Let $A_j : L^2(\Omega) \to L^2(\Omega)$, $j = \overline{1, \infty}$, $\{A_j\}_{j=1}^{+\infty}$ is a sequence of linear bounded operators such that

$$\frac{1}{\alpha - \delta} \sum_{j=1}^{+\infty} e^{\delta h_j} \| A_j \| < 1, \text{ for a } \delta \text{ such that } \alpha > \delta > 0. \tag{2.55}$$

In these hypotheses, the conditions (2.1) – (2.4) are satisfied with

$$X = L^2(\Omega), \mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega) \text{ and } B_j = 0, \; j = \overline{1, +\infty}.$$

The relations (2.6), 2.7) become

$$B^{-1}(s) = I, \tag{2.56}$$

$$\mathcal{A}(s) = [((s + \alpha)I - \Delta)^{-1}] \sum_{j=1}^{+\infty} e^{-sh_j} A_j. \tag{2.57}$$

We define

$$\omega_0 = \sup \{\mathrm{Re}\, s \mid \exists (I - \mathcal{A}(s))^{-1}\}. \tag{2.58}$$

From relations (2.54), (2.55) and (2.57) it results that

$$\omega_0 \leqslant -\delta < 0. \tag{2.59}$$

One can easily remark that the hypotheses of Corollary 2.1 are satisfiey and therefore the trivial solution of the problem (2.53) is exponentialld stable, for $\Phi \in C_-$.

The following type of operators A_j satisfies the above condition

$$(A_j f)(x) = \int_{(\Omega)} K_j(x, y) f(y)\, dy, \quad \forall f \in L^2(\Omega),$$

with

$$K_j(.,.) \in L^2(\Omega \times \Omega),$$

and

$$\sum_{j=1}^{+\infty} \left(\iint_{(\Omega \times \Omega)} | K_j^2(x, y) | \, dx\, dy \right)^{\frac{1}{2}} \cdot e^{\delta h_j} < \alpha - \delta,$$

for δ such that $\alpha > \delta > 0$.

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ON CERTAIN INTEGRAL INEQUALITIES FOR PARTIAL DIFFERENTIAL AND INTEGRAL EQUATIONS

BY

B. G. PACHPATTE

1. Introduction. In the theory of hyperbolic partial differential equations, once the question of existence and uniqueness has been disposed of, we turn to a more precise study of the analytic character of the solutions. It frequently happens that a method which works very efficiently to establish existence and uniqueness does not yield other properties of the solution in any ready fashion. As it seems to be a current trend to use some kinds of inequalities and variational principles involving functions and their derivatives to study the qualitative properties of solutions of partial differential equations, it is not surprising that the interest in partial differential and integral inequalities has been revived in the last few years (See, [1] – [9], [13] – [20]). The basic aim of the present paper is to establish a number of partial integral inequalities which can be used in some applications in the theory of hyperbolic partial differential and integral equations of the more general type.

2. **Main Results.** In this section we state and prove some useful partial integral inequalities which are the two independent variable generalizations of some of the integral inequalitiesr ecently established by Pachpatte [10, 11]. An elementary method used by Snow [17] to obtain a generalization of Gronwall's inequality in two independent variables will be used to establish our results.

A useful two independent variable generalization of the integral inequality recently established by Pachpatte [10, Theorem 1] is embodied in the following theorem.

Theorem 1. *Suppose* $\Phi(x, y)$, $b(x, y)$, $c(x, y)$ *and* $p(x, y)$ *are real-valued nonnegative continuous functions defined on a domain Dw hich lies in the first quadrant of* (x, y) *plane and let the function* $a(x, y)$ *be positive, continuous and nondecreasing in both the variables and defined on a domain D. Let* $P_0 (x_0, y_0)$ *and* $P(x,y)$ *be two points in D such that* $(x - x_0)(y - y_0) > 0$ *and let* $R \subset D$ *be the rectangular region whose opposite corners are the points* P_0 *and* P. *Let* $v(s, t\,; x, y)$ *be the solution of the characteristic initial value problem*

$$L[v] = v_{st} - b(s, t)\, [c(s, t) + p(s, t)]\, v = 0, \; v(s, y) = v(x, t) = 1, \tag{1}$$

and let D^+ *be a connected subdomain of D which contains P and on which* $v \geqslant 0$ (see, Fig. 1). *Then for* $(x, y) \in D$ *there exist* $R \subset D^+$ *and* $v(s,t\,; x, y) > 0$ *on* R, *and* $\Phi(x, y)$ *satisfies*

$$\Phi(x,y) \leqslant a(x,y) + b(x,y)\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\Phi(s,t)dsdt + \right.$$

(2)

$$\left. + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)b(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t} p(\xi,\eta)\Phi(\xi,\eta)d\xi d\eta\right) dsdt\right),$$

and $\Phi(x,y)$ *satisfies also*

$$\Phi(x,y) \leqslant a(x,y)\left[1 + b(x,y)\left(\int_{x_0}^{x}\int_{y_0}^{y}\left\{c(s,t) + b(s,t)c(s,t)\right.\right.\right.$$

(3)

$$\left.\left.\left.\cdot\left(\int_{x_0}^{s}\int_{y_0}^{t}[c(\xi,\eta)+p(\xi,\eta)]v(\xi;\eta;s,t)\, d\xi d\eta\right)\right\} dsdt\right)\right].$$

The *proof* of this theorem is obtained by reducing the integral inequality to a differential inequality and then integrating it by Riemann's method for hyperbolic partial differential equations. The function $v(s,t;x,y)$ involved in theorem is a Riemann function relative to the point $P(x,y)$ for the selfadjoint operator L. The existence and continuity of the Riemann function is well known and may be proved by the method of successive approximations (see, [8]).There is such a function and a domain D^+on which $v \geqslant 0$ since $v=1$ on the vertical and horizontal lines through P and since v is continuous.

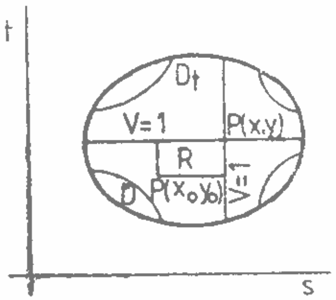

Figure 1

Proof. Since $a(x,y)$ is positive, nondecreasing, we observe from (2) that

$$\frac{\Phi(x,y)}{a(x,y)} \leqslant 1 + b(x,y)\left(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\frac{\Phi(s,t)}{a(s,t)} dsdt + \right.$$

(4)

$$\left. + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)b(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t} p(\xi,\eta)\frac{\Phi(\xi,\eta)}{a(\xi,\eta)} d\xi d\eta\right) dsdt\right).$$

Define a function $u(x,y)$ *such that*

$$u(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\frac{\Phi(s,t)}{a(s,t)} dsdt +$$

$$+ \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)b(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t} p(\xi,\eta)\frac{\Phi(\xi,\eta)}{a(\xi,\eta)} d\xi d\eta\right) dsdt, \quad u(x,y_0) = u(x_0,y) = 0;$$

then we have

$$u_{xy}(x,y) = c(x,y)\frac{\Phi(x,y)}{a(x,y)} + c(x,y)b(x,y)\left(\int_{x_0}^{x}\int_{y_0}^{y} p(\xi,\eta)\frac{\Phi(\xi,\eta)}{a(\xi,\eta)} d\xi d\eta\right),$$

which in view of (4) implies

$$u_{xy}(x,y) \leqslant c(x,y) + b(x,y)c(x,y)\left\{u(x,y) + \right.$$

(5)

$$\left. + \int_{x_0}^{x}\int_{y_0}^{y} p(\xi,\eta)[1 + b(\xi,\eta)u(\xi,\eta)]\, d\xi d\eta\right\}.$$

If we put

(6) $$r(x,y) = u(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} p(\xi,\eta)[1 + b(\xi,\eta)u(\xi,\eta)]\, d\xi d\eta,$$

$$r(x,y_0) = r(x_0,y) = 0,$$

then we obtain

(7) $$r_{xy}(x,y) = u_{xy}(x,y) + p(x,y)\ \ [1 + b(x,y)u(x,y)]\,.$$

Using (5) and the fact that $u(x,y) \leqslant r(x,y)$ from (6) in (7), we have

(8) $$L[r] = r_{xy}(x,y) - b(x,y)[c(x,y) + p(x,y)]r(x,y) \leqslant c(x,y) + p(x,y).$$

The operator L is self-adjoint and hyperbolic. For any twice continuously differentiable r and v, the operator L satisifies the identity

(9) $$vL[r] - rL[v] = -(rv_y)_x + (vr_x)_y.$$

Let P_0 and P be any points as in Theorem and label the directed sides and corners of the rectangle R as shown in Fig. 2.

Using s and t as the independent variables, we integrate the identity (9) over R and use Green's theorem to obtain

$$\iint_R (vL[r] - rL[v])\, dsdt = -\int_{c_1+c_2+c_3+c_4} (vr_s ds + rv_t dt) = -\int_{c_1+c_3} vr_s ds - \int_{c_2+c_4} r\, v_t\, dt.$$

This holds for any functions in $C^2(\overline{R})$.

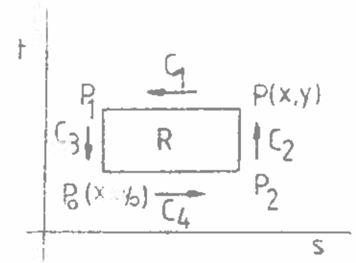

Figure 2

For the particular function r defined earlier we have $r=0$ on C_2 and $r=r_t=0$ on C_4; so the right hand side reduces to

$$(10)\qquad -\int_{c_1} vr_s ds - \int_{c_3} r\, v_t\, dt.$$

Now suppose v satisfies

$$(11)\qquad L[v]=v_{st}-b(s,t)\,[c(s,t)+p(s,t)]\,v=0,$$

$$(12)\qquad v=1 \text{ on } c_1,$$

$$(13)\qquad v_t=0 \text{ on } c_3.$$

Then it follows from (11)–(13) and $v\in C(\overline{R})$ that

$$(14)\qquad v=1 \text{ on } c_2.$$

Since $v\geqslant 0$ on R and $r(P_1)=0$, by using (8), (12), we obtain

$$r(x,y)\leqslant \int_{x_0}^{x}\int_{y_0}^{y}[c(s,t)+p(s,t)]\,v(s,t\,;x,y)\,dsdt.$$

Substituting this value of $r(x,y)$ in (5), we obtain

$$u_{xy}(x,y)\leqslant c(x,y)+b(x,y)c(x,y)\int_{x_0}^{x}\int_{y_0}^{y}[c(s,t)+p(s,t)]\,v(s,t\,;x,y)dsdt,$$

which implies the estimate for $u(x,y)$ such that

$$u(x,y)\leqslant \int_{x_0}^{x}\int_{y_0}^{y}\Big\{c(s,t)+b(s,t)c(s,t)\cdot$$
$$\cdot\int_{x_0}^{s}\int_{y_0}^{t}[c(\xi,\eta)+p(\xi,\eta)]v(\xi,\eta\,;s,t)\,d\xi d\eta\Big\}\,dsdt.$$

Now substituting this bound on $u(x,y)$ in (4) we obtain the desired bound in (3).

We next establish the following two independent variable generalization of the integral inequality recently established by this author in [11, Theorem 1] which can be used in some applications.

Theorem 2. *Suppose $\Phi(x,y)$, $b(x,y)$, $c(x,y)$ and $a(x,y)$ be as defined in Theorem 1. Let $P_0(x_0,y_0)$ and $P(x,y)$ be two points in a domain D such that $(x-x_0)(y-y_0)\geqslant 0$ and let $R\subset D$ be the rectangular region whose opposite corners are the points P_0 and P. Let $v(s,t\,;x,y)$ be the characteristic initial value problem*

$$(15)\qquad L[v]=v_{st}-b(s,t)c(s,t)\,v=0,\quad v(s,y)=v(x,t)=1,$$

and let D^+ be a connected subdomain of D which contains P and on which $v\geqslant 0$ (See, Fig. 1). *Then for $(x,y)\in D$ there exist $R\subset D^+$ and $v(s,t\,;x,y)>0$ on R and $\Phi(x,y)$ satisfies*

$$(16)\qquad \Phi(x,y)\leqslant a(x,y)+b(x,y)\int_{x_0}^{x}\int_{y_0}^{y}c(s,t)\Phi(s,t)\,dsdt;$$

$\Phi(x,y)$ satisfies also

$$(17)\qquad \Phi(x,y)\leqslant a(x,y)\Big[1+b(x,y)\int_{x_0}^{x}\int_{y_0}^{y}c(s,t)v(s,t\,;x,y)\,dsdt\Big].$$

The proof of this theorem follows by the similar argument as in the proof of Theorem 1. We omit the details.

We now apply Theorem 1 and 2 to establish the following useful and more general inequalities.

Theorem 3. *Let $\Phi(x,y)$, $b(x,y)$, $c(x,y)$, $p(x,y)$ and $q(x,y)$ be real-valued nonnegative continuous functions defined on a domain D as mentioned in Theorem 1. Let $W(r(x,y))$ be a positive, continuous, monotonic, nondecreasing and submultiplicative function for $r(x,y)>0$, $(x,y)\in D$ and $W_y(r(x,y))=\frac{\partial}{\partial y}W(r(x,y))\geqslant 0$ for $r(x,y)\geqslant 0$. Let $P_0(x_0,y_0)$ and $P(x,y)$ be as in Theorem 1. Let $v(s,t\,;x,y)$ be the solution of the characteristic initial value problem (1) and let D^+ be a connected subdomain of D which contains P and on which $v\geqslant 0$(see, fig. 1). Then for $(x,y)\in D$ there exist $R\subset D^+$ and $v(s,t\,;x,y)>0$ on R and $\Phi(x,y)$ satisfies*

$$\Phi(x,y)\leqslant M+b(x,y)\Big(\int_{x_0}^{x}\int_{y_0}^{y}c(s,t)\,\Phi(s,t)\,ds\,dt+$$

$$(18)\qquad +\int_{x_0}^{x}\int_{y_0}^{y}c(s,t)\,b(s,t)\Big(\int_{x_0}^{s}\int_{y_0}^{t}p(\xi,\eta)\Phi(\xi,\eta)\,d\xi d\eta\Big)\,ds\,dt\Big)+$$

$$+\int_{x_0}^{x}\int_{y_0}^{y}q(s,t)W(\Phi(s,t))dsdt,$$

where $M > 0$ *is a constant, then for* $0 \leqslant x \leqslant x_1$, $0 \leqslant y \leqslant y_1$; $\Phi(x, y)$ *satisfies also*

$$\Phi(x, y) \leqslant \Omega^{-1}\Bigg[\Omega(M) + \int_{x_0}^{x}\int_{y_0}^{y} q(s,t)\, W\Bigg((1 + b(s,t)\Bigg(\int_{x_0}^{s}\int_{y_0}^{t}\Big\{c(\xi,\eta) + b(\xi,\eta)c(\xi,\eta)\cdot$$

$$\tag{19}\cdot\Big(\int_{x_0}^{\xi}\int_{y_0}^{\eta}[c(m,n) + p(m,n)]v(m,n;\xi,\eta)\,dm\,dn\Big)\Big\}\,d\xi\,d\eta\Bigg)\Bigg)ds\,dt\Bigg] \times$$

$$\times\Bigg[1 + b(x,y)\Bigg(\int_{x_0}^{x}\int_{y_0}^{y}\Big\{c(s,t) + b(s,t)c(s,t)\cdot$$

$$\cdot\Big(\int_{x_0}^{s}\int_{y_0}^{t}[c(\xi,\eta) + p(\xi,\eta)]\,v(\xi,\eta;s,t)\,d\xi\,d\eta\Big)\Big\}ds\,dt\Bigg)\Bigg],$$

where

$$\tag{20}\Omega(r) = \int_{r_0}^{r}\frac{ds}{W(s)}, \quad r \geqslant r_0 > 0,$$

Ω^{-1} *is the inverse function of* Ω, *and*

$$\Omega(M) + \int_{x_0}^{x}\int_{y_0}^{y} q(s,t)W\Bigg((1 + b(s,t)\Bigg(\int_{x_0}^{s}\int_{y_0}^{t}\Big\{c(\xi,\eta) + b(\xi,\eta)c(\xi,\eta)\cdot$$

$$\cdot\Big(\int_{x_0}^{\xi}\int_{y_0}^{\eta}[c(m,n) + p(m,n)]\,v(m,n;\xi,\eta)\,dm\,dn\Big)\Big\}\,d\xi\,d\eta\Bigg)\Bigg)ds\,dt \in \operatorname{Dom}\Omega^{-1}\Bigg),$$

for all x, y *lying in the subintervals* $0 \leqslant x \leqslant x_1$, $0 \leqslant y \leqslant y_1$ *of real-numbers.*

Proof. Define

$$\tag{21} a(x,y) = M + \int_{x_0}^{x}\int_{y_0}^{y} q(s,t)W(\Phi(s,t))ds\,dt, \quad a(x,y_0) = a(x_0,y) = M,$$

then (18) can be restated as

$$\Phi(x,y) \leqslant a(x,y) + b(x,y)\Bigg(\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\Phi(s,t)\,ds\,dt +$$

$$+ \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)b(s,t)\Big(\int_{x_0}^{s}\int_{y_0}^{t} p(\xi,\eta)\Phi(\xi,\eta)\,d\xi\,d\eta\Big)ds\,dt\Bigg).$$

Since $a(x, y)$ is positive, nondecreasing, we have from Theorem 1

$$\Phi(x,y) \leqslant a(x,y)\Bigg[1 + b(x,y)\Bigg(\int_{x_0}^{x}\int_{y_0}^{y}\Big\{c(s,t) + b(s,t)c(s,t)\cdot$$

$$\tag{22}\cdot\Big(\int_{x_0}^{s}\int_{y_0}^{t}[c(\xi,\eta) + p(\xi,\eta)]\,v(\xi,\eta;s,t)\,d\xi\,d\eta\Big)\Big\}ds\,dt\Bigg)\Bigg].$$

Further

$$W(\Phi(x,y)) \leqslant W(a(x,y))\,W\Bigg(1 + b(x,y)\Bigg(\int_{x_0}^{x}\int_{y_0}^{y}\Big\{c(s,t) + b(s,t)c(s,t)\cdot$$

$$\cdot\Big(\int_{x_0}^{s}\int_{y_0}^{t}[c(\xi,\eta) + p(\xi,\eta)]\,v(\xi,\eta;s,t)\,d\xi\,d\eta\Big)\Big\}ds\,dt\Bigg)\Bigg),$$

since W is submultiplicative. Hence

$$\frac{q(x,y)W(\Phi(x,y))}{W(a(x,y))} \leqslant q(x,y)W\Bigg(1 + b(x,y)\Bigg(\int_{x_0}^{x}\int_{y_0}^{y}\Big\{c(s,t) + b(s,t)c(s,t)\cdot$$

$$\cdot\Big(\int_{x_0}^{s}\int_{y_0}^{t}[c(\xi,\eta) + p(\xi,\eta)]v(\xi,\eta;s,t)\,d\xi\,d\eta\Big)\Big\}ds\,dt\Bigg)\Bigg).$$

Because of (21), this reduces to

$$\frac{a_{xy}(x,y)}{W(a(x,y))} \leqslant q(x,y)W\Bigg(1 + b(x,y)\Bigg(\int_{x_0}^{x}\int_{y_0}^{y}\Big\{c(s,t) + b(s,t)c(s,t)\cdot$$

$$\tag{23}\cdot\Big(\int_{x_0}^{s}\int_{y_0}^{t}[c(\xi,\eta) + p(\xi,\eta)]v(\xi,\eta;s,t)\,d\xi\,d\eta\Big)\Big\}ds\,dt\Bigg)\Bigg).$$

From (23) we observe that

$$\frac{W(a(x,y))\,a_{xy}(x,y)}{W^2(a(x,y))} \leqslant q(x,y)W\Bigg(1 + b(x,y)\Bigg(\int_{x_0}^{x}\int_{y_0}^{y}\Big\{c(s,t) + b(s,t).c(s,t)\cdot$$

$$\cdot\Big(\int_{x_0}^{s}\int_{y_0}^{t}[c(\xi,\eta) + p(\xi,\eta)]v(\xi,\eta;s,t)\,d\xi\,d\eta\Big)\Big\}ds\,dt\Bigg)\Bigg) + \frac{a_x(x,y)W_y(a(x,y))}{W^2(a(x,y))}.$$

i.e.

$$\frac{\partial}{\partial y}\left(\frac{(a_x(x,y)}{W(a(x,y))}\right) \leqslant q(x,y)W\left(1+b(x,y)\left(\int_{x_0}^{x}\int_{y_0}^{y}\left\{c(s,t)+b(s,t)c(s,t)\cdot\right.\right.\right.$$

$$\left.\left.\left.\cdot\left(\int_{x_0}^{s}\int_{y_0}^{t}[c(\xi,\eta)+p(\xi,\eta)]v(\xi,\eta;s,t)d\xi d\eta\right)\right\}dsdt\right)\right).$$

Keeping x fixed in the above inequality, set $y=t$ and integrate with respect to t from y_0 to y to obtain

$$\frac{a_x(x,y)}{W(a(x,y))} \leqslant \int_{y_0}^{y} q(x,t)W\left(1+b(x,t)\left(\int_{x_0}^{x}\int_{y_0}^{t}\left\{c(s,\eta)+b(s,\eta)c(s,\eta)\cdot\right.\right.\right. \tag{24}$$

$$\left.\left.\left.\cdot\left(\int_{x_0}^{s}\int_{y_0}^{\eta}[c(\xi,n)+p(\xi,n)]v(\xi,n;s,\eta)d\xi dn\right)\right\}dsd\eta\right)\right)dt.$$

From (20) and (24) we observe that

$$\Omega_x(a(x,y)) \leqslant \int_{y_0}^{y} q(x,t)W\left(1+b(x,t)\left(\int_{x_0}^{x}\int_{y_0}^{t}\left\{c(s,\eta)+b(s,\eta)c(s,\eta)\cdot\right.\right.\right.$$

$$\left.\left.\left.\cdot\left(\int_{x_0}^{s}\int_{y_0}^{\eta}[c(\xi,n)+p(\xi,n)]v(\xi,n;s,\eta)\,d\xi dn\right)\right\}ds\,d\eta\right)\right)dt.$$

Now keeping y fixed in the above inequality, set $x=s$ and then integrate with respect to s from x_0 to x to obtain the estimate

$$\Omega(a(x,y)) \leqslant \Omega(M)+\int_{x_0}^{x}\int_{y_0}^{y}q(s,t)W\left(1+b(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t}\left\{c(\xi,\eta)+b(\xi,\eta)c(\xi,\eta)\cdot\right.\right.\right. \tag{25}$$

$$\left.\left.\left.\cdot\left(\int_{x_0}^{\xi}\int_{y_0}^{\eta}[c(m,n)+p(m,n)]v(m,n;\xi,\eta)\,dmdn\right)\right\}d\xi\,d\eta\right)\right)ds\,dt.$$

Now substituting the bound on $a(x,y)$ from (25) in (22) we obtain the desired bound in (19).

Theorem 4. *Let $\Phi(x,y)$, $b(x,y)$, $c(x,y)$ and $p(x,y)$ be real-valued nonnegative continuous functions defined on a domain D as mentioned in Theorem 1. Let $W(r(x,y))$ $W_y(r(x,y))$, $P_0(x,y)$ and $P(x,y)$ be as in Theorem 3. Let $v(s,t;x,y)$ be the solution of the characteristic initial value problem (15) and let D^+ be a connected subdomain of D which contains P and on which $v\geqslant 0$ (see, Fig. 1). Then for $(x,y)\in D$ there exist $R\subset D^+$ and $v(s,t;x,y)>0$ on R and $\Phi(x,y)$ satisfies*

$$\Phi(x,y) \leqslant M + b(x,y)\int_{x_0}^{x}\int_{y_0}^{y}c(s,t)\Phi(s,t)\,dsdt + \int_{x_0}^{x}\int_{y_0}^{y}p(s,t)W(\Phi(s,t))\,dsdt, \tag{26}$$

where $M>0$ is a constant, for $0\leqslant x\leqslant x_2$, $0\leqslant y\leqslant y_2$; $\Phi(x,y)$ satisfies also

$$\Phi(x,y) \leqslant \Omega^{-1}\left[\Omega(M)+\int_{x_0}^{x}\int_{y_0}^{y}p(s,t)W\left(1+b(s,t)\cdot\right.\right. \tag{27}$$

$$\left.\left.\cdot\left(\int_{x_0}^{s}\int_{y_0}^{t}c(\xi,\eta)v(\xi,\eta;s,t)\,d\xi d\eta\right)\right)dsdt\right]\cdot\left[1+b(x,y)\left(\int_{x_0}^{x}\int_{y_0}^{y}c(s,t)v(s,t;x,y)dsdt\right)\right],$$

where Ω, Ω^{-1} are as defined in Theorem 3, and

$$\Omega(M)+\int_{x_0}^{x}\int_{y_0}^{y}p(s,t)W\left(1+b(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t}c(\xi,\eta)v(\xi,\eta;s,t)d\xi d\eta\right)\right)dsdt \subset \mathrm{Dom}\,(\Omega^{-1}),$$

for all x, y lying in the subintervals $0\leqslant x\leqslant x_2$, $0\leqslant y\leqslant y_2$ of real numbers.

The details of the proof of this theorem follows by an argument similar to that in the proof of Theorem 3, by making use of Theorem 2. We omit the details.

We note that there is no essential difficulty in obtaining two independent variable generalizations of the integral inequalities recently established by this author in [12]. Since this translation is quite straightforward in view of the results of this paper, we leave it for the reader to fill in where needed (see, [13, 14]).

In concluding this paper we note that the integral inequality established in Theorem 1 can be used to study the stability, boundedness, continuous dependence and other problems of the solutions of Volterra integral equations of the form

$$u(x,y)=f(x,y)+\int_{x_0}^{x}\int_{y_0}^{y}K[x,y,s,t,u(s,t)]dsdt \tag{28}$$

$$+\int_{x_0}^{x}\int_{y_0}^{y}h\left[x,y,s,t,\int_{x_0}^{s}\int_{y_0}^{t}K_0(s,t,\xi,\eta,u(\xi,\eta))d\xi d\eta\right]dsdt$$

under some suitable conditions on the functions involved in (38). The details of these results are very close to that given in [13]–[16] and hence we omit the details. In our future papers we wish to illustrate further applications of other inequalities to some problems in the theory of partial differential and integral equations of a more general type.

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L'HOMOGENEISATION D'UN PROBLEME NON-LINEAIRE

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Dans ce qui suit on considère une équation elliptique non linéaire, de type Navier-Stokes, dont les coefficients a_{ijkh} sont Y periodiques, Y étant un parallélipipède de R^3. On fait l'homogénéisation de cette équation et l'on démontre un théorème de convergence, bien que le terme nonlinéaire soit présent.

1. Introduction. On considère l'équation :

$$(1)\qquad \begin{cases} u_k \dfrac{\partial u_i}{\partial X_k} + \dfrac{\partial}{\partial X_j}(p\delta_{ij} - \sigma_{ij}) = f_i \text{ dans } \Omega \subset R^3, \\ \operatorname{div} \mathbf{u} = 0, \ \mathbf{u}/_{\partial\Omega} = 0, \end{cases}$$

où u est la vitesse, p est la pression, $\sigma_{ij} = a_{ijkh}(x)\varepsilon_{kh}(u)$,

$$\varepsilon_{kh}(\mathbf{u}) = \left(\frac{\partial u_h}{\partial X_k} + \frac{\partial u_k}{\partial X_h}\right), \ (\exists)\alpha > 0 \text{ et } a_{ijkh}\xi_{ij}\xi_{kh} \geq \alpha\xi_{ij}\xi_{kh},$$

$$a_{ijkh} = Y\text{-}périodiques, \ a_{ijkh} \in L_\infty(Y), \ a_{ijkh} = a_{ijhk} = a_{khij}$$

En définissant : $a^{\varepsilon}_{ijkh}(x) = a_{ijkh}(x/\varepsilon)$, (1) se transforme en une famille de problèmes pour $\varepsilon \to 0$. Alors nous avons le *problème* ε

$$(2)\qquad \begin{cases} u^{\varepsilon}_k \dfrac{\partial u^{\varepsilon}_i}{\partial X_k} + \dfrac{\partial}{\partial X_i}(p^{\varepsilon}\delta_{ij} - \sigma^{\varepsilon}_{ij}) = f_i, \text{ dans } \Omega\ , \\ \operatorname{div} \mathbf{u}^{\varepsilon} = 0, \ \mathbf{u}^{\varepsilon}/_{\partial\Omega} = 0. \end{cases}$$

En définissant $\xi^{\varepsilon}_{ij} = a^{\varepsilon}_{ijkh}\varepsilon_{kh}(\mathbf{u}^{\varepsilon})$, (2) devient :

$$(2')\qquad \left(u^{\varepsilon}_k \frac{\partial u^{\varepsilon}_i}{\partial X_k}, w_i\right) + (\xi^{\varepsilon}_{ij}, \varepsilon_{ij}(\mathbf{w})) - (p^{\varepsilon} \operatorname{div} \mathbf{w}) = (f_i, w_i),$$

$(\forall)$ $\mathbf{w} \in \mathring{w}^1_2(\Omega)$, où $(.,.)$ est le produit scalaire dans $L_2(\Omega)$.

Les conditions d'existence et unicité pour (2) sont démontrées dans [3]. L'équation homogéneisée est obtenue par les procédées classiques comme dans [2]. On suppose que $\mathbf{u}^{\varepsilon}$, p^{ε}, $\sigma^{\varepsilon}_{ij}$ ont un développement de la forme :

$$\mathbf{u}^{\varepsilon} = \mathbf{u}^0(x, y) + \varepsilon\mathbf{u}^1(x, y) + \dots$$