Proposition 5.16. *Si* Γ *est compacte, alors* $s_{\Gamma\Delta}(\Gamma)=\Delta$.

Démonstration. Γ étant compacte, chaque $\mathbf{F}\in\Delta$ a un point d'accumulation sur Γ, soit $\mathbf{F}_1$. Il existe alors un ultrafiltre plus fin que $\mathbf{F}$ et $\mathbf{F}_1$.

Corollaire 5.17. *Si* Γ *est* T_2 *alors il existe une fonction*

$$\alpha_{\Gamma\Delta}: t_{\Gamma\Delta}(\Gamma)\to\Gamma$$

telle que $\alpha_{\Gamma\Delta}^{-1}(\Gamma')=t_{\Gamma\Delta}(\Gamma')$, $\forall\Gamma'\subseteq\Gamma$.

Le résultat suivant a été utilisé dans [6] :

Proposition 5.18. *Si* Γ *est compact et* $A\subseteq X$ *alors :*

$$\Delta\cap\mathcal{C}\mathbf{O}(A)\subseteq s_{\Gamma\Delta}(\Gamma\cap\mathcal{C}\mathbf{O}(A))$$

Démonstration. Soit $\mathbf{F}\in\Delta\cap\mathcal{C}\mathbf{O}(A)$ donc $A\notin\mathbf{F}$. Il existe un ultrafiltre $\mathbf{U}$ plus fin que $\mathbf{F}$ tel que $\mathcal{C}A\in\mathbf{U}$. Γ étant compact, il existe $\mathbf{F}_1\in\Gamma$ tel que $\mathbf{U}$ soit plus fin que $\mathbf{F}_1$. Alors $A\notin\mathbf{F}_1$, ce qui prouve que $\mathbf{F}_1\in\Gamma\cap\mathcal{C}\mathbf{O}(A)$.

L'application $t_{\Gamma\Delta}$ permet, dans certain cas, la comparaison des solutions du problème de Dirichlet, relative à deux bords. On considère, comme exemple, le cas simplifié suivant. Supposons [3] X localement compact et $\mathbf{S}$ un ensemble de fonctions $u: X\to(-\infty, +\infty]$, i.s.c. Pour chaque bord Γ et chaque fonction $f:\Gamma\to[-\infty, +\infty]$ soit :

$$H_f^{\Gamma}(x)=\inf\{u(x)\mid u\in\mathbf{S},\ \liminf_{\mathbf{F}} u\geq f(\mathbf{F}),\ \forall\mathbf{F}\in\Gamma\}\quad(x\in X).$$

Supposons Γ compact et soit Δ un bord, tel que $t_{\Gamma\Delta}(\Gamma)=\Delta$. Alors on a l'application $\alpha_{\Gamma\Delta}:\Delta\to\Gamma$ et l'inégalité suivante est valable :

$$H_f^{\Delta}\circ\alpha_{\Gamma\Delta}\leqslant H_f^{\Gamma}.$$

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Faculté de Mathematiques
Université de Iași
6 600 — Iași
R. S. Roumanie

Analele științifice ale Universității „Al. I. Cuza" din IAȘI
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SUBGRADIENTS OF CONVEX FUNCTIONS ON RIEMANNIAN MANIFOLDS *)

BY

CONSTANTIN N. UDRIȘTE

Suppose M is a complete connected finite-dimensional Riemannian manifold. Let A be a totally convex set in M and $f: A\to R$ be a convex function. The set T_x^*A of cotangent vectors to A at a point x is a convex cone in T_x^*M. Let y be a generic point in A and $\gamma_{xy}(t)$, $t\in[0, 1]$, be a geodesic arc such that $\gamma_{xy}(0)=x$, $\gamma_{xy}(1)=y$, $\dot\gamma_{xy}(0)=X_x\in T_xA$. We denote by Γ_{xy} the set of all geodesic arcs joining the points x and y.

1. Definition. *A 1-form* $\omega_x\in T_x^*A$ *is called the subgradient of* f *at* x *if* $f(y)\geq f(x)+\omega_x(\dot\gamma_{xy}(0))$, $\forall y\in A$, $\forall\gamma_{xy}\in\Gamma_{xy}$.

2. Definition. *The set of all subgradients of* f *at* x *is called the subdifferential of* f *at* x *and is denoted by* $\partial f(x)$. *The multiform map* $\partial f: x\to\partial f(x)$ *is called the subdifferential of* f.

3. *Remark.* If f is differentiable at x, then $\omega_x=df(x)$.

4. Theorem. *The set* $\partial f(x)\subset T_x^*A$ *contains at least one element.*

Proof. Since f is convex on A, we have

$$f(y)-f(x)\geq\frac{f(\gamma_{xy}(t))-f(x)}{t}\geq Df(x;\ X_x),$$

where $t>0$ and $Df(x;\ X_x)$ is the unilateral, directional derivative [7] of f at x with respect to $\dot\gamma_{xy}(0)=X_x$. But T_x^*A is a convex set in T_x^*M and $X_x\to Df(x;\ X_x)$ is a convex function [7]. Therefore [8] there exists $\omega_x\in T_x^*M$ such that $\omega_x(X_x)\leqslant Df(x;\ X_x)$ and hence $f(y)-f(x)\geqslant\omega_x(X_x)$ i.e. $\omega_x\in\partial f(x)$.

5. Theorem. *The set* $\partial f(x)$ *is convex and compact.*

Proof. Let ω_x^1, $\omega_x^2\in\partial f(x)$. Then

$$f(y)\geq f(x)+\omega_x^1(\dot\gamma_{xy}(0)),\ f(y)\geq f(x)+\omega_x^2(\dot\gamma_{xy}(0)),\quad\forall y\in A,\ \forall\gamma_{xy}\in\Gamma_{xy}.$$

Multiplying the first relation by 1-s, the second by $s\in[0, 1]$, and adding, we find $f(y)\geq f(x)+(1-s)\,\omega_x^1+s\omega_x^2)\,(\dot\gamma_{xy}(0))$, $\forall y\in A$. So $(1-s)\,\omega_x^1+s\omega_x^2\in\partial f(x)$ and hence $\partial f(x)$ is convex.

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We shall show that $\partial f(x)$ is closed and bounded, i.e. compact. Let $\{\omega_x^n\} \subset \partial f(x)$ be a sequence of subgradients convergent to a covector ω_x. Suppose $\omega_x \notin \partial f(x)$, i.e., $\exists \varepsilon > 0$. $\exists z \in M$, $\exists \gamma_{xz}$, such that $f(z) + \varepsilon = f(x) + \omega_x(\dot\gamma_{xz}(0))$. On the other hand $f(z) \geq f(x) + \omega_x^n(\dot\gamma_{xz}(0))$. By subtraction we get $\varepsilon \leq (\omega_x - \omega_x^n)(\dot\gamma_{xz}(0)) \leq \|\omega_x - \omega_x^n\| \, \|\dot\gamma_{xz}(0)\|$. It follows $\|\omega_x - \omega_x^n\| \geq \dfrac{\varepsilon}{\|\dot\gamma_{xz}(0)\|} > 0$, $\forall_n \in N$, which contradicts the fact that $\{\omega_x^n\}$ converges to ω_x. Hence $\omega_x \in \partial f(x)$ and $\partial f(x)$ is closed.

Let $\omega_x \in \partial f(x)$, i.e. $\omega_x(\dot\gamma_{xy}(0)) + f(x) \leq f(y)$ or $\|\omega_x\| \, \|\dot\gamma_{xy}(0)\| \cos\theta + f(x) \leq f(y)$, $\forall y \in A$, $\forall \gamma_{xy} \in \Gamma_{xy}$. It follows that $\|\omega_x\|$ is finite, and hence $\partial f(x)$ is closed.

6. Theorem. *$0_x \in \partial f(x)$ iff x is a minimizing point of f.*

Proof. $0_x \in \partial f(x)$ iff $f(y) \geq f(x) + 0_x(\dot\gamma_{xy}(0))$, $\forall y \in A$, $\forall \gamma_{xy} \in \Gamma_{xy}$, i.e., iff x is a minimizing point of f.

7. Theorem. *$\omega_x \in \partial f(x)$ iff $Df(x\,;\,X_x) \geq \omega_x(X_x)$, $\forall X_x \in T_x A$.*

Proof. Suppose $\omega_x \in \partial f(x)$, i.e. $f(y) \geq f(x) + \omega_x(\dot\gamma_{xy}(0))$, $\forall y \in A$, $\forall \gamma_{xy} \in \Gamma_{xy}$. Let $\gamma_{xy}(t)$, $t \in [0, 1]$ be a geodesic arc such that $\dot\gamma_{xy}(0) = X_x$. Substituting y by $\gamma_{xy}(st)$, $s > 0$, we find

$$\frac{f(\gamma_{xy}(st)) - f(x)}{s} \geq \omega_x(X_x)\,,$$

and taking the limit as $s \to 0$, we conclude that $Df(x\,;\,X_x) \geq \omega_x(X_x)$.

Conversely, the relations

$$Df(x\,;\,X_x) \geq \omega_x(X_x)\,,\ f(y) - f(x) \geq \frac{f(\gamma_{xy}(t)) - f(x)}{t} \geq Df(x\,;\,X_x)$$

imply $f(y) - f(x) \geq \omega_x(X_x)$ and hence $\omega_x \in \partial f(x)$.

8. Corollary. x is a minimizing point of f iff $Df(x\,;\,X_x) \geq 0$, $\forall X_x \in T_x A$.

9. Theorem. *Let A be an open totally convex set. The function $f : A \to R$ is convex iff whatever $x \in A$, there exists $\omega_x \in T_x^* A$ so that, $\forall y \in A$, $\forall \gamma_{xy} \in \Gamma_{xy}$, we have $f(y) - f(x) \geq \omega_x(\dot\gamma_{xy}(0))$.*

Proof. Suppose that for each $x \in A$, there exists $\omega_x T_x^* A$ so that, $\forall y \in A$, $\forall \gamma_{xy} \in \Gamma_{xy}$, $f(y) - f(x) \geq \omega_x(\dot\gamma_{xy}(0))$. Similar arguments to those in the Proof of Theorem 3 in [5] show that f is convex on A.

Conversely, the convexity of $f : A \to R$ and the above Theorem 4 provide the existence of $\omega_x \in T_x^* M$ so that $f(y) - f(x) \geq \omega_x(\dot\gamma_{xy}(0))$.

10. *Remark.* Let A be an open totally convex set of sufficiently small diameter. The function defined by $y \to \omega_x(\dot\gamma_{xy}(0))$, $y \in A$, is linear affine (i.e. convex and concave).

11. Theorem. *A convex function on an open totally convex set of sufficiently small diameter is the superior of a family of linear affine functions.*

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Department of Mathematics I
Polytechnic Institute
Bucharest, Roumania