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ON THE UNIFORM MATRIX SUMMABILITY OF FOURIER SERIES

BY

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In this paper we generalize the theorem of S a x e n a for the uniform matrix summability.

1. Let $f(x)$ be a 2π-periodic function and Lebesgue integrable in the interval $(-\pi, \pi)$. Let its Fourier series be given by

(1.1) $$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

We shall use the following notations:

$$\Phi(t) = \Phi(x, t) = f(x+t) + f(x-t) - 2f(x),$$

$$\Phi(t) = \int_0^t |\Phi(u)|\, du, \qquad \Delta\lambda_{n,k} = \lambda_{n,k} - \lambda_{n,k+1}, \qquad \tau \neq \left[\frac{1}{t}\right],$$

where $[\lambda]$ denotes the integral part of λ.

2. Let $(\lambda_{n,k})$ $(n = 0, 1, 2, 3, \ldots;\ k = 0, 1, 2, \ldots)$ be a triangular matrix of real or complex numbers and let

(2.1) $$\sigma_n = \sum_{k=0}^{n} \lambda_{n,k} U_k.$$

A series Σu_n with partial sum U_n is said to be summable (A) to U if the sequence $\{\sigma_n\}$ tends to a finite limit U as $n \to \infty$.

The necessary and sufficient conditions for (A) to be regular are that

(i) there is a constant M such that

(2.2) $$\sum_{k=0}^{\infty} |\lambda_{n,k}| < M \text{ for every } n.$$

(ii) for every k

(2.3) $$\lim_{n\to\infty} \lambda_{n,k} = 0$$

(2.4) (iii) $$\lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k} = 1$$

3. The object of this paper is to introduce the concept of uniform matrix summability which we define as follows:
Let

$$u_0(x)+u_1(x)+u_2(x)+\dots \tag{3.1}$$

be any infinite series and

$$U_\nu(x)=u_0(x)+u_1(x)+u_2(x)+\dots+u_\nu(x).$$

Let $(\lambda_{n,k})$ $(n=0, 1, 2,\dots;\ k=0, 1, 2,\dots)$ be a triangular matrix of real or complex numbers.

If there exists a function $U=U(x)$ such that

$$\sum_{k=0}^{n} \lambda_{n,k}\,\{U_k(x)-U(x)\}=O(1) \tag{3.2}$$

uniformly in a set E in which $U=U(x)$ is bounded, then we shall say that the series (3.1) is summable (Λ) uniformly in E to the sum U.

The conditions for the regularity of the method of uniform (Λ) summability defined by (3.2) are the same as in the case of ordinary (Λ) summability, because they are independent of x.

Particular cases of (3.2):

(i) uniform harmonic summability (3)
when

$$\lambda_{n,k}=\begin{cases}\dfrac{1}{(n-k+1)\log(n+1)}, & k\leqslant n,\\ 0 & k>n.\end{cases}$$

and (ii) uniform Nörlund summability (4)
when

$$\lambda_{n,k}=\begin{cases}\dfrac{p_{n-k}}{P_n} & k\leqslant n,\\ 0 & k>n.\end{cases}$$

In this paper we take $\{\lambda_{n,k}\}_{k=0}^{n}$ to be a real nonnegative non-decreasing sequence with respect to k.

4. In 1965 Saxena [3] established the following theorem for the uniform harmonic summability of Fourier series:

Theorem A : *If*

$$\Phi(t)=o\left(\frac{t}{\log 1/t}\right),$$

uniformly in a set E in which $f(x)$ is bounded, as $t\to+0$, then the series (1.1) *is summable by harmonic means uniformly in E to the sum $f(x)$.*

In the present paper, we generalize Theorem *A* by replacing the special sequence of coefficients

$$\lambda_{n,k}=\begin{cases}\dfrac{(n-k+1)^{-1}}{\log(n+1)}, & k\leqslant n\\ 0, & k>n.\end{cases}$$

by a more general sequence of coefficients and establish the following theorem

Theorem : *If*

$$\Phi(t)=o\left(\frac{t}{\log 1/t}\right)$$

uniformly in a set E as $t\to+0$, in which $f(x)$ is bounded, then the series (1.1) *is summable (Λ) uniformly in E to the sum $f(x)$.*

5. We require the following lemmas for the proof of our theorem:

Lemma 1. *Suppose that f and g have period 2π, that f is L and g is V, that λ and α are real, and that*

$$-\pi\leqslant a\leqslant b\leqslant\lambda, \text{ then}$$

$$J(a, b, \alpha, \lambda)=\int_a^b f(\theta+\alpha)g(\theta)\,e^{-\lambda i\theta}\,d\theta\to 0$$

uniformly in a, b and α when $|\lambda|\to\infty$.

Lemma 2. [2]: If $\{\lambda_{n,k}\}_{k=0}^{n}$ *is a non-negative and non-decreasing sequence with respect to k, then for $0\leqslant a<b\leqslant\infty$, $0\leqslant t\leqslant\pi$ and for any n*

$$\left|\sum_{k=a}^{b}\lambda_{n,n-k}\,e^{i(n-k)t}\right|<\frac{B\lambda_{n,n-\tau}}{t}$$

where B is a constant and $\tau=[1/t]$.

Lemma 3 [2]. *If $\{\lambda_{n,k}\}_{k=0}^{n}$ is a non-negative, non-decreasing sequence with respect to k, then as $n\to\infty$*

$$\lambda_{n,k}=\left(\frac{1}{n-k+1}\right)\text{since}\sum_{k=0}^{n}\lambda_{n,k}=1 \text{ as } n\to\infty,\ \text{uniformly for all } k\leqslant n, \text{so that}$$

$$\lambda_{n,0}=O\left(\frac{1}{n}\right).$$

6. *Proof of the theorem*: It is well known that

$$S_n(x)-f(x)=\frac{1}{2\pi}\int_0^\pi \Phi(t)\,\frac{\sin(n+\frac12)t}{\sin\frac12 t}\,dt=\frac{1}{2\pi}\left(\int_0^\delta+\int_\delta^\pi\right)\Phi(t)\,\frac{\sin(n+\frac12)t}{\sin\frac12 t}\,dt=$$

$$=\frac{1}{2\pi}\int_0^\delta \Phi(t)\,\frac{\sin(n+\frac12)t}{\sin\frac12 t}\,dt+\frac{1}{2\pi}\int_\delta^\pi f(x+t)\,\frac{\sin(n+\frac12)t}{\sin\frac12 t}\,dt+$$

(6.1) $$+\frac{1}{2\pi}\int_{\delta}^{\pi} f(x-t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt-\frac{f(x)}{\pi}\int_{\delta}^{\pi}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt.=$$

$$=\frac{1}{2\pi}\int_{0}^{\delta}\Phi(t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt+I_1+I_2+I_3\text{ (say)}.$$

(6.2) Now $I_1=o(1)$ *uniformly in* E *by lemma* 1,

$$\text{Again } I_2=\frac{1}{2\pi}\int_{\delta}^{\pi} f(x-t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt=\frac{1}{2\pi}\int_{-\delta}^{-\pi} f(x+t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt=$$

(6.3) $$=\frac{1}{2\pi}\int_{-\pi}^{-\delta} f(x+t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt=0(1)\text{ uniformly in } E \text{ by lemma } 1$$

Lastly

$$I_3=\frac{-f(x)}{\pi}\int_{\delta}^{\pi}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt,\text{ or } |I_3|=\frac{|f(x)|}{\pi}\int_{\delta}^{\pi}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt\leqslant$$

$$\leqslant\frac{M}{\pi}\left|\int_{\delta}^{\pi}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt\right|.$$

since $|f(x)|\leqslant M$ for every $x\in E$ where M is some constant. Therefore.

(6.4) $I_3=o(1)$ uniformly in E, by lemma 1

Combining (6.1), (6.2), (6.3) and (6.4) we get,

$$S_n(x)-f(x)=\frac{1}{2\pi}\int_{0}^{\delta}\Phi(t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt+o(1)\text{ uniformly in } E \text{ and therefore,}$$

$$\sum_{k=0}^{k}\lambda_{n,k}\{S_k(x)-f(x)\}=\sum_{k=0}^{n}\lambda_{n,k}\frac{1}{2\pi}\int_{0}^{\delta}\Phi(t)\frac{\sin(k+\frac{1}{2})t}{\sin\frac{1}{2}t}dt+o(1)\text{ uniformly in } E.$$

$$=\int_{0}^{\delta}\Phi(t).\frac{1}{2\pi}\sum_{k=0}^{n}\lambda_{n,k}\frac{\sin(k+\frac{1}{2})t}{\sin\frac{1}{2}t}dt+o(1)\text{ uniformly in } E.$$

$$=\int_{0}^{\delta}\Phi(t)J_n(t)dt+o(1)\quad\text{uniformly in } E.$$

where

$$J_n(t)=\frac{1}{2\pi}\sum_{k=0}^{n}\lambda_{n,k}\frac{\sin(k+\frac{1}{2})t}{\sin\frac{1}{2}t}=\left(\int_{0}^{\frac{1}{n}}+\int_{\frac{1}{n}}^{\delta}\right)\Phi(t)J_n(t)dt+o(1)\text{ uniformly in } E.$$

(6.5) $$=L_1+L_2+o(1)\text{ uniformly in } E \text{ (say)}$$

(i) Now uniformly in $0<t\leqslant 1/n$

$$|J_n(t)|=\frac{1}{2\pi}\left|\sum_{k=0}^{n}\lambda_{n,k}\frac{\sin(2k+1)\frac{1}{2}t}{\sin\frac{1}{2}t}\right|\leqslant\frac{1}{2\pi}\sum_{k=0}^{n}|\lambda_{n,k}|(2k+1).$$

Applying Abel's lemma we get

$$J_n(t)\leqslant\frac{1}{2\pi}\left[\sum_{k=0}^{n-1}\left\{\left(\sum_{v=0}^{k}|\lambda_{n,v}|\right)|2k+1-2k-3|\right\}+(2n+1)\sum_{v=0}^{n}|\lambda_{n,v}|\right]<$$

$$<\frac{1}{2\pi}\left[2.O\left(\frac{1}{n}\right)+2M(n-1)+(2n+1)M\right]=O(n)\text{ as } n\to\infty.$$

(ii) Now for $1/n\leqslant t\leqslant\delta<\pi$.

$$|J_n(t)|=\frac{1}{2\pi}\left|\sum_{k=0}^{n}\lambda_{n,k}\frac{\sin(2k+1)\frac{1}{2}t}{\sin\frac{1}{2}t}\right|=\frac{1}{2\pi}\left|\sum_{k=0}^{n}\lambda_{n,n-k}\frac{\sin(2n-2k+1)\frac{1}{2}t}{\sin\frac{1}{2}t}\right|=$$

$$=O\left(\frac{\lambda_{n,n-\tau}}{t^2}\right)\text{ by lemma 2.}$$

Therefore,

(6.6) $$L_1=\int_{0}^{\frac{1}{n}}\Phi(t)J_n(t)dt=O\left(\int_{0}^{\frac{1}{n}}|\Phi(t)||J_n(t)|dt\right)=O(n).\int_{0}^{\frac{1}{n}}|\Phi(t)|dt=$$

$$=O\left(n.o\left(\frac{n^{-1}}{\log n}\right)\right)\text{ uniformly in } E=o(1)\text{ uniformly in } E \text{ as } n\to\infty.$$

and

$$L_2=\int_{\frac{1}{n}}^{\delta}\Phi(t)J_n(t)dt=O\left(\int_{\frac{1}{n}}^{\delta}|\Phi(t)||J_n(t)|dt\right)=O\left(\int_{\frac{1}{n}}^{\delta}|\Phi(t)|\frac{\lambda_{n,n-\tau}}{t^2}dt\right).$$

$$\text{But}\int_{\frac{1}{n}}^{\delta}\frac{|\Phi(t)|}{t^2}\lambda_{n,n-\tau}dt=\left[\Phi(t).\frac{\lambda_{n,n-\tau}}{t^2}\right]_{\frac{1}{n}}^{\delta}+2\int_{\frac{1}{n}}^{\delta}\Phi(t)\frac{\lambda_{n,n-\tau}}{t^3}dt-$$

$$-\int_{\frac{1}{n}}^{\delta}\frac{\Phi(t)}{t^2}d(\lambda_{n,n-\tau})=\left\{o(1)+o\left(\frac{1}{\log n}\right)+o\left(\int_{\frac{1}{n}}^{\delta}\frac{\lambda_{n,n-\tau}}{t^2\log 1/t}dt\right)+\right.$$

$$+o\left(\int_{\frac{1}{n}}^{\delta}\frac{d(\lambda_{n,\,n-\tau})}{t\log 1/t}\right)\Bigg]\text{uniformly in } E.$$

$$=\left[o(1)+o\left(\frac{1}{\log n}\right)+o\left(\int_{\frac{1}{\delta}}^{n}\frac{\lambda_{n,\,n-[x]}}{\log x}\,dx\right)+o\left(\int_{\frac{1}{\delta}}^{n}\frac{xd\lambda_{n,\,n[x]}}{\log x}\right)\right]\text{uniformly in } E,$$

$$=\left[o(1)+o\left(\frac{1}{\log n}\right)+o\left(\left\{\int_{[\delta^{-1}]}^{a}+\int_{a}^{a+1}+\ldots+\int_{k}^{k+1}+\ldots+\int_{n-1}^{n}\right\}\frac{\lambda_{n,\,n-[x]}}{\log x}\,dx\right)\right.$$

$$\left.+o\left(\left\{\int_{[\delta^{-1}]}^{a}+\int_{a}^{a+1}+\ldots+\int_{k}^{k+1}+\ldots+\int_{n-1}^{n}\right\}\frac{xd\lambda_{n,\,n-[x]}}{\log x}\right)\right) \text{ uniformly in } E.$$

where $a=[\delta^{-1}]+1$

$$=\left[o(1)+o\left(\frac{1}{\log n}\right)+o\left(\sum_{k=a-1}^{n-1}\frac{\lambda_{n,\,n-k}}{\log k}\right)+0\left(\sum_{k=a}^{n}\frac{k}{\log k}\,\Delta\lambda_{n,\,n-k}\right)\right]\text{uniformly in } E.$$

$$=\left[o(1)+o\left(\frac{1}{\log n}\right)+o\left(\sum_{k=0}^{n}\lambda_{n,\,n-k}\right)+o\left(\frac{n}{\log n}\right)\sum_{k=a}^{n}\lambda_{n,\,n-k}-\lambda_{n,\,n-k+1}\}\right]$$

uniformly in E.

$$=\left[o(1)+o\left(\frac{1}{\log n}\right)+o\left(\sum_{k=0}^{n}\lambda_{n,\,k}\right)+o\left(\frac{n}{\log n}\right)\{\lambda_{n,\,0}-\lambda_{n,n-a+1}\}\right]$$

uniformly in E.

$$=[o(1)+o(1)+o(1)+o(1)]\text{ uniformly in } E.$$

as $n\to\infty$. by regularity condition and lemma 3

$$=o(1)\text{ uniformly in } E. \tag{6.7}$$

Now combining (6.1), (6.2) and ... (6.7), we get the proof of the theorem.

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NOTION DE SPECTRE DANS UNE CATEGORIE QUELCONQUE (I)

PAR

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L'objet de cette note est d'étendre, dans le cadre d'une catégorie quelconque (éventuellement additive), certaines constructions fondamentales en géométrie algébrique [4]: spectre, anneau de fractions, localisé, etc. Dans le §1, on définit un foncteur contravariant, noté Sp, sur une catégorie quelconque, à valeurs dans TOP. Lorsque la catégorie est additive, on peut définir, sur chaque espace topologique Sp A, des faisceaux de groupes abéliens $\mathbf{O}_{AB}$. Cette construction est acomplie en §6. On utilise de façon essentielle un certain foncteur Q_+, introduit au §2. Un rôle important joue la notion d'objet complet, qui est détaillée en §3. §4 et §5 contiennent des application à **TOP** (et à **ENS**, comme cas particulier), tandis que §7 traite l'exemple classique de la catégorie A-mod. Un dernier paragraphe contient une brève disscution du comportement des constructions précédentes, par l'action des foncteurs.

Les trois derniers paragraphes paraîtront séparément. Cette note est le développement de la conférence présentée à Cape-Town 1976 [7].

Pour tout ce qui concerne les catégories, nous renvoyons à [5], [8], [9]. $\mathbf{C}$ sera une catégorie quelconque. Pour un objet $A\in\mathrm{Ob}\,\mathbf{C}$, on note $\mathbf{P}(A)$ la classe des sous-objets de de A. $\mathbf{P}(A)$ sera ordonnée par la relation d'inclusion : $A'\subseteq A''$ si et seulement si, il existe un monomorphisme $A'\to A''$, compatible avec les monomorphismes $A'\to A$ et $A''\to A$. Si $\mathbf{C}$ possède des limites projectives finies, on note $A'\cap A''$ la limite projective du diagramme :

$$\begin{array}{ccc} & A' & \\ & \downarrow & \qquad A',\,A''\in\mathbf{P}(A) \\ A''\to & A & \end{array}$$

et $u^{-1}(B')$ la limite projective du diagramme :

$$\begin{array}{ccc} & B' & \\ & \downarrow & \qquad B'\in\mathbf{P}(B) \\ A\xrightarrow{u} & B & \end{array}$$

On note $u\,|_{A'}$ la composition $A'\to A\xrightarrow{u}B$.

Un ensemble ordonné sera identifié avec la catégorie correspondente. Une application $f: X\to Y$ entre deux espaces topologiques sera appellée homéomorphisme sur son image, si $f: X\to f(X)$ est un homéomorphisme. $A\setminus B$ désignera la différence d'ensembles ; $\complement A$ sera le complémentaire par rapport à un ensemble qui sera clair du contexte.