^0_2\}$, it follows $L^0_{e^0_1}(r_1) \subseteq L^0_{e^0_2}(r_2) \subseteq L^0_{e^0_2}(r_3)$. Similarly, by taking $e^0_3 := \min\{e^0_2, e^0_3\}$ it follows : $L^0_{e^0_1}(r_1) \subseteq L^0_{r^0_2}(r_2) \subseteq L^0_{e^0_3}(r_3) \subseteq L^0_{e^0_3}(r_4)$. We get then (2.7) and (2.8) by induction. (2.9) is also proved by induction. For the first indices we get :

$$L^0_e(r_1) \subseteq L^0_e(r_2), \quad 0 \leqslant e \leqslant e^0_1 \Rightarrow r_1(a) \geqslant r_2(a), \quad \forall a \in L^0_{e^0_1}(r_2),$$

$$L^0_e(r_2) \subseteq L^0_e(r_3), \quad 0 \leqslant e \leqslant e^0_2 \Rightarrow r_2(a) \geqslant r_3(a), \quad \forall a \in L^0_{e^0_2}(r_3),$$

$$L^0_e(r_3) \subseteq L^0_e(r_4), \quad 0 \leqslant e \leqslant e^0_3 \Rightarrow r_3(a) \geqslant r_4(a), \quad \forall a \in L^0_{e^0_3}(r_4),$$

. . . .

Taking into account relations (2.7) also, it follows (2.9).

Lemma 2.3. *Consider the function:*

$$(2.10)\qquad r(a) = \begin{cases} \lim\limits_{i\to\infty} r_i(a) & a \in (\bigcup\limits_i L^0_{e^0_i}(r_i)) \cup (\bigcap\limits_i L^1_{e^1_i}(r_i)), \\ \textit{convenient (for instance } (b_0+b_1)/2) & \textit{otherwise (eventually).} \end{cases}$$

Then r will satisfy the lub *conditions for the $<r_i>$ chain in F.*

Proof. For $a \in \bigcup\limits_i L^0_{e^0_i}(r_i)$, $e^0_i \in D^0$, as seen in lemma 2.2, $r(a)$ is well defined. For $a_i \in \bigcap\limits_i L^1_{e^1_i}(r_i)$, $e^1_i \in D^1$, we obtain $r_1(a) \leqslant r_2(a) \ldots \leqslant r_n(a) \leqslant \ldots$ so that r, defined by (2.10) is well defined.

The lub conditions

$$(2.11)\qquad r_i \overset{e}{\precsim} r, \quad \forall i \in \mathbf{N}, \text{ and}$$

(2.12) for any r' satisfying the property (2.11), we have $r \overset{e}{\precsim} r'$, will be verified with lemma 2.2, where e^0_i and e^1_i are determined likewise.

Remarks: *2.5.* It can be proved that if $e^0_*=\lim e^0_i$ and $e^1_*=\lim e^1_i$, with $<e^0_i>$and$<e^1_i>$ as in lemma 2.2,

$$(2.13)\qquad \bigcup_i L^0_{e^0_i}(r_i) \subseteq L^0_{e^0_*}(r) \text{ and } L^1_{e^1_*}(r) \subseteq \bigcap_i L^1_{e^1_i}(r_i).$$

In general, we can get equalities in (2.13) if the sequences $<e^0_i>$ and $<e^1_i>$ are adequately chosen. The inclusions (2.13) will supply indications as to the measure of the set of arguments for function r in the neighbourhood of 0 and 1. It is to be noticed that function r has suitable properties in the sense of **Def. 2.1.**

2.6. If the two sequences $<e^0_i>$ and $<e^1_i>$ in lemma 2.2 satisfy:

$$(2.14)\qquad e^0_*>0 \text{ and } e^1_*>0,$$

then these limits can uniformly satisfy lub conditions (2.11) and (2.12).

2.7. We note that function r is unique to an e-equality. If condition (2.14) is absent then r is unique since the e-equality becomes an equality on „0" or „1".

Definitions 2.3. *A chain fulfilling condition* (2.14) *is called an uniform chain.*

2.4. *An uniform chain for which the conditions*: $\mu(L^0_{e^0_*}(r))=\infty$ *and* $L^1_{e^1_*}(r)$ *is bounded, are satisfied is called a well-approximating chain.*

In the set F which has been ordered in this manner, we have as the least element the function class $r:\mathbf{R}\to[0,\ 1]$ for which $b_0 \leqslant r(a) \leqslant b_1$, $\forall a \in \mathbf{R}$, and we denote it by r_{bs}. Of course $r_{bs} \overset{e}{\precsim} r$, $\forall r \in F$.

We shall associate commutative operations $\oplus$ and $\odot$ with the set F. It is desired that operations $\oplus$ and $\odot$ generalize the corresponding operations with real constants in the following sense:

$$\text{if } r_i(a) = \begin{cases} 0 & \text{for } a \neq a_i \\ 1 & \text{for } a = a_i \end{cases}, \quad i=1,2,$$

$$a_i \in \mathbf{R}, \text{ then } r_1 \oplus r_2(a) = \begin{cases} 0 & \text{for } a \neq a_1+a_2, \text{ and analogous} \\ 1 & \text{for } a = a_1+a_2 \end{cases}$$

for $\odot$. We shall assume that F, with the above mentioned operations, forms a commutative semiring structure and that r_{bs} is the neutral element for $\oplus$. We shall also assume that the operations are compatible with the ordering structure in F (i.e. are monotonous), so that:

$$(2.15)\qquad r_1 \overset{e}{\precsim} r_2 \Rightarrow r_1 \oplus r \overset{e}{\precsim} r_2 \oplus r \text{ and } r_1 \odot r \overset{e}{\precsim} r_2 \odot r, \quad \forall r_1, r_2, r \in F.$$

We denote by $(\hat{F}, \precsim)$ the partial ordered set of the classes of equivalent states. We assume that the operations $\oplus$ and $\odot$ preserve the classes $\hat{r} \in \hat{F}$ and also that $\hat{F}$ is a commutative semiring with the neutral element $\hat{r}_{bs}$. The monotony of operations $\oplus$ and $\odot$ in $\hat{F}$ is also preserved. In $\hat{F}$ equality $\hat{r}_1 = \hat{r}_2$ means double inclusion as in the case of sets.

§3. **Hoare's and Floyd's assignment rules.** We shall extend the notions used in denotational semantics as synthesized by De Bakker by using instead of V (**Def.** 1.3), the set of fuzzy constants F, with the properties specified in §2.

Definition 3.1. *A state is an element* $\sigma \in \Sigma = Ivar \to F$, *where Ivar is a set of variables as defined in* 1.1.

We shall now define in connection with the set of states the following preordering and equivalence relations:

Definitions. 3.2. $\sigma_1 \overset{e}{\precsim} \sigma_2$, *if* $\sigma_1(x) \overset{e}{\precsim} {}_2(x)$, $\forall x \in Ivar$.

3.3. $\sigma_1 \overset{e}{\doteq} \sigma_2$, *if* $\sigma_1 \overset{e}{\precsim} \sigma_2$ *and* $\sigma_2 \overset{e}{\precsim} \sigma_1$.

3.4. *The state (denoted „bs") for which* $bs(x) \overset{e}{\doteq} r_{bs}$, $\forall x \in Ivar$, *will be called bottom state.* We shall adopt the definitions from [1], with the following mention: not all definitions will be absolutely necessary to this paper and they are not the only versions possible.

Definitions *for fuzzy expressions (Fexp), boolean expressions (Bfexp) statements (Fstat)*:

3.5. $s ::= r/s/s_1 \oplus s_2/\ldots/$*if* b *then* s_1 *else* s_2 *fi*.

3.6. $b ::= true/false/s_1 \overset{e}{\precsim} s_2/s_1 \overset{e}{\doteq} s_2/\ldots/\overline{b}/b_1 \Rightarrow b_2/\exists x\ [b]$.

3.7. $S ::= x := s/S_1;\ S_2/$*if* b *then* S_1 *else* S_2 *fi*. We shall define as meaning for the fuzzy expressions and assertions (boolean expressions) the mappings:

$V: Fexp \to (\Sigma \to F)$ and $T: Bfexp \to (\Sigma \to W)$, with Σ as in **Def. 3.1** and W as in **Def. 1.4** in the following way:

3.8. $V(r)\,(\sigma)=r,$

$V(x)\,(\sigma)=\sigma(x),$

$V(\mathbf{s}_1\oplus \mathbf{s}_2)\,(\sigma)=V(\mathbf{s}_1)\,(\sigma)\oplus V(\mathbf{s}_2)\,(\sigma),$

. . . . . .

$V(\textit{if}\ \mathbf{b}\ \textit{then}\ \mathbf{s}_1\ \textit{else}\ \mathbf{s}_2\ \textit{fi})\,(\sigma)=$

$$=\begin{cases} V(\mathbf{s}_1)\,(\sigma),\ \textit{if}\ T\,(\mathbf{b})\,(\sigma)=tt\\ V(\mathbf{s}_2)\,(\sigma),\ \textit{if}\ T\,(\mathbf{b})\,(\sigma)=ff.\end{cases}$$

3.9. $T\,(\textit{true})\,(\sigma)=tt,$

$T\,(\textit{false})\,(\sigma)=ff,$

$T(\mathbf{s}_1 \overset{e}{\precsim} \mathbf{s}_2)\,(\sigma)=(V(\mathbf{s}_1)\,(\sigma) \overset{e}{\precsim} V(\mathbf{s}_2)\,(\sigma)),$

$T(\mathbf{s}_1 \overset{e}{=} \mathbf{s}_2)\,(\sigma)=(V(\mathbf{s}_1)\,(\sigma) \overset{e}{=} V(\mathbf{s}_2)\,(\sigma)),$

. . . . . .

$$T(\overline{\mathbf{b}})\,(\sigma)=\begin{cases} ff, & \textit{if}\quad T(\mathbf{b})\,(\sigma)=tt\\ tt, & \textit{if}\quad T(\mathbf{b})\,(\sigma)=ff,\end{cases}$$

$$T(\mathbf{b}_1\Rightarrow \mathbf{b}_2)\,(\sigma)=\begin{cases} tt, \text{ if } T(\mathbf{b}_1)\,(\sigma)=ff \text{ or } T(\mathbf{b}_2)\,(\sigma)=tt\\ ff, \textit{ in the contrary case,}\end{cases}$$

$T(\exists x[\mathbf{b}])\,(\sigma)=tt,$ *if there exists* $r\in F$ *such that*

$T(\mathbf{b})\,(\sigma\{r/x\})=t,$ *where the state* $\sigma\{r/x\}$ *is defined in* 3.10.

3.10. *Let us consider* $\sigma\in\Sigma,\ x\in Ivar,\ r\in F$. *We shall call the version state of* σ *the state denoted* $\sigma\{r/x\}$ *satisfying*:

(i) $\sigma\,\{r/x\}\,(x)=r$

(ii) $\sigma\,\{r/x\}\,(y)=\sigma(y)$, *for* $x\not\equiv y$,

(„$\equiv$" is the syntactical identity).
The meaning of a statement is the mapping $M: Fstat\to(\Sigma\to\Sigma)$ *satisfying*:

3.11. $M(x:=\mathbf{s})\,(\sigma)=\sigma\,\{V(\mathbf{s})\,(\sigma)/x\}$,

$M(S_1\,;\,S_2)\,(\sigma)=M(S_2)\,(M(S_1)\,(\sigma)),$

$$M(\textit{if}\ \mathbf{b}\ \textit{then}\ S_1\ \textit{else}\ S_2\ \textit{fi})\,(\sigma)=\begin{cases} M(S_1)\,(\sigma),\ \textit{if}\\ T(\mathbf{b})\,(\sigma)=tt,\\ M(S_2)\,(\sigma),\ \textit{if}\\ T(\mathbf{b})\,(\sigma)=ff.\end{cases}$$

An important class of semantic objects is the class of correctness formulae, F form :

$$3.12.\ \ \mathbf{f}::=\mathbf{b}/S_1 \overset{e}{\precsim} S_2/S_1 \overset{e}{=} S_2/\ \{\mathbf{p}\}\ S\ \{\mathbf{q}\}/\mathbf{f}_1\wedge \mathbf{f}_2.$$

In 3.12 **b**, **p**, **q** are assertions ; **p** is named a precondition, and **q** a postcondition. The meaning of the formulae is given by the mapping $T: Fform\to\to(\Sigma\to W)$, with the properties :

3.13. *For* **b**, T *has similar properties to* 3.9.

$T(S_1 \overset{e}{\precsim} S_2)\,(\sigma)=tt$ *if* $M(S_1)\,(\sigma) \overset{e}{\precsim} M(S_2)\,(\sigma),$

$T(S_1 \overset{e}{=} S_2)\,(\sigma)=tt$ *if* $M(S_1)\,(\sigma) \overset{e}{=} M(S_2)\,(\sigma),$

$T(\{\mathbf{p}\}\ S\ \{\mathbf{q}\})\,(\sigma)=tt$ *if* $T(\mathbf{p})\,(\sigma)=tt$ *and* $\sigma'=M(S)\,(\sigma)$ *implies* $T(\mathbf{q})\,(\sigma')=tt,$

$T(\mathbf{f}_1\wedge \mathbf{f}_2)\,(\sigma)=tt$ *if* $T(\mathbf{f}_1)\,(\sigma)=tt$ *and* $T(\mathbf{f}_2)\,(\sigma)=tt.$

3.14. *A correctness formula* **f** *is called valid if* $T(\mathbf{f})\,(\sigma)=tt$, *for all* $\sigma\in\Sigma$.

Finally, let us remake the definitions of substitution (where $\mathbf{s}[t/x]$ signifies that in expression s, the variable x is replaced by expression **t**) :

3.15. $r[\mathbf{t}/x]\equiv r\ ;\ x[\mathbf{t}/x]\equiv \mathbf{t}\ ;\ y[\mathbf{t}/x]\equiv y\ \ (x\not\equiv y)\ ;$

$(\mathbf{s}_1\oplus \mathbf{s}_2)\ [\mathbf{t}/x]\equiv \mathbf{s}_1[\mathbf{t}/x]\oplus \mathbf{s}_2[\mathbf{t}/x]$; *if* **b** *then* $\mathbf{s}_1$ *else* $\mathbf{s}_2$ *fi* $[\mathbf{t}/x]\equiv$ *if* $\mathbf{b}[\mathbf{t}/x]$ *then* $\mathbf{s}_1\,[\mathbf{t}/x]$ *else* $\mathbf{s}_2\,[\mathbf{t}/x]$ *fi* ;
true $[\mathbf{t}/x]\equiv$ *true* ; *false* $[\mathbf{t}/x]\equiv$ *false* ;

$(\mathbf{s}_1 \overset{e}{\precsim} \mathbf{s}_2)\ [\mathbf{t}/x]\equiv(\mathbf{s}_1[\mathbf{t}/x] \overset{e}{\precsim} \mathbf{s}_2\,[\mathbf{t}/x])\ ;$

$(\mathbf{s}_1 \overset{e}{=} \mathbf{s}_2)\ [\mathbf{t}/x]\equiv(\mathbf{s}_1[\mathbf{t}/x] \overset{e}{=} \mathbf{s}_2[\mathbf{t}/x])\ ;$

$(\overline{\mathbf{b}})\ \ [\mathbf{t}/x]\equiv\overline{\mathbf{b}[\mathbf{t}/x]}\ ;\ (\mathbf{b}_1\Rightarrow \mathbf{b}_2)\ \ [\mathbf{t}/x]\equiv(\mathbf{b}_1[\mathbf{t}/x]\Rightarrow \mathbf{b}_2[\mathbf{t}/x])\ ;$

$\exists y[\mathbf{b}]\ [\mathbf{t}/x]\equiv$

(i) $\exists y\ [\mathbf{b}]$, *if* $x\equiv y$

(ii) $\exists y\ [\mathbf{b}\ [\mathbf{t}/x]]$, if $x\not\equiv y$ *and* y *is not free in* **t**,

(iii) $\exists y'\ [\mathbf{b}[y'/y]\ [\mathbf{t}/x]]$, *if* $x\not\equiv y$ *and* y *is free in* **t**, $y'\not\equiv x$ *and* y' *is not free in* **b** *or* **t**.

(x in expression **b** (or **t**) has a bound occurence if it occurs in a subexpression of **b** in the form $\exists x\ [\mathbf{b}']$. Any other occurence of x in **b** (or **t**) is called free). In what follows we will use the same notations for the variables to which we assign values (of the type) $\hat{r}\in\hat{F}$. We extend all the definitions for the set $\hat{\Sigma}$ by similar definitions to **Def. 3.1 – 3.13.** These definitions are easily obtained by taking care to use only operations which preserve the equivalence classes on operations with elements.

3.16. $\hat{\sigma}(x)=\widehat{\sigma(x)}, \ \forall x \in Ivar.$

The variant of a class of states is:

3.17. $\hat{\sigma}\{\hat{r}/x\}=\widehat{\sigma\{r/x\}}$, *where* $r \in \hat{r}$.

We have analogous properties to those in the case of the variant of a state. We define the mappings: $\mathcal{V}: Fexp \to (\hat{\Sigma} \to \hat{F})$, $\mathcal{T}: Bfexp \to (\hat{\Sigma} \to W)$, $\mathcal{M}: Fstat \to (\hat{\Sigma} \to \hat{\Sigma})$, by:

3.18. $\mathcal{V}(\mathrm{s})\,(\hat{\sigma})=\widehat{V(\mathrm{s})\,(\sigma)}$ (V *as in* **Def. 3.8**)

3.19. $\mathcal{T}(\mathrm{p})\,(\hat{\sigma})=tt$ if $T(\mathrm{p})\,(\sigma)=tt$, $\forall \sigma \in \hat{\sigma}$ (T *as in* **Def. 3.9**)

3.20. $\mathcal{M}(S)\,(\hat{\sigma})=\widehat{M(S)\,(\sigma)}$ (M *as in* **Def. 3.10**).

We recall that the operations $\oplus$ and $\odot$ in F preserve the classes in $\hat{F}$. We can give now:

Theorem 3.1. *For the assertions* p *and expressions* s *and* t *for which we have*:

$$(3.21)\quad T(\mathrm{p}\,[\mathrm{s}/x])\,(\sigma)=T(\mathrm{p})\,(M(x:=\mathrm{s})\,(\sigma)),\ \forall \sigma \in \Sigma,\ \textit{and}$$

$$(3.22)\quad V(\mathrm{t}[\mathrm{s}/x])\,(\sigma)=V(\mathrm{t})\,(M(x:=\mathrm{s})\,(\sigma)),\ \forall \sigma \in \Sigma,$$

the Hoare's and Floyd's assignment rules are valid:

$$(3.23)\quad \{\mathrm{p}[\mathrm{s}/x]\}\ x:=\mathrm{s}\ \{\mathrm{p}\}$$

$$(3.24)\quad \{\mathrm{p}\}\ x:=\mathrm{s}\ \{\exists y\,[\mathrm{p}\,[y/x] \wedge x \stackrel{e}{=} \mathrm{s}[y/x]]\},$$ *where* y *is a variable not free in* p *or* s.

Proof. We specify from the beginning the (partial) correctness formulae (3.23) and (3.24) have the significance that they are valid not only for a state σ but also for all the states equivalent to σ. With the new definitions and assuming (3.21) and (3.22), the proof parallels the course of the proof for theorem 2.19 from [1], the set of states being replaced by the set $\hat{\Sigma}$.

Let us show, for instance, the Floyd's assignment rule and the equivalence with the formulation of (3.24). From the adequate definitions, (3.24) will be true if from $\mathcal{T}(\mathrm{p})\,(\hat{\sigma})=tt$ and $\hat{\sigma}'=\mathcal{M}(x:=\mathrm{s})\,(\hat{\sigma})=\hat{\sigma}\,\{\mathcal{V}(\mathrm{s})\,(\hat{\sigma})/x\}$ results that there exists $\hat{r} \in \hat{F}$ so that $\mathcal{T}(\mathrm{p}[y/x] \wedge x=\mathrm{s}[y/x])\,(\hat{\sigma}'\,\{\hat{r}/y\})=tt$. (The equality „$x=\mathrm{s}[y/x]$" is in the sense of equality for classes of equivalent states). Let $\hat{\sigma} \in \hat{\Sigma}$ be arbitrary with $\hat{\sigma}(x)=\hat{r}$. We note $\hat{\sigma}''=\hat{\sigma}\{\hat{r}/y\}$. From the

analogous definitions of **Def. 3.13**, it remains to show $\mathcal{T}(\mathrm{p}[y/x])\,(\hat{\sigma}'')=tt$ and $\mathcal{T}(x=\mathrm{s}[y/x])\,(\hat{\sigma}'')=tt$. We have: $\mathcal{T}(\mathrm{p}[y/x])\,(\hat{\sigma}'')=\mathcal{T}(\mathrm{p})\,(\hat{\sigma}''\,\{\mathcal{V}(y)\,(\hat{\sigma}'')/x\})=\mathcal{T}(\mathrm{p})\,(\hat{\sigma}''\{\hat{\sigma}''\,(y)/x\})=\mathcal{T}(\mathrm{p})\,(\hat{\sigma}''\{r/x\})$ because $\hat{\sigma}''\,(y)=\hat{\sigma}'\{\hat{r}/y\}\,(y)=\hat{r}$ and $\hat{\sigma}''\,\{\hat{r}/x\}=\hat{\sigma}'\,\{\hat{r}/y\}\,\{\hat{r}/x\}=\hat{\sigma}\,\{\mathcal{V}(\mathrm{s})\,(\hat{\sigma})/x\}\,\{\hat{r}/y\}\,\{\hat{r}/x\}=\hat{\sigma}\,\{\hat{r}/y\}\,\{\hat{r}/x\}$, therefore: $\mathcal{T}(\mathrm{p})\,(\hat{\sigma}\,\{\hat{r}/y\}\,\{\hat{r}/x\})=$ (because y is not free in p or s) $=\mathcal{T}(\mathrm{p})\,(\hat{\sigma}\,\{\hat{r}/x\})=\mathcal{T}(\mathrm{p})(\hat{\sigma}\,\{\hat{\sigma}(x)/x\})=\mathcal{T}(\mathrm{p})\,(\hat{\sigma})=tt$. $\mathcal{T}(x=\mathrm{s}\,[y/x])\,(\hat{\sigma}'')=(\mathcal{V}(x)\,(\hat{\sigma}'')=\mathcal{V}(\mathrm{s}[y/x])\,(\hat{\sigma}''))$, and considering the two members one by one: $\mathcal{V}(x)\,(\hat{\sigma}'')=\widehat{V(x)\,(\sigma'')}=\widehat{V(\mathrm{s})\,(\sigma)}=\mathcal{V}(\mathrm{s})\,(\hat{\sigma})$, and $\mathcal{V}\,(\mathrm{s}[y/x])\,(\hat{\sigma}'')=\mathcal{V}(\mathrm{s})\,(\hat{\sigma}''\,\{\hat{\sigma}''(y)/x\})=$ $=$(according to the previous observations)$=\mathcal{V}(\mathrm{s})\,(\hat{\sigma})$.

According to the corresponding definitions it follows that we have the Floyd's assignment rule for the set $\hat{\Sigma}$. The equivalence with (3.24) is immediate for the reason that $T(\mathrm{s}_1=\mathrm{s}_2)\,(\hat{\sigma})=tt$ iff $T(\mathrm{s}_1 \stackrel{e}{=} \mathrm{s}_2)\,(\sigma)=tt$, $\forall \sigma \in \hat{\sigma}$.

Remarks 3.1. The conditions (3.21) and (3.22) are special cases of the formula (T2) in [2], where the statement is the assignment one. The significance is that during processing, assignment which causes a change of state, and substitution, which is a syntactical operation and leaves state unchanged, are intervertible yielding the same truth-function.

3.2. The conditions (3.21) and (3.22) are satisfied for those types of assertions which we have discussed.

The fulfilment of these conditions and of e-equalities are sufficient for the proof of the theorem, if we do not use the set $\hat{\Sigma}$.

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