where matrices $T_{11}(x)$ and $T_{22}(x)$ are regular for all $x \in R$ as it simply follows from the assumption (3.2). It is easily seen that the operator D is completely continuous. Thus, by a well-known theorem, the operator $K'(a^0, b^0)$ is a linear homeomorphism because it is a one-to-one mapping and also a sum of a linear homeomorphism and a completely continuous operator. This completes the proof of Lemma 3.2.

6. An additional result. Let the functions f_i^k satisfy, $i=1,\dots,n$,

$$f_i^1(t, -x, -u, v) = -f_i^1(t, x, u, v), \tag{6.1}$$

$$f_i^2(t, -x, -u, v) = f_i^2(t, x, u, v). \tag{6.2}$$

Theorem 6.1. *Let the assumptions (3.1), (3.2), (6.1), and (6.2) be satisfied. Then there exists a unique* $a^0 \in C_n^1$, $\|a^0\|_{C_n^1} \leqq R_0$, *and two continuous mappings* $u=u(\varepsilon)$, $v=v(\varepsilon)$ *of an interval* $[-\varepsilon_0, \varepsilon_0]$, $\varepsilon_0>0$, *into* $C_n^1(R^2)$ *such that, for every* ε, *the functions* $u=u(\varepsilon)$, $v=v(\varepsilon)$ *satisfy the equations (1.1) – (1.8) and*

$$u(0)\,(t, x) = a^0(t+x) - a^0(t-x), \qquad v(0)\,(t, x) = a^0(t+x) + a^0(t-x).$$

The proof of this theorem can be performed by using the technique of the preceding sections. If for $w \in \mathcal{C}_n^1([0, 2\pi])$, two operators E and O are introduced by

$$(Ew)\,(t, x) = \frac{1}{2}\,(w(t, x) + w(t, -x)), \qquad (Ow)\,(t, x) = \frac{1}{2}\,(w(t, x) - w(t, -x)),$$

and if two functions $w \in \mathcal{C}_1^n([0, 2\pi])$ and $a \in \mathcal{C}_n^1$ satisfy $i=1,\dots,n$,

$$-w_i(t, x) + 2a_i(t+x) + \frac{\varepsilon}{2}\int_0^t \sum_{\varkappa=1}^{2} F_i^{\varkappa}\,(Ow, Ew)\,(\tau, x+t-\tau)\,d\tau = 0, \tag{6.3}$$

$$t \in [0, 2\pi], \quad x \in R,$$

and

$$\sum_{\varkappa=1}^{2}\int_0^{2\pi} F_i^{\varkappa}(Ow, Ew)\,(t, x-t)\,dt = 0, \quad x \in R, \tag{6.4}$$

then $u=Ow$, $v=Ew$, a, $b(\xi)=-a(-\xi)$, and ε satisfy $G(u, v, a, b, \varepsilon)=0$. Hence the problem is reduced to solving (6.3) and (6.4) which can be achieved by applying the methods used in the preceding sections.

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ON THE EXPONENTIAL STABILITY OF THE SOLUTIONS OF A CLASS OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DELAY

BY

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1. Preliminaries. In [5] we have studied the existence, uniqueness and representation of the solutions of the following problem:

$$\frac{d}{dt}\left[x(t) - \sum_{j=1}^{\infty} B_j x(t-h_j)\right] = Ax(t) - \sum_{j=1}^{\infty} A_j x(t-h_j), \ t \geqslant 0, \tag{1.1a}$$

$$x(t) = \Phi(t), \quad t \in (-\infty\,; 0], \ \Phi \in C, \tag{1.1b}$$

where C is the Banach space defined as

$$C = C[R, X] = \{\Phi \mid \Phi : (-\infty\,; 0] \to X,\ \Phi - a \textit{ continuous function},\ \exists \lim_{t\to-\infty} \Phi(t) \in X\}$$

with the norm

$$\|\Phi\| = \sup_{t\in R} |\Phi(t)|,$$

X being a Banach space with the norm $|\cdot|$.

In the following, we need the space:

$AC_0 = \{\Phi \in C \mid \Phi$ *is absolutely continuous and* $\Phi' \in C_-\}$. In [5] we have also proved that, in the hypotheses:

(1.2) $A : \mathcal{D}(A) \subset X \to X$ *is the infinitesimal generator of a* (C_0)*-semigroup* $\{T(t),\ t \geqslant 0\} \subset L(X)$;

$$\{B_j\}_{j=1}^{\infty} \subset L(X),\ \textit{range } B_j \subset \mathcal{D}(A),\ AB_j \in L(X), \quad \forall j \in N\,; \tag{1.3}$$

$$\sum_{j=1}^{\infty} \|B_j\| < +\infty\,; \tag{1.4}$$

$$\sum_{j=1}^{\infty} \|AB_j\| < +\infty\,; \tag{1.5}$$

$$\{A_j\}_{j=1}^{\infty} \subset L(X), \quad \sum_{j=1}^{\infty} \|A_j\| < +\infty\,; \tag{1.6}$$

(1.7) $\{h_j\}_{j=1}^{\infty}$ *is a strict increasing sequence of real positive numbers, every solution of the problem (1.1a) – (1.1b) satisfies the integral equation*

$$x(t,0,\Phi)=\sum_{j=1}^{\infty}B_jx(t-h_j,0,\Phi)+T(t)\left[\Phi(0)-\sum_{j=1}^{\infty}B_j\Phi(-h_j)\right]+$$

(1.8a)
$$+\int_0^t T(t-\sigma)\left[\sum_{j=1}^{\infty}(A_j-AB_j)x(\sigma-h_j,0,\Phi)\right]d\sigma.$$

and the initial condition

(1.8b)
$$x(t,0,\Phi)=\Phi(t),\quad \Phi\in C_-,\quad t\in(-\infty\,;0].$$

We remark that every solution $x(t)$ of the problem (1.8a) – (1.8b), for which there exists $Ax(t)$, is a solution of the problem (1.1a) – (1.1b).

In this paper we show that if $\{T(t)\,;t\geq 0\}$ is an analytic semigroup of negative type and $\Phi\in AC_0$ or $\Phi\in C_-$ then, in certain conditions on $\{A_j\}_{j=1}^{\infty}$ and $\{B_j\}_{j=1}^{\infty}$, the trivial solution of (1.8a) – (1.8b) is exponentially stable or uniformly stable respectively.

For the proof of the main results of this paper we need the following definitions and theorems.

Definition 1 ([5]). *Every continuous solution of the problem (1.8a) – (1.8b) is named a „weak" solution for the problem (1.1a) – (1.8b).*

Theorem 1.1. ([5]). *In the hypotheses (1.2) – (1.7), for each $\Phi\in C^1$ the system (1.8a) – (1.8b) has a unique solution which belongs to $C=C(R,X)=\{u\mid u:R\to X,\ u$ continuous$\}$.*

For the problem (1.8a) – (1.8b) we consider the following equation

(1.9)
$$\left[\left(I-\sum_{j=1}^{+\infty}B_je^{-sh_j}\right)-(sI-A)^{-1}\sum_{j=1}^{+\infty}(A_j+AB_j)\,e^{-sh_j}\right]\bar{S}(s)=$$
$$=(sI-A)^{-1},\ s\in\mathcal{C},\ \operatorname{Re}s>\rho.$$

In [5] we have also proved that the hypotheses of Theorem 1.1 assure the existence of the function $S(t)$, $t\in R$ such that

(1.10)
$$S(t)=\begin{cases}\mathcal{L}^{-1}(\bar{S})\,(t), & t\geq 0\\ 0, & t<0,\end{cases}$$

(1.11)
$$\|S(t)\|\leq M\,e^{at},\qquad t\geq 0,$$

where $\mathcal{L}^{-1}(\bar{S})\,(t)$ is the inverse of the Laplace transform and $\|\,.\|$ means the norm in $L(X)$.

Theorem 1.2 ([5]). *If $\Phi\in AC_0$, then the solution of the problem (1.8a) (1.8b) may be written as*

$$x(t,0,\Phi)=\left[S(t)-\sum_{j=1}^{+\infty}S(t-h_j)B_j\right]\Phi(0)+$$

(1.12)
$$+\sum_{j=1}^{+\infty}\int_{-h_j}^{0}S(t-\sigma-h_j)\,[A_j\Phi(\sigma)+B_j\Phi'(\sigma)]\,d\sigma,\quad t\geq 0,$$
$$x(t,0,\Phi)=\Phi(t),\quad t<0.$$

Definition 1.2 ([5]). *For $j=1,2,\ldots$ we define the map*

$$R_j:[0\,;+\infty)\to[C_-,X]$$

as

(1.13)
$$R_j(t)\Phi=\mathcal{L}^{-1}\left[s\bar{S}(s)B_je^{-sh_j}\int_{-h_j}^{0}e^{-s\sigma}\Phi(\sigma)d\sigma\right](t),$$

where $[C_-,X]$ is the Banach space of linear bounded operators from C_- into X.

Theorem 1.3 ([5]). *If $\Phi\in C_-$, then the solution of the problem (1.8a) – (1.8b) can be written in the form*

$$x(t,0,\Phi)=S(t)\left[\Phi(0)-\sum_{j=1}^{+\infty}B_j\Phi(-h_j)\right]+$$

(1.14)
$$+\sum_{j=1}^{+\infty}\int_{-h_j}^{0}S(t-\sigma-h_j)\,A_j\Phi(\sigma)\,d\sigma+\sum_{j=1}^{+\infty}R_j(t)\Phi,\ t\geq 0.$$
$$x(t,0,\Phi)=\Phi(t),\quad t<0.$$

2. **The main result.** We consider the problem (1.1a) – (1.1b) and we suppose that the following conditions are fulfiled:

(2.1)
$$A:\mathcal{D}(A)\subset X\to X$$

is the infinitesimal generator of an analytic semigroup of negative type $\{T(t),t\geq 0\}\subset L(X)$; (therefore there exists the constants $M_1\geq 1$ and $\omega>0$ such that $\|T(t)\|\leq M_1e^{-\omega t}$, $t\in R_+$);

(2.2)
$$\{A_j\}_{j=1}^{+\infty}\subset L(X),\ \ \exists\delta>0\ \textit{such that}\ \sum_{j=1}^{+\infty}\|A_j\|\,e^{\delta h_j}<+\infty\,;$$

(2.3)
$$\{B_j\}_{j=1}^{+\infty}\subset L(X),\ \text{range}\ B_j\subset\mathcal{D}(A),\ \ \forall j\in N,\ \ \exists\delta>0$$

such that

$$\sum_{j=1}^{+\infty}\|B_j\|\,e^{\delta h_j}<1\,;$$

(2.4)
$$\{AB_j\}_{j=1}^{+\infty}\subset L(X),\ \ \exists\delta>0\ \text{such that}\ \sum_{j=1}^{+\infty}\|AB_j\|\,e^{\delta h_j}<+\infty\,;$$

(2.5)
$$\{h_j\}_{j=1}^{+\infty}\ \text{satisfies the condition (1.7)}.$$

It is obvious that (2.1) – (2.5) imply the validity of the condition (1.2) – (1.7).

Then, the problem (1.1a) – (1.1b) can be reduced to a problem of the form (1.8a) – (1.8b) for which we have considered the equation (1.9).

With the notations

(2.6)
$$B(s)=\left(I-\sum_{j=1}^{+\infty}B_je^{-sh_j}\right),$$

(2.7) $$\mathcal{a}(s)=(sI-A)^{-1}\sum_{j=1}^{+\infty}(A_j+AB_j)\,e^{-sh_j},$$

the equation (1.9) becomes

(2.8) $$[B(s)-\mathcal{a}(s)]\,\tilde{S}(s)=(sI-A)^{-1},\ \operatorname{Re}s>-\omega_1,$$

where

$$\omega_1=\min\{\delta,\omega\}.$$

Let

(2.9) $$\omega_0\overset{\text{def}}{=}\sup\{\operatorname{Re}s\mid\ \not\exists[I-\mathcal{a}(s)\,B^{-1}(s)]^{-1}\}.$$

In [5] we have proved that $\omega_0<+\infty$.

Because in the following, only the sign of ω_0 is important, we shall consider that $\omega_0\geqslant-\omega_1$. Then, taking into account (2.9), equation (2.8) becomes

(2.10) $$\tilde{S}(s)=B^{-1}(s)[I-\mathcal{a}(s)B^{-1}(s)]^{-1}(sI-A)^{-1},\ \operatorname{Re}s>\omega_0.$$

Using Theorem 1.1, relation (1.10) and some results from [5], we can write

(2.11) $$S(t)=\int_{(c)}e^{st}B^{-1}(s)\ [I-\mathcal{a}(s)B^{-1}(s)]^{-1}(sI-A)^{-1}ds,$$

where

$$\int_{(c)}=\lim_{r\to+\infty}\frac{1}{2\pi i}\int_{c-ir}^{c+ir}\qquad\text{for } c \text{ fixed large enough.}$$

For the proof of the main resu'ts, we shall need the following two lemmas.

Lemma 2.1. *In the hypotheses (2.1)–(2.5), the equality*

(2.12) $$\int_{(c)}e^{st}B^{-1}(s)\ [I-\mathcal{a}(s)\ B^{-1}(s)]^{-1}(sI-A)^{-1}ds=$$
$$=\int_{(\alpha)}e^{st}B^{-1}(s)\ [I-\mathcal{a}(s)\ B^{-1}(s)]^{-1}(sI-A)^{-1}ds,\quad \forall\alpha\in R\ \textit{with}\ \omega_0<\alpha\leqslant c,$$

holds.

Proof. Denoting

(2.13) $$H(s)=e^{st}B^{-1}(s)[I-\mathcal{a}(s)B^{-1}(s)]^{-1}(sI-A)^{-1}$$

and integrating $H(s)$ on $L_1M_1L_2M_2$, where

$$L_1=\{c+i\tau\mid-\nu\leqslant\tau\leqslant\nu\},\qquad L_2=\{\alpha+i\tau\mid-\nu\leqslant\tau\leqslant\nu\},$$
$$M_1=\{\sigma+i\nu\mid\alpha\leqslant\sigma\leqslant c\},\qquad M_2=\{\sigma-i\nu\mid\alpha\leqslant\sigma\leqslant c\}$$

we obtain

(2.14) $$\int_{L_1M_1L_2M_2}H(s)ds=0$$

(because $H(s)$ is analytic in the domain $L_1M_1L_2M_2$).

If we prove that $\int_{M_1}H(s)ds$ and $\int_{M_2}H(s)ds$ tend to 0 for $\nu\to+\infty$, then the proof of the Lemma 2.1 is completed.

From relation (2.3), it results that

$$\left\|\sum_{j=1}^{+\infty}B_j\,e^{-sh_j}\right\|\leqslant\sum_{j=1}^{+\infty}\|B_j\|\,e^{\delta h_j}<1,\quad\forall s\in\mathcal{C}\ \text{with}\ \operatorname{Re}s>\omega_0\geqslant-\omega_1.$$

Therefore, there exists $B^{-1}(s)$ and, moreover the inequality

(2.15) $$\|B^{-1}(s)\|\leqslant K,\quad\operatorname{Re}s>\omega_0\quad(\operatorname{Re}s\geqslant\alpha>\omega_0)$$

is satisfied.

Because $T(t)$ is an analytic semigroup, it results that ([1])

(2.16) $$\|(sI-A)^{-1}\|\leqslant\frac{M_1}{|s+\omega|},\quad\operatorname{Re}s>-\omega.$$

From (2.2)–(2.4) and (2.15)–(2.16), it results that

(2.17) $$\left\|(sI-A)^{-1}\sum_{j=1}^{+\infty}(A_j+AB_j)\,e^{-sh_j}B^{-1}(s)\right\|\leqslant$$
$$\leqslant\left\|(sI-A)^{-1}\right\|\cdot\left\|\sum_{j=1}^{+\infty}(A_j+AB_j)\,e^{-sh_j}\right\|\cdot\left\|B^{-1}(s)\right\|\leqslant\frac{M_1}{|s+\omega|}\cdot K\cdot K_1,\ \operatorname{Re}s>\omega_0.$$

Taking into account (2.7) and (2.17), we obtain

(2.18) $$\|\mathcal{a}(s)\ B^{-1}(s)\|\leqslant d<1,\ \text{for}\ \nu=\operatorname{Im}s\geqslant\nu_0,$$

(for a ν_0 large enough) and $\operatorname{Re}s\geqslant\alpha>\omega_0$.

From (2.18) it results that for $\nu\geqslant\nu_0$ there exists $[I-\mathcal{a}(s)B^{-1}(s)]^{-1}$ and moreover, the inequality

(2.19) $$[I-\mathcal{a}(s)B^{-1}(s)]^{-1}\leqslant K_2,\ \operatorname{Re}s\geqslant\alpha,\ \operatorname{Im}s=\nu\geqslant\nu_0$$

holds.

Using inequalities (2.15), (2.16) and (2.19), from (2.13) we obtain

(2.20) $$\|H(s)\|\leqslant e^{t\operatorname{Re}s}\left[K\cdot K_2\cdot\frac{M_1}{|s+\omega|}\right]\leqslant\frac{e^{ct}KK_2M_1}{\sqrt{(\alpha+\omega)^2+\nu^2}}\quad\forall s\in\mathcal{C}$$

which $\operatorname{Re}s\geqslant\alpha>\omega_0$, $\operatorname{Im}s=\nu\geqslant\nu_0$, where from it results easily that

(2.21) $$\left\|\int_{M_1}H(s)ds\right\|\leqslant\frac{e^{ct}KK_2M_1}{\sqrt{(\alpha+\omega)^2+\nu^2}}\,(c-\alpha)\to0,\ \text{for}\ \nu\to+\infty.$$

In the same way, we obtain

$$\left\| \int_{M_1} H(s)ds \right\| \to 0, \quad \text{for } \nu \to +\infty.$$

Lemma 2.2. *In the hypotheses* (2.1) – (2.5), *for any* $\alpha > \omega_0$, $S(t)$ *satisfies the estimate*

$$(2.22) \qquad \| S(t) \| = \left\| \int_{(\alpha)} e^{st} B^{-1}(s)[I - \mathcal{A}(s)B^{-1}(s)]^{-1}(sI - A)^{-1} ds \right\| \leqslant K_3 e^{\alpha t}, \ t \geqslant 0.$$

Proof. Let us remark that we can write

$$(2.23) \qquad \begin{aligned} B^{-1}(s)[I - \mathcal{A}(s)B^{-1}(s)]^{-1}(sI - A)^{-1} &= B^{-1}(s)(sI - A)^{-1} + \\ &+ B^{-1}(s)[I - \mathcal{A}(s)B^{-1}(s)]^{-1}[\mathcal{A}(s)B^{-1}(s)](sI - A)^{-1}, \end{aligned}$$

for $\operatorname{Re} s \geqslant \alpha > \omega_0$.

Using the relations (2.15) – (2.17) and (2.19), we obtain

$$(2.24) \qquad \begin{aligned} &\| B^{-1}(s)[I - \mathcal{A}(s)B^{-1}(s)]^{-1}[\mathcal{A}(s)B^{-1}(s)](sI - A)^{-1} \| \leqslant \\ &KK_2 \cdot \frac{KK_1M_1}{|s + \omega|} \cdot \frac{M_1}{|s + \omega|} = \frac{K^2K_1K_2M_1^2}{|s + \omega|^2}, \end{aligned}$$

for $\operatorname{Re} s \geqslant \alpha > \omega_0$ and $\operatorname{Im} s \geqslant \nu_0$. Therefore

$$\int_{(\alpha)} e^{st} B^{-1}(s)[I - \mathcal{A}(s)B^{-1}(s)]^{-1}[\mathcal{A}(s)B^{-1}(s)](sI - A)^{-1} ds$$

is absolutely convergent.

Using hypothesis (2.3) and the arguments from [5], it results that

$$(2.25) \qquad B^{-1}(s) = \left(I - \sum_{j=1}^{+\infty} B_j e^{-sh_j}\right)^{-1} = I + \sum_{k=1}^{+\infty}\left(\sum_{j=1}^{+\infty} B_j e^{-sh_j}\right)^k = I + \sum_{k=1}^{+\infty} e^{-\omega_k s} H_k$$

where

$$\omega_k = \sum_{i=1}^{+\infty} n_i^{(k)} h_i \qquad n_i^{(k)} \in N \cup \{0\}, i = \overline{1, \infty}$$

(a formal series because only a finite number of $n_i^{(k)}$ are positive) and, moreover, we have

$$(2.26) \qquad \sum_{k=1}^{+\infty} e^{\delta\omega_k} \| H_k \| < +\infty.$$

From relations (2.25) and (2.26) and the properties of the Laplace transform, it results

$$(2.27) \qquad \begin{aligned} &\left\| \int_{(\alpha)} e^{st} B^{-1}(s)(sI - A)^{-1} ds \right\| \leqslant \sum_{k=1}^{+\infty} \| H_k \| \cdot \| T(t - \omega_k) \| + \\ &+ \| T(t) \| \leqslant \sum_{k=1}^{+\infty} \| H_k \| M_1 e^{\alpha(t - \omega_k)} + M_1 e^{\alpha t} \leqslant M_1 e^{\alpha t}\left(1 + \sum_{k=1}^{+\infty} e^{-\alpha\omega_k} \| H_k \|\right). \end{aligned}$$

if $\alpha > \omega_0 \geqslant -\omega_1 = \max\{-\omega, -\delta\}$.

Therefore

$$(2.28) \qquad \left\| \int_{(\alpha)} e^{st} B^{-1}(s)(sI - A)^{-1} ds \right\| \leqslant K_4 e^{\alpha t}, \quad t \geqslant 0.$$

Using relations (2.23) – (2.24) and (2.28), one can easily obtain the following inequality

$$\begin{aligned} &\left\| \int_{(\alpha)} e^{st} B^{-1}(s)[I - \mathcal{A}(s)B^{-1}(s)]^{-1}(sI - A)^{-1} ds \right\| \leqslant \left\| \int_{(\alpha)} e^{st} B^{-1}(s)(sI - A)^{-1} ds \right\| + \\ &+ \left\| \int_{(\alpha)} e^{st} B^{-1}(s)[I - \mathcal{A}(s)B^{-1}(s)]^{-1}[\mathcal{A}(s)B^{-1}(s)](sI - A)^{-1} ds \right\| \leqslant \\ &\leqslant K_4 e^{\alpha t} + K_5 e^{\alpha t} = K_3 e^{\alpha t}, \quad t \geqslant 0, \end{aligned}$$

where

$$K_5 = \int_{(\alpha)} \frac{K^2K_1K_2M_1^2}{|s + \omega|^2} ds, \quad K_3 = K_4 + K_5.$$

Remark 2.1. In the hypotheses (2.1) – (2.5), $S(t)$, has an exponential growth of a negative type if and only if $\omega_0 < 0$.

Theorem 2.1. *In the hypotheses* (2.1) – (2.5), *if* $\Phi \in AC_0$ *and* $\omega_0 < 0$, *then the trivial solution of the problem* (1.8a) – (1.8b) *is exponentially stable.*

Proof. In [5] we have proved that, on the basis of the hypotheses of the Theorem 1.2, which in our case are fulfiled, the solution of the problem (1.8a) – (1.8b) can be written as

$$(2.29) \qquad \begin{aligned} x(t, 0, \Phi) &= \left[S(t) - \sum_{j=1}^{+\infty} S(t - h_j) B_j\right] \Phi(0) + \\ &+ \sum_{j=1}^{+\infty} \int_{-h_j}^{0} S(t - \sigma - h_j)[A_j\Phi(\sigma) + B_j \Phi'(\sigma)] d\sigma, \ t \geqslant 0, \end{aligned}$$

where from using Lemma 2.2, it results that

$$\begin{aligned} &| x(t, 0, \Phi) | \leqslant K_3 e^{-\beta t}\left(1 + \sum_{k=1}^{+\infty} \| B_j \| e^{\beta h_j}\right) \| \Phi \|_{AC_0} + \\ &+ K_3 \sum_{j=1}^{+\infty} \int_{-h_j}^{0} e^{-\beta t} e^{\beta\sigma}[e^{\beta h_j} \| A_j \| \cdot | \Phi(\sigma) | + e^{\beta h_j} \| B_j \| \cdot | \Phi'(\sigma) |] \, d\sigma, t \geqslant 0. \end{aligned}$$

with $0 > -\beta > \omega_0 \geqslant -\omega_1 = \max\{-\delta, -\omega\}$.

Therefore

$$|x(t,0,\Phi)| \leqslant K_6 e^{-\beta t}\|\Phi\|_{AC_0} + $$
$$+K_2\|\Phi\|_{AC_0}\int_{-\infty}^{0} e^{-\beta t} e^{\beta\sigma}\left[\sum_{j=1}^{+\infty} e^{\beta h_j}(\|A_j\|+\|B_j\|)\right]d\sigma =$$
$$=K_6 e^{-\beta t}\|\Phi\|_{AC_0} + e^{-\beta t}\|\Phi\|_{AC_0}K_7 = K_8 e^{-\beta t}\|\Phi\|_{AC_0}, \quad t \geqslant 0.$$

Theorem 2.2. *In the hypotheses (2.1)–(2.5), if $\Phi \in C_-$ and $\omega_0 < 0$, then the trivial solution of the problem (1.8a)–(1.8b) is uniformly stable.*

Proof. Taking into account Theorem 1.3, for $\Phi \in C_-$, the solution of the problem (1.8a)–(1.8b) can be written as

$$x(t,0,\Phi) = S(t)\left[\Phi(0) - \sum_{j=1}^{+\infty} B_j\Phi(-h_j)\right] +$$
$$(2.30) \qquad + \sum_{j=1}^{+\infty}\int_{-h_j}^{0} S(t-\sigma-h_j)A_j\Phi(\sigma)d\sigma + \sum_{j=1}^{+\infty} R_j(t)\Phi, \quad t \geqslant 0.$$

Denoting

$$(2.31) \qquad Q(t) = \mathcal{L}^{-1}\{B^{-1}(s)\,[(I-\mathcal{A}(s)\,B^{-1}(s))^{-1} - I]\}(t), \quad t \geqslant 0,$$

we obtain without difficulty relations ([5]) :

$$(2.32) \qquad \frac{d}{dt}[S(t)B_j] = S(t)AB_j + Q(t)B_j, \ t \geqslant 0.$$

$$R_j(t)\Phi = \int_{-h_j}^{0}\left(\frac{d}{dt}S(t-\sigma-h_j)B_j\right)\Phi(\sigma)\,d\sigma + B_j\Phi(t-h_j)(1-\mathcal{H}(t-h_j)) +$$
$$(2.33) \qquad + \sum_{k=1}^{+\infty}\mathcal{H}(t-\omega_k)(1-\mathcal{H}(t-h_j-\omega_k))H_kB_j\Phi(t-h_j-\omega_k), \quad t \geqslant 0,$$

where

$$\mathcal{H}(t) = \begin{cases} 1, & t \geqslant 0, \\ 0, & t < 0. \end{cases}$$

Therefore

$$R_j(t)\Phi = \int_{-h_j}^{0} S(t-\sigma-h_j)AB_j\Phi(\sigma)\,d\sigma + \int_{-h_j}^{0} Q(t-\sigma-h_j)B_j\Phi(\sigma)\,d\sigma +$$
$$+B_j\Phi(t-h_j)(1-\mathcal{H}(t-h_j)) +$$
$$\sum_{k=1}^{+\infty}\mathcal{H}(t-\omega_k)\,(1-\mathcal{H}(t-h_j-\omega_k))H_kB_k\Phi(t-h_j-\omega_k), \quad \text{for } t \geqslant 0.$$

Then, according to Lemma 2.2, for $0 > -\beta > \omega_0$, we have

$$\|R_j(t)\Phi\| \leqslant \|\Phi\|\cdot e^{-\beta t}\cdot K_8\cdot\|AB_j\|e^{\beta h_j}\cdot\frac{1}{\alpha}(1-e^{-\beta h_j}) + \|B_j\|\cdot\|\Phi\| +$$
$$(2.34) \qquad + \left(\sum_{k=1}^{+\infty}\|H_k\|\right)\|\Phi\|\cdot\|B_j\| + \int_{-h}^{0}\|Q(t-\sigma-h_j)\|\cdot\|B_j\|\cdot|\Phi(\sigma)|\,d\sigma, \quad t \geqslant 0.$$

From (2.31) it results

$$(2.35) \qquad Q(t) = \int_{(c)} e^{st}\{B^{-1}(s)[(I-\mathcal{A}(s)B^{-1}(s))^{-1} - I]\}\,ds\ ,$$

for c large enough.

In the same way, as in Lemma 2.1 we shall prove that for $\alpha > \omega_0$ we have

$$\int_{(c)} e^{st}\{B^{-1}(s)[(I-\mathcal{A}(s)\ B^{-1}(s))^{-1} - I]\}\,ds =$$
$$(2.36) \qquad = \int_{(\alpha)} e^{st}\{B^{-1}(s)[(I-\mathcal{A}(s)B^{-1}(s))^{-1} - I]\}\,ds.$$

We consider the domain $L_1M_1L_2M_2$, in which the map

$$(2.37) \qquad U(s) = e^{st}\{B^{-1}(s)[(I-\mathcal{A}(s)B^{-1}(s))^{-1} - I]\}$$

is analytic and therefore

$$\int_{L_1M_1L_2M_2} U(s)\,ds = 0.$$

Because

$$(2.38) \qquad B^{-1}(s)[(I-\mathcal{A}(s)B^{-1}(s))^{-1} - I] = B^{-1}(s)\mathcal{A}(s)B^{-1}(s)(I-\mathcal{A}(s)B^{-1}(s))^{-1}$$

for $\mathrm{Re}\, s \geqslant \alpha$, using (2.15), (2.17) and (2.19), from (2.38) we obtain

$$(2.39) \qquad \|B^{-1}(s)[(I-\mathcal{A}(s)B^{-1}(s))^{-1} - I]\| \leqslant \frac{K^2K_1K_2M_1}{|s+\omega|},$$

for $\mathrm{Re}\ s \geqslant \alpha$ and $\mathrm{Im}\ s \geqslant \nu_0$.

Therefore

$$(2.40) \qquad \|U(s)\| \leqslant \frac{K^2K_1K_2M_1}{|s+\omega|}e^{t} \quad \mathrm{Re}\, s \geqslant \alpha\,,\ \mathrm{Im}\, s \geqslant \nu_0,$$

where from we obtain

$$\left\|\int_{M_1} U(s)ds\right\| \leqslant \frac{e^{ct}K^2K_1K_2M_1}{\sqrt{(\alpha+\omega)^2+\nu^2}}(c-\alpha)$$

which tends to zero as $\nu \to +\infty$.

Ín the same way we obtain

$$\left\| \int_{M_1} U(s)ds \right\| \text{ tends to zero as } \nu \to +\infty.$$

Ín the following, we shall prove that, in hypotheses (2.1) – (2.5), for $\alpha > \omega_0$, we have

$$\left\| \int_{(\alpha)} U(s)\, ds \right\| \leqslant k_1 e^{\alpha}, \ t \geqslant 0. \tag{2.41}$$

Because

$$\begin{aligned} B^{-1}(s)[(I - \mathcal{A}(s)B^{-1}(s))^{-1} - I] &= B^{-1}(s)\mathcal{A}(s)B^{-1}(s) + \\ &+ B^{-1}(s)(\mathcal{A}(s)B^{-1}(s))^2(I - \mathcal{A}(s)B^{-1}(s))^{-1}, \end{aligned} \tag{2.42}$$

using once more relations (2.15), (2.17) and (2.19) we obtain

$$\| B^{-1}(s)(\mathcal{A}(s)B^{-1}(s))^2(I - \mathcal{A}(s)B^{-1}(s))^{-1} \| \leqslant \frac{K^3K_1^2K^2M_1^2}{|s+\omega|^2}, \tag{2.43}$$

$$\mathrm{Re}\, s \geqslant \alpha, \ \mathrm{Im}\, s \geqslant \nu_0,$$

and, therefore

$$\left\| \int_{(\alpha)} e^{st} B^{-1}(s)\mathcal{A}(s)B^{-1}(s))^2(I - \mathcal{A}(s)B^{-1}(s))^{-1}ds \right\| \leqslant k_2 e^{\alpha}, \ \text{for} \quad t \geqslant 0. \tag{2.44}$$

Ín the same way as in [5], using relations (2.2) – (2.4), one can easily obtain :

$$\mathcal{L}^{-1}[B^{-1}(s)\, \mathcal{A}(s) B^{-1}(s)](t) = \sum_{k=1}^{+\infty} \mathcal{H}(t - \omega_k)\hat{H}_k^1 T(t - \omega_k)\hat{H}_k^2, \tag{2.45}$$

$$\sum_{k=1}^{+\infty} e^{\delta\omega_k} \| \hat{H}_k^1 \| \cdot \| \hat{H}_k^2 \| < +\infty. \tag{2.46}$$

Using hypothesis (2.1) and relation (2.46) we obtain

$$\left\| \| \mathcal{L}^{-1}[B_{(\alpha)}^{-1}\, \mathcal{A}(s)B^{-1}(s)]\,(t) \right\| \leqslant e^{-\delta t} \sum_{k=1}^{+\infty} e^{\delta\omega_k} \| \hat{H}_k^1 \| \cdot \| \hat{H}_k^2 \| \leqslant k_3 e^{\alpha t}, \quad \text{for } t \geqslant 0. \tag{2.47}$$

Then, from (2.42), (2.44) and (2.47), it results that

$$\left\| \int_{(\alpha)} e^{st} B^{-1}(s)[(I - \mathcal{A}(s)B^{-1}(s))^{-1} - I]\, ds \right\| \leqslant (k_2 + k_3)\, e^{\alpha t}, \quad t \geqslant 0. \tag{2.48}$$

Using (2.36) and (2.41) with $0 > -\beta > \omega_0$, we obtain

$$\| Q(t) \| \leqslant k_1 e^{-\beta t}, \quad t \geqslant 0. \tag{2.49}$$

From (2.34), on the basis of the relation (2.49), we obtain

$$\| R_k(t)\Phi \| \leqslant m_1 \| AB_j \| e^{\beta h_j} e^{-\beta t} + \| B_j \| \, \| \Phi \| \, m_2 + \frac{1}{\beta} \cdot e^{-\beta t} \| \Phi \| e^{\beta h_j} \| B_j \|, \tag{2.50}$$

with

$$m_1 = \frac{K_3}{\beta}, \quad m_2 = 1 + \sum_{k=1}^{+\infty} \| H_k \|.$$

Because $0 > -\beta > \omega_0 \geqslant -\omega_1 = \max\{-\omega, -\delta\}$, using (2.3), (2.4) and (2.50) it results that

$$\sum_{j=1}^{+\infty} \| R_j(t)\Phi \| \leqslant (m_3 e^{-\beta t} + m_4) \| \Phi \|. \tag{2.15}$$

with

$$m_3 = m_1 \sum_{j=1}^{+\infty} \| AB_j \| e^{\beta h_j} + \frac{1}{\beta} \sum_{j=1}^{+\infty} e^{\beta h_j} \| B_j \|, \quad m_4 = m_2 \sum_{j=1}^{+\infty} \| B_j \|.$$

Using the relation (2.51) and Lemma 2.2, from the relation (2.30) it results that

$$| x(t, 0, \Phi) | \leqslant m_5 \cdot e^{-\beta t} \| \Phi \| + e^{-\beta t} \| \Phi \| m_6 + (m_3 e^{-\beta t} + m_4) \| \Phi \|, \quad t \geqslant 0,$$

or, in other words,

$$| x(t, 0, \Phi) | \leqslant (m_5 + m_6 + m_3)\, e^{-\beta t} \| \Phi \| + m_4 \| \Phi \|, \quad t \geqslant 0. \tag{2.52}$$

with

$$m_5 = \left(1 + \sum_{j=1}^{+\infty} \| B_j \| \right) K_3, \quad m_6 = \frac{K_3}{\beta} \sum_{j=1}^{+\infty} e^{\beta h_j} \| A_j \|.$$

Corollary 2.1. *If $B_j = 0$, $\forall j \in N$, from (2.52), it results that*

$$| x(t, 0, \Phi) | \leqslant (m_3 + K_3)\, e^{-\beta t} \| \Phi \|, \tag{2.53}$$

because, in this case, $R_j(\cdot) \equiv 0$, $\forall_j \in N$. Therefore, in this hypotheses (2.1) – (2.5), if $B_j = 0$, $\forall_j \in N$ and $\omega_0 < 0$, it results that the trivial solution of the problem (1.8a) – (1.8b) is exponentially stable.

Examples. Let us consider the Cauchy problem

$$\frac{\partial}{\partial t} u(t, x) = \Delta u(t, x) - \alpha u(t, x) + \sum_{j=1}^{+\infty} A_j u(t - h_j, x), \quad \alpha > 0, \quad t \geqslant 0, \tag{2.53}$$

$$u(t,.) = \Phi\,(t,.) \in L^2(\Omega), \ t \leqslant 0, \Omega \subset R^n,$$

$$u(t,.) \in L^2(\Omega).\ \Delta\,; H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega) \to L^2(\Omega),\ t \geqslant 0,$$

where Ω is a bounded domain the boundary of which is smooth enough.

For this problem $A = \Delta - \alpha I$ and (see [1])

$$\| (\lambda I - A)^{-1} \| = \| ((\lambda + \alpha)\, I - \Delta)^{-1} \| \leqslant \frac{1}{|\lambda + \alpha|}, \quad \mathrm{Re}\, \lambda > -\alpha. \tag{2.54}$$

(Therefore A is the infinitesimal generator of an analytic semigroup of type $-\alpha < 0$).

Let $A_j : L^2(\Omega) \to L^2(\Omega)$, $j = \overline{1, \infty}$, $\{A_j\}_{j=1}^{+\infty}$ is a sequence of linear bounded operators such that

(2.55) $$\frac{1}{\alpha - \delta} \sum_{j=1}^{+\infty} e^{\delta h_j} \| A_j \| < 1, \text{ for a } \delta \text{ such that } \alpha > \delta > 0.$$

In these hypotheses, the conditions (2.1) – (2.4) are satisfied with

$$X = L^2(\Omega), \mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega) \text{ and } B_j = 0, \ j = \overline{1, +\infty}.$$

The relations (2.6), 2.7) become

(2.56) $$B^{-1}(s) = I,$$

(2.57) $$\mathcal{A}(s) = [((s+\alpha)I - \Delta)^{-1}] \sum_{j=1}^{+\infty} e^{-sh_j} A_j.$$

We define

(2.58) $$\omega_0 = \sup \{\operatorname{Re} s \mid \exists (I - \mathcal{A}(s))^{-1}\}.$$

From relations (2.54), (2.55) and (2.57) it results that

(2.59) $$\omega_0 \leqslant -\delta < 0.$$

One can easily remark that the hypotheses of Corollary 2.1 are satisfiey and therefore the trivial solution of the problem (2.53) is exponentialld stable, for $\Phi \in C_-$.

The following type of operators A_j satisfies the above condition

$$(A_j f)(x) = \int_{(\Omega)} K_j(x, y) f(y)\, dy, \quad \forall f \in L^2(\Omega),$$

with

$$K_j(.,.) \in L^2(\Omega \times \Omega),$$

and

$$\sum_{j=1}^{+\infty} \left(\int\int_{(\Omega\times\Omega)} | K_j^2(x, y) | \, dx\, dy \right)^{\frac{1}{2}} \cdot e^{\delta h_j} < \alpha - \delta,$$

for δ such that $\alpha > \delta > 0$.

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ON CERTAIN INTEGRAL INEQUALITIES FOR PARTIAL DIFFERENTIAL AND INTEGRAL EQUATIONS

BY

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1. Introduction. In the theory of hyperbolic partial differential equations, once the question of existence and uniqueness has been disposed of, we turn to a more precise study of the analytic character of the solutions. It frequently happens that a method which works very efficiently to establish existence and uniqueness does not yield other properties of the solution in any ready fashion. As it seems to be a current trend to use some kinds of inequalities and variational principles involving functions and their derivatives to study the qualitative properties of solutions of partial differential equations, it is not surprising that the interest in partial differential and integral inequalities has been revived in the last few years (See, [1]–[9], [13]–[20]). The basic aim of the present paper is to establish a number of partial integral inequalities which can be used in some applications in the theory of hyperbolic partial differential and integral equations of the more general type.

2. **Main Results.** In this section we state and prove some useful partial integral inequalities which are the two independent variable generalizations of some of the integral inequalitiesr ecently established by Pachpatte [10, 11]. An elementary method used by Snow [17] to obtain a generalization of Gronwall's inequality in two independent variables will be used to establish our results.

A useful two independent variable generalization of the integral inequality recently established by Pachpatte [10, Theorem 1] is embodied in the following theorem.

Theorem 1. *Suppose $\Phi(x, y)$, $b(x, y)$, $c(x, y)$ and $p(x, y)$ are real-valued nonnegative continuous functions defined on a domain Dw hich lies in the first quadrant of (x, y) plane and let the function $a(x, y)$ be positive, continuous and nondecreasing in both the variables and defined on a domain D. Let $P_0(x_0, y_0)$ and $P(x,y)$ be two points in D such that $(x - x_0)(y - y_0) > 0$ and let $R \subset D$ be the rectangular region whose opposite corners are the points P_0 and P. Let $v(s, t\,; x, y)$ be the solution of the characteristic initial value problem*

(1) $$L[v] = v_{st} - b(s, t)\,[c(s, t) + p(s, t)]\, v = 0,\ v(s, y) = v(x, t) = 1,$$

and let D^+ be a connected subdomain of D which contains P and on which $v \geqslant 0$ (see, Fig. 1). *Then for $(x, y) \in D$ there exist $R \subset D^+$ and $v(s,t\,; x, y) > 0$ on R, and $\Phi(x, y)$ satisfies*