Analele ştiinţifice ale Universităţii „Al. I. Cuza" din IAŞI
Tomul XXVI, s. I a, 1980, f. 2

BOUNDEDNESS AND OSCILLATION OF SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

BY

CH. G. PHILOS and V.A. STAIKOS

In this paper we treat the boundedness or unboundedness of the solutions of differential equations with deviating argument as well as the damped or unbounded oscillations. The differential equations considered are of the form

$$(E)\qquad [r_{n-1}(t)\,[r_{n-2}(t)\,[\ldots[r_2(t)\,[r_1(t)\,x'(t)]']'\ldots]']']' + a(t)\,F(x[g(t)])=b(t),\ t\geqslant t_0,$$

where $r_k(k=1,\ldots,n-1)$, a, g and b are continuous real-valued functions on the interval $[t_0, \infty)$ and F is a continuous real-valued function on the real line **R**. Without mentioning it any further, the functions r_k $(k=1,\ldots,n-1)$ will be supposed positive on $[t_0, \infty)$ and the function g such that

$$\lim_{t\to\infty} g(t)=\infty.$$

We consider only such solutions $x(t)$ of the differential equation (E) which are defined for all large t. Sufficient smoothness for the existence of such solution of (E) will be assumed without mention. The oscillatory character is considered in the usual sense, i.e. a continuous real-valued function defined on an interval of the form $[T, \infty)$ is said to be *oscillatory* if the set of its zeros is unbounded above, and otherwise it is said to be *nonoscillatory*.

For the sake of brevity, the notations $D_r^{(i)}x$ $(i=0, 1,\ldots, n)$ for the so called *r-derivatives* of the function x are introduced as follows

$$D_r^{(0)}x=x,\quad D_r^{(k)}x=r_k(D_r^{(k-1)}x)'\ (k=1,\ldots,n-1)\ \text{ and }\ D_r^{(n)}x=(D_r^{(n-1)}x)'.$$

Then the differential equation under consideration can be written in the form

$$(E)\qquad (D_r^{(n)}x)(t)+a(t)F(x[g(t)])=b(t).$$

For such differential equations involving the operator $D_r^{(n)}$ there has been an increasing interest on the oscillation and the asymptotic behavior. As one can see in the book of C o p p e l [1; Theorem 2 and Lemma 6, pp. 91 and 93], the class of operators $D_r^{(n)}$ properly contains all disconjugate operators L defined by $Lx=x^{(n)}+p_1x^{(n-1)}+\ldots+p_{n-1}x'+p_nx$ (p_i continuous functions).

The results of this paper are presented as four theorems. Theorem 1 establishes sufficient conditions for the boundedness of all solutions of the differential equation (E) when this equation is of retarded type. Theorem 2 concerns also retarded differential equations of the form (E) and says that the assumptions of Theorem 1 are sufficient for all oscillations to be damped. For the general case of differential equations of the form (E) with deviating argument, Theorem 3 ensures the nonoscillation of the bounded solutions (i.e. the unboundedness of the oscillations) and Theorem 4 gives a sufficient condition for the unboundedness of all solutions. Note that in Theorems 1 and 2 it is supposed that $|F(y)|\leq K|y|$ for all y, where K is a positive constant, while Theorems 3 and 4 hold without this assumption on the function F. Numerous examples are inserted in the text, to illustrate the relevance of the theorems. Also, some examples are given, which provide the answers to some natural questions which arise when reading the text.

The results obtained here generalize and improve some recent results of Singh [2] and this is illustrated by appropriate examples. For related results, we refer to [3] and to the references in [2].

Theorem 1. *Let the differential equation* (E) *be of retarded type, i.e.*

$$g(t)\leq t \text{ for every } t\geq t_0,$$

and suppose that there exists a positive constant K *such that*

$$|F(y)|\leq K|y| \text{ for all } y.$$

Moreover, let the conditions (C_1) *and* (C_2) *below be satisfied:*

$$(C_1)\qquad \lim_{t\to\infty}\int_{t_0}^{t\equiv s_0}\frac{1}{r_1(s_1)}\int_{t_0}^{s_1}\frac{1}{r_2(s_2)}\cdots\int_{t_0}^{s_{n-2}}\frac{1}{r_{n-1}(s_{n-1})}\,ds_{n-1}\ldots ds_2ds_1<\infty.$$

$$(C_2)\qquad \int^{\infty}|a(t)|\,dt<\infty \quad\text{and}\quad \int^{\infty}|b(t)|\,dt<\infty.$$

Then all solutions of the equation (E) *are bounded.*

Proof. Let x be a solution on an interval $[T_0,\infty)$, $T_0\geq t_0$, of the equation (E) and let $T\geq T_0$ be chosen so that

$$g(t)\geq T_0 \text{ for every } t\geq T.$$

Because of (C_2), we have

$$0\leq A_T\equiv\int_T^{\infty}|a(t)|\,dt<\infty \quad\text{and}\quad 0\leq B_T\equiv\int_T^{\infty}|b(t)|\,dt<\infty.$$

Furthermore, for any $k=1,\ldots,n-1$ and every $t\geq T$ we define

$$R_k(t\,;T)=\int_T^{t\equiv s_0}\frac{1}{r_1(s_1)}\int_T^{s_1}\frac{1}{r_2(s_2)}\cdots\int_T^{s_{k-1}}\frac{1}{r_k(s_k)}\,ds_k\ldots ds_2ds_1.$$

After some manipulations, one can easily derive from (C_1) that

$$0<R_k(\infty\,;T)\equiv\lim_{t\to\infty}R_k(t\,;T)<\infty\quad(k=1,\ldots,n-1).$$

Now, it is easy to verify the generalized Taylor formula

$$x(t)=x(T)+\sum_{k=1}^{n-1}(D_r^{(k)}x)(T)\,R_k(t\,;T)+$$

$$+\int_T^{t\equiv s_0}\frac{1}{r_1(s_1)}\int_T^{s_1}\frac{1}{r_2(s_2)}\cdots\int_T^{s_{n-2}}\frac{1}{r_{n-1}(s_{n-1})}\int_T^{s_{n-1}}(D_r^{(n)}x)(s)\,ds\,ds_{n-1}\ldots ds_2ds_1$$

for all $t\geq T$. Thus, , for $t\geq T$ we obtain

$$|x(t)|\leq|x(T)|+\sum_{k=1}^{n-1}|(D_r^{(k)}x)(T)|R_k(t\,;T)+$$

$$+\int_T^{t\equiv s_0}\frac{1}{r_1(s_1)}\int_T^{s_1}\frac{1}{r_2(s_2)}\cdots\int_T^{s_{n-2}}\frac{1}{r_{n-1}(s_{n-1})}\int_T^{s_{n-1}}|(D_r^{(n)}x)(s)|\,ds\,ds_{n-1}\ldots ds_2ds_1$$

$$\leq|x(T)|+\sum_{k=1}^{n-1}|(D_r^{(k)}x)(T)|\,R_k(t\,;T)+R_{n-1}(t\,;T)\int_T^t|(D_r^{(n)}x)(s)|\,ds$$

$$\leq|x(T)|+\sum_{k=1}^{n-1}|(D_r^{(k)}x)(T)|\,R_k(t;T)+R_{n-1}(t;T)\int_T^t|b(s)|\,ds+$$

$$+R_{n-1}(t\,;T)\int_t^T|a(s)|\,|F(x[g(s)])|\,ds$$

$$\leq|x(T)|+\sum_{k=1}^{n-1}|(D_r^{(k)}x)(T)|\,R_k(t\,;T)+R_{n-1}(t;T)\int_T^t|b(s)|\,ds+$$

$$+KR_{n-1}(t\,;T)\int_T^t|a(s)|\,|x[g(s)]|\,ds$$

$$\leq|x(T)|+\sum_{k=1}^{n-1}|(D_r^{(k)}x)(T)|\,R_k(\infty\,;T)+R_{n-1}(\infty\,;T)\,B_T+$$

$$+KR_{n-1}(\infty\,;T)\int_T^t|a(s)|\,|x[g(s)]|\,ds.$$

Hence, there exist two constants α_1, β with $\alpha_1 \geqslant 0$, $\beta > 0$ such that

$$|x(t)| \leqslant \alpha_1 + \beta \int_T^t |a(s)| \, |x[g(s)]| \, ds \quad \text{for all } t \geq T.$$

Next, we consider a $\tau \geqq T$ such that $g(t) \geqslant T$ for every $t \geqslant \tau$. Then for $t \geqq \tau$ we get

$$|x[g(t)]| \leqslant \alpha_1 + \beta \int_T^{g(t)} |a(s)| \, |x[g(s)]| \, ds \leqslant \alpha_1 + \beta \int_T^t |a(s)| \, |x[g(s)]| \, ds =$$

$$= \alpha_1 + \beta \int_T^\tau |a(s)| \, |x[g(s)]| \, ds + \beta \int_\tau^t |a(s)| \, |x[g(s)]| \, ds.$$

Thus, we have

$$|x[g(t)]| \leqq \alpha + \beta \int_\tau^t |a(s)| \, |x[g(s)]| \, ds \quad \text{for all } t \geqq \tau,$$

where α is a nonnegative constant. Therefore, by the Gronwall inequality, we derive

$$|x[g(t)]| \leqslant \alpha \exp\left[\beta \int_T^t |a(s)| \, ds\right] \leqslant \alpha \exp(\beta A_T)$$

for every $t \geqslant \tau$. Hence, the function $x \circ g$ is bounded and consequently, by $\lim_{t \to \infty} g(t) = \infty$, the solution x is also bounded.

Theorem 2. *Let the differential equation* (E) *be of retarded type and suppose that there exists a positive constant* K *such that*

$$|F(y)| \leqq K |y| \quad \textit{for all } y.$$

Moreover, let the conditions (C_1) *and* (C_2) *be satisfied.*

Then all oscillatory solution of the equation (E) *tend to zero at* ∞.

Proof. Let x be an oscillatory solution on an interval $[T_0, \infty)$, $T_0 \geqslant t_0$, of the equation (E). By Theorem 1, x is bounded. The boundedness of x and the continuity of F ensure that the function $F \circ x$ is bounded on $[T_0, \infty)$. Let S be a positive bound of $F \circ x$. Furthermore, we define

$$R(t) = \int_{t_0}^{t=s_0} \frac{1}{r_1(s_1)} \int_{t_0}^{s_1} \frac{1}{r_2(s_2)} \cdots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \, ds_{n-1} \ldots ds_2 ds_1, \quad t \geqslant t_0.$$

By (C_1), we have that

$$0 < R(\infty) \equiv \lim_{t \to \infty} R(t) < \infty.$$

Now, let ε be an arbitrary positive number. Because of (C_2) and the fact that $\lim_{t \to \infty} g(t) = \infty$, we can choose a $T \geqslant T_0$ so that

$$\int_T^\infty |a(t)| \, dt < \frac{\varepsilon}{2SR(\infty)}, \quad \int_T^\infty |b(t)| \, dt < \frac{\varepsilon}{2R(\infty)},$$

and $g(t) \geqslant T_0$ for every $t \geqslant T$.

Next, we observe that the functions $D_r^{(i)} x$ $(i = 0, 1, \ldots, n-1)$ are oscillatory, since x is oscillatory. So, we can consider τ_i $(i = 0, 1, \ldots, n-1)$ with $\tau_0 \geqslant \tau_1 \geqslant \ldots \geqslant \tau_{n-1} \geqslant T$ and

$$(D_r^{(i)} x)(\tau_i) = 0 \quad (i = 0, 1, \ldots, n-1).$$

Then it is easy to verify that for $t \geqq \tau_0$

$$x(t) = \int_{\tau_0}^{t=s_0} \frac{1}{r_1(s_1)} \int_{\tau_1}^{s_1} \frac{1}{r_2(s_2)} \cdots \int_{\tau_{n-2}}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{\tau_{n-1}}^{s_{n-1}} (D_r^{(n)} x)(s) \, ds ds_{n-1} \ldots ds_2 ds_1.$$

Thus, for every $t \geqq \tau_0$ we obtain

$$|x(t)| \leqslant \int_{\tau_0}^{t=s_0} \frac{1}{r_1(s_1)} \int_{\tau_1}^{s_1} \frac{1}{r_2(s_2)} \cdots \int_{\tau_{n-2}}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{\tau_{n-1}}^{s_{n-1}} |(D_r^{(n)} x)(s)| \, ds ds_{n-1} \ldots ds_2 ds_1$$

$$\leqslant R(t) \int_T^t |(D_r^{(n)} x)(s)| \, ds \leqq R(t) \int_T^t |b(s)| \, ds + R(t) \int_T^t |a(s)| \, |F(x[g(s)])| \, ds$$

$$\leqslant R(\infty) \int_T^\infty |b(s)| \, ds + R(\infty) S \int_T^\infty |a(s)| \, ds < R(\infty) \frac{\varepsilon}{2R(\infty)} + R(\infty) S \frac{\varepsilon}{2SR(\infty)} = \varepsilon.$$

We have thus proved that for every positive number ε it holds $|x(t)| < \varepsilon$ *for* all large t. This means that

$$\lim_{t \to \infty} x(t) = 0$$

and the proof of the theorem is complete.

We choose to give below four examples to illustrate the relevance of Theorems 1 and 2.

Example 1. The retarded differential equation

$$\left[\frac{1}{t^2} [t^3 x'(t)]'\right]' + \frac{1}{t^{3/2}} x(\sqrt{t}) = \frac{1}{t^2}, \quad t \geqslant 1$$

has the bounded (nonoscillatory) solution $x(t) = 1/t$. By Theorem 1, every (oscillatory or nonoscillatory) solution of this equation is bounded.

Example 2. By Theorem 1, all solutions of the retarded equation

$$[t^2[t^{3/2}x'(t)]']' + \frac{1}{t^{5/4}}\, x(\sqrt{t}) = \frac{1}{t^2}, \quad t \geqslant 1$$

are bounded. For example, this equation admits $x(t) = 1/t^{3/2}$ as a bounded (nonoscillatory) solution.

Example 3. The retarded equation

$$\left[\frac{1}{t}[e^t x'(t)]'\right]' + e^{-t-2\pi}\, x(t-\pi) = e^{-t}\left[\left(\frac{2}{t} - \frac{1}{t^2}\right)\sin t + \left(\frac{4}{t} + \frac{3}{t^2}\right)\cos t\right] -$$

$$- e^{-3t}\sin t, \quad t \geqslant 1$$

has the oscillatory solution $x(t) = e^{-2t}\sin t$ for which we have $\lim_{t\to\infty} x(t) = 0$. Theorem 2 establishes that all oscillatory solutions $x(t)$ of this equation tend to zero as $t \to \infty$. Moreover, by Theorem 1, all nonoscillatory solutions of the above equation are bounded.

Example 4. Theorem 2 ensures that all oscillatory solutions $x(t)$ of the retarded equation

$$[e^t[t^{3/2}x'(t)]']' + e^{-t-2\pi}\, x(t-\pi) = -e^{-3t}\sin t + e^{-t}\left[\left(t^{5/2} + 6t^{1/2} - \frac{3}{2}t^{-1/2}\right)\sin t + \right.$$

$$\left. + \left(7t^{3/2} - \frac{21}{2}t^{1/2} + \frac{3}{4}t^{-1/2}\right)\cos t\right], \quad t \geqslant 1$$

tend to zero as $t \to \infty$. For example, $x(t) = e^{-2t}\sin t$ is such a solution. Moreover, from Theorem 1 it follows that all nonoscillatory solutions of the above equation are bounded.

The following example shows that the conditions of Theorem 1 are not sufficient in order to have $\lim_{t\to\infty} x(t) = 0$ for the nonoscillatory solutions $x(t)$ of the differential equation (E).

Example 5. All assumptions of Theorem 1 hold in the case of the differential equation

$$\left[\frac{1}{t^2}[t^5 x'(t)]'\right]' + \frac{1}{t^{3/2}}\, x(\sqrt{t}) = \frac{1}{t^{3/2}} + \frac{1}{t^2}, \quad t \geqslant 1.$$

On the other hand, this equation has the (bounded) nonoscillatory solution $x(t) = 1 + 1/t$ for which we have $\lim_{t\to\infty} x(t) = 1$.

Theorem 3. *Suppose that*:

(C_3) *There exists an integer* m, $1 \leqslant m \leqslant n-1$, *such that the assumption* $(H_1[m])$ *below is satisfied*:

$(H_1[m])$ *We have*:

$$\int^{\infty} |a(t)|\, dt < \infty \quad \textit{and} \quad \int^{\infty} b(t)\, dt = \pm\infty,$$

if $m = n-1$; a *is not identically zero on* $[t_0, \infty)$.

$$\lim_{t\to\infty} \int_{t_0}^{t=s_m} \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{t_0}^{s_{n-1}} |a(s)|\, ds\, ds_{n-1} \ldots ds_{m+2}\, ds_{m+1} < \infty$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t=s_m} \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{t_0}^{s_{n-1}} b(s)\, ds\, ds_{n-1} \ldots ds_{m+2}\, ds_{m+1} = \pm\infty,$$

if $m < n-1$.

Then all bounded solutions of the differential equation (E) *are nonoscillatory*.

Theorem 4. *Suppose that*:

(C_4) *There exists an integer* m, $0 \leqslant m \leqslant n-1$, *such that the assumption* $(H_1[m])$ *and the following one are satisfied*:

$(H_2[m])$ *If* $m > 0$, *then*

$$\int^{\infty} \frac{dt}{r_i(t)} = \infty \qquad (i = 1, \ldots, m).$$

Then all solutions of the differential equation (E) *are unbounded*.

Proof of Theorems 3 *and* 4. Let x be a bounded solution on an interval $[T_0, \infty)$, $T_0 \geqslant t_0$, of the equation (E) and let $T \geqslant T_0$ be such that $g(t) \geqslant T_0$ for every $t \geqslant T$.

Because of the boundedness of x and the continuity of F, we have

$$S = \sup\{|F(x[g(t)])| : t \geqslant T\} < \infty.$$

Now, suppose, that $m = n-1$. For every $t \geqslant T$, we get

$$(D_r^{(n-1)} x)(t) - (D_r^{(n-1)} x)(T) = \int_T^t (D_r^{(n)} x)(s)\, ds = \int_T^t b(s)\, ds - \int_T^t a(s) F(x[g(s)])\, ds.$$

Thus, for $t \geqslant T$ we obtain

$$|(D_r^{(n-1)} x)(t)| + |(D_r^{(n-1)} x)(T)| \geqslant \left|\int_T^t b(s) ds\right| - \int_T^t |a(s)|\, |F(x[g(s)])|\, ds \geqslant$$

$$\geqslant \left|\int_T^t b(s) ds\right| - S\int_T^t |a(s)|\, ds.$$

Hence, by $(H_1[n-1])$, we derive that

$$\lim_{t\to\infty} |(D_r^{(n-1)} x)(t)| = \infty.$$

Next, let us consider the case where $m<n-1$. For any $k=m+1,\dots,n-1$ and every $t\geqslant T$ we define

$$R_{mk}(t\,;\,T)=\int_T^{t=s_m}\frac{1}{r_{m+1}(s_{m+1})}\int_T^{s_{m+1}}\frac{1}{r_{m+2}(s_{m+2})}\cdots\int_T^{s_{k-1}}\frac{1}{r_k(s_k)}\,ds_k\dots ds_{m+2}\,ds_{m+1}.$$

Then, after some manipulations, from $(H_1[m])$ we can derive that

$$\lim_{t\to\infty}R_{mk}(t\,;\,T)<\infty\qquad(k=m+1,\dots,n-1).$$

Furthermore, for every $t\geqslant T$ we obtain

$$(D_r^{(m)}x)(t)-(D_r^{(m)}x)(T)-\sum_{k=m+1}^{n-1}(D_r^{(k)}x)(T)\,R_{mk}(t\,;\,T)=$$

$$=\int_T^{t=s_m}\frac{1}{r_{m+1}(s_{m+1})}\int_T^{s_{m+1}}\frac{1}{r_{m+2}(s_{m+2})}\cdots\int_T^{s_{n-2}}\frac{1}{r_{n-1}(s_{n-1})}\int_T^{s_{n-1}}(D_r^{(n)}x)(s)\,dsds_{n-1}\dots ds_{m+2}ds_{m+1}$$

$$=\int_T^{t=s_m}\frac{1}{r_{m+1}(s_{m+1})}\int_T^{s_{m+1}}\frac{1}{r_{m+2}(s_{m+2})}\cdots\int_T^{s_{n-2}}\frac{1}{r_{n-1}(s_{n-1})}\int_T^{s_{n-1}}b(s)\,dsds_{n-1}\dots ds_{m+2}ds_{m+1}$$

$$-\int_T^{t=s_m}\frac{1}{r_{m+1}(s_{m+1})}\int_T^{s_{m+1}}\frac{1}{r_{m+2}(s_{m+2})}\cdots\int_T^{s_{n-2}}\frac{1}{r_{n-1}(s_{n-1})}\int_T^{s_{n-1}}a(s)F(x[g(s)])\,dsds_{n-1}\dots ds_{m+2}ds_{m+1}.$$

Therefore for $t\geqslant T$ we get

$$|(D_r^{(m)}x)(t)|+|(D_r^{(m)}x)(T)|+\sum_{k=m+1}^{n-1}|(D_r^{(k)}x)(T)|\,R_{mk}(t\,;\,T)\geqslant$$

$$\geqslant\left|\int_T^{t=s_m}\frac{1}{r_{m+1}(s_{m+1})}\int_T^{s_{m+1}}\frac{1}{r_{m+2}(s_{m+2})}\cdots\int_T^{s_{n-2}}\frac{1}{r_{n-1}(s_{n-1})}\int_T^{s_{n-1}}b(s)\,dsds_{n-1}\dots ds_{m+2}\,ds_{m+1}\right|-$$

$$-S\int_T^{t=s_m}\frac{1}{r_{m+1}(s_{m+1})}\int_T^{s_{m+1}}\frac{1}{r_{m+2}(s_{m+2})}\cdots\int_T^{s_{n-2}}\frac{1}{r_{n-1}(s_{n-1})}\int_T^{s_{n-1}}|a(s)|\,dsds_{n-1}\dots ds_{m+2}ds_{m+1}.$$

This, because of $(H_1[m])$, gives

(*) $$\lim_{t\to\infty}|(D_r^{(m)}x)(t)|=\infty.$$

We have thus proved that (*) holds in both cases where $m=n-1$ or $m<n-1$. If $D_r^{(m)}x$ is oscillatory, then (*) is obviously a contradiction. Since the oscillation of x implies the oscillation of $D_r^{(m)}x$, the proof of Theorem 3

is complete. Furthermore, if $D_r^{(m)}x$ is nonoscillatory, then from (*) and $(H_2[m])$ it follows that $\lim_{t\to\infty}x(t)=\pm\infty$, which contradicts the boundedness of x. This completes the proof of Theorem 4.

We now give four examples to demonstate the significance of Theorems 3 and 4.

Example 6. The differential equation

$$[t^{5/2}[tx'(t)]']'+\frac{1}{t^{5/2}}x(\sqrt{t})=\frac{1}{t^3}+\frac{1}{2t^{1/2}},\quad t\geqslant 1$$

has the bounded nonoscillatory solution $x(t)=1/t$. By Theorem 3, all bounded solutions of this equation are nonoscillatory.

Example 7. Theorem 3 ensures that all bounded solutions of the equation

$$[e^t[te^tx'(t)]']'+e^{t/2}x(t/2)=1-e^t,\quad t\geqslant 1$$

are nonoscillatory. For example, $x(t)=e^{-t}$ is such a solution.

Example 8. By Theorem 4, all solution of the equations

$$[e^{-t/2}x'(t)]'+\frac{3}{2}e^{-\frac{t}{2}+\frac{\pi}{2}}x\left(t-\frac{\pi}{2}\right)=e^{-t/2}\left(2-\frac{1}{2}\sin t\right),\quad t\geqslant 1$$

are unbounded. For example, this equation admits $x(t)=e^t(1+\sin t)$ as an unbounded (oscillatory) solution.

Example 9. The equation

$$[e^t[te^{-t}x'(t)]']'+e^{t/2}x(t/2)=2e^t,\quad t\geqslant 1$$

admits the unbounded (nonoscillatory) solution $x(t)=e^t$. Theorem 4 establishes the unboundedness of all solutions of this equation.

Theorem 3 says that, under (C_3), all bounded solution of the differential equation (E) are nonoscillatory. This condition is not sufficient for the nonoscillation of all (bounded or unbounded) solutions of the equation (E). Indeed, the equation of Example 8 is subject to the condition (C_3) while it admits $x(t)=e^t(1+\sin t)$ as an (unbounded) oscillatory solution. Let us consider the differential equation with deviating arguments

$(\hat{E})$ $$(D_r^{(n)}x)(t)+a(t)\,\hat{F}(x[g_1(t)],\dots,x[g_\nu(t)])=b(t),$$

where F is a continuous real-valued function on R^ν and $g_j\,(j=1,\dots,\nu)$ are continuous real-valued functions on $[t_0,\infty)$ such that

$$\lim_{t\to\infty}g_j(t)=\infty\quad(j=1,\dots,\nu).$$

Theorems 3 and 4 also remain valid for the differential equation $(\hat{E})$. This is due to the fact that, if x is a bounded solution of the equation $(\hat{E})$, then $\hat{F}\circ(x\circ g_1,\dots,x\circ g_\nu)$ is also bounded.

By Theorem 1, conditions (C_1) and (C_2) are sufficient for the boundedness of the solutions of the equation (E), under the following assumptions:

(A_1) *The differential equation* (E) *is of retarded type,*

(A_2) *There exists a positive constant* K *such that*

$$|F(y)| \leq K|y| \text{ for all } y.$$

It is noteworthy that for the boundedness of at least the nonoscillatory solutions the assumptions (A_1) and (A_2) are essential, i.e. none of them can be removed without damage, as it is illustrated by the following two examples.

Example 10. The differential equation

$$\left[\frac{1}{t^2}[t^5 x'(t)]'\right]' - \frac{10}{t^2}x(t^3) = 0, \quad t \geq 1$$

is subject to the conditions (A_2), (C_1) and (C_2), but this equation has the unbounded nonoscillatory solution $x(t)=t$. This is due to the fact that the above equation is of advanced type, which is the case when assumption (A_1) fails.

Example 11. Conditions (C_1) and (C_2) are satisfied for the retarded equation

$$[t^2 x'(t)]' - \frac{2}{t^{3/2}} x(\sqrt{t}) = 0, \qquad t \geq 1.$$

But this equation admits $x(t)=t$ as an unbounded nonoscillatory solution due to the fact that the assumption (A_2) fails.

Theorem 1 says also that: Under the condition (A_1), (A_2), (C_1) and (C_2), all oscillatory solutions of the differential equation (E) are bounded. It remains here an open question whether the assumption (A_1) or (A_2) can be removed. It seems that the answer to this question is „not", but the authors fail to give the necessary examples to demonstrate this. From the proof of Theorem 2 it follows that for the bounded oscillatory solutions of the equation (E) no use of the assumptions (A_1) or (A_2) is made. So, we obtain that: Conditions (C_1) and (C_2) are sufficient for all bounded oscillatory solutions of (E) to vanish at ∞. Hence, the above question is equivalent to: „Can the assumption (A_1) or (A_2) be removed from Theorem 2?".

Before closing the paper, we note that Theorems 2, 3 and 4 generalize and improve some recent results of S i n g h [2]. Singh has proved Theorem 2 with the condition

$$(C_1^*) \qquad \begin{cases} \liminf\limits_{t\to\infty} [r_1(t)/t^{n-\beta}] > 0 \text{ for some } \beta \in [0, 1), \\ \liminf\limits_{t\to\infty} r_i(t) > 0 \ (i=2,\dots,n-1), \text{ when } n > 2. \end{cases}$$

in place of (C_1). It is easy to verify that (C_1^*) implies (C_1). On the other hand, it is possible to have condition (C_1) valid while (C_1^*) fails. For example, for the differential equations of Examples 3 and 4, the result of Singh cannot be applied while Theorem 2 can be applied.

It is obvious that the condition (C_3) is satisfied if

$$(C_3^*) \qquad \int^{\infty} |a(t)|\, dt < \infty \quad \text{and} \quad \int^{\infty} b(t)\, dt = \pm\infty.$$

Singh has proved that the condition (C_3^*) is sufficient for the nonoscillation of the bounded solutions (i.e. for the unboundedness of the oscillatory solutions) of the differential equation (E). This Singh's result is covered by Theorem 3. Moreover, Theorem 3 is applicable in some cases in which the above result of Singh cannot be applied, as for example in the case of the differential equation of Example 7.

Singh has established that, under the condition

$$(C_4^*) \qquad \int^{\infty} |a(t)|\, dt < \infty \quad \text{and} \quad \int^{\infty} b(t)\, dt = \pm\infty.$$

$$r_i \ (i=1,\dots, n-1) \text{ are bounded.}$$

all solutions of the differential equation (E) are unbounded. It is clear that (C_4^*) implies (C_4), but these conditions are not equivalent, as it follows by Example 9, where Singh's result cannot be applied.

REFERENCES

1. C o p p e l W.A. — *Disconjugacy*, Lecture Notes in Mathematics 220, Springer-Verlag, Berlin, (1971).

2. S i n g h B. — *Necessary and sufficient conditions for eventually vanishing oscillatory solutions of functional equations with small delays*. Internat. J. Math. and Math. Sci. 1 (1978), pp. 269 — 283.

3. S t a i k o s V.A. and P h i l o s Ch. G. — *Non-oscillatory phenomena and damped oscillations*. J. Nonlinear Anal. 2 (1978), pp. 197 — 210.

Received 2.11.1980

Department of Mathematics
University of Ioannina
Ioannina, GREECE