Nous avons :

$$\left(b_j \frac{\partial u_1^3}{\partial x_j} - b_j \frac{\partial u_2^3}{\partial x_j},\ u_1 - u_2 \right) = \int_\Omega b_j \frac{\partial}{\partial x_j} (u_1^3 - u_2^3)(u_1 - u_2) =$$

$$= \int_\Omega b_j (u_1 - u_2)(u_1^2 + u_1 u_2 + u_2^2) \frac{\partial}{\partial x_j} (u_1 - u_2) \leqslant$$

$$\leqslant C_1 \|b_j\|_{L_\infty} \|u_1 - u_2\|^2_{\mathring{W}^1_2} (\|u_1\|^2_{L_4} + \|u_1\|_{L_4} \|u_2\|_{L_4} + \|u_2\|^2_{L_4}).$$

Donc, en prenant $w = u_1 - u_2$ dans (24), on a :

$$\|u_1 - u_2\|^2_{\mathring{W}^1_2} (\alpha - \|b_j\|_{L_\infty} \{\|u_1\|^2_{L_4} + \|u_1\|_{L_4} \|u_2\|_{L_4} + \|u_2\|^2_{L_4}\}) \leqslant 0. \tag{25}$$

Mais nous avons (23) et :

$$\|u_i\|_{L_4} \leqslant K \cdot \|u_i\|_{\mathring{W}^1_2} \leqslant K \frac{\|f\|_{L_2}}{\alpha}.$$

Donc la condition :

$$3K^2 \|f\|^2_{L_2} \cdot \|b_j\|_{L_\infty} \leqslant \alpha^3.$$

nous assure l'unicité de la solution de (22).

L'homogénéisation est alors obtenue par les mêmes procédés, en utilisant la propriété (21), le fait que l'équation en ε^{-1} qui correspond à (4) est la même que celle du cas linéaire, et l'unicité de la solution de (19).

BIBLIOGRAPHIE

1. L i o n s, T. L., B e n s s o u s s a n, P a p a n i c o l a o u — *Asimptotic analysis for periodic structures*. North-Holland, 1978.
2. S a n c h e z-P a l e n c i a, E. — *Non homogeneous media and vibration theory*. Lectures Notes in Physics, nr. 127, 1980.
3. T e m a m, R. — *Navier-Stokes Equations*, North-Holland, 1977.
4. L i o n s J. L. — *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.

Received 28. X. 1980

I N C R E S T
Bd. Păcii 220
11538 București

Analele științifice ale Universității „Al. I. Cuza" din Iași
Tomul XXVII, s. I a, 1981 f. 2

APPROXIMATE SOLUTION OF NONLINEAR VOLTERRA INTEGRAL EQUATIONS

BY

P. VAȚĂ

Starting with the Newton-Kantorovich method for the approximation of the solution of a nonlinear Volterra integral equation which gives a cvasilinearized algorithm for the determination of the successive approximations, using a variant of the Sokolov method, we obtain an algorithm by means of which the functional corrections may be determined in a simpler way than in the original Sokolov method or other variants, such as, for example, the one in [2].

In a certain way the rapidity of the convergence is also estimated.

I. Presentation of the method. Let

$$y(x) = f(x) + \int_a^x K[x, t; y(t)]\, dt, \quad x \in [a, b], \tag{1}$$

be the Volterra integral equation, where $f(x) \in C([a, b], R)$, $K : D \to R$ is continuous and has derivative with respect to y in D,

$$D = \{(x, t, y) : a \leqslant t \leqslant x \leqslant b,\ |y - f| \leqslant m,\ m > 0\}. \tag{2}$$

As it is known, using the Newton-Kantorovich method, we have the following algorithm for the approximation of solution of the equation (1) :

$$\begin{cases} y_{n+1}(x) = f(x) + \displaystyle\int_a^x \{K[x, t; y_n(t)] - K'_y[x, t; y_n(t)]\, y_n(t) + \\ \quad + K'_y[x, t; y_n(t)]\, y_{n+1}(t)\}\, dt; \\ y_0(x) = f(x) \end{cases} \tag{3}$$

In the paper [2], N. A. S v a r i c h e v s k a y a and M. A. Y a g u b o v showed that if $y_n(x)$ satisfies a condition of the form

$$-m + y_0 \leqslant y_n \leqslant y_0 + m, \quad m > 0,$$

then $y_{n+1}(x)$ satisfies a similar condition.

If for each $n = 0, 1, 2, \ldots$ we denote.

$$(4) \qquad \begin{cases} P_n(x) = f(x) + \int\limits_a^x \{K[x, t;\ y_n(t)] - K'_y\ [x, t;\ y_n(t)]\ y_n(t)\}\ dt\ ; \\ F[x, t;\ y_n(t)] = K'_y[x, t;\ y_n(t)]\ , \end{cases}$$

then the equations (3) will be written as

$$(5) \qquad \begin{cases} y_{n+1}(x) = P_n(x) + \int\limits_a^x F[x, t;\ y_n(t)]\ y_{n+1}(t)\ dt\ ; \\ y_0(x) = f(x)\ . \end{cases}$$

As it is obvious, the approximations y_{n+1}, for each n, are solutions of the linear integral equation

$$(6) \qquad z(x) = P_n(x) + \int\limits_a^x F[x, t;\ y_n(t)]\ z(t)\ dt, \quad x \in [a, b].$$

In order to approximate the solution of this equation we shall use the Sokolov method in a convenient and easier form.

We consider the approximations $z_k(x)$ of the solution $z(x)$ of equation (6) in the form

$$(7) \qquad z_k(x) = P_n(x) + \alpha_k Q_n(x) + \int\limits_a^x F[x, t;\ y_n(t)]\ z_{k-1}(t)\ dt, \quad k = 1, 2, \ldots$$

where

$$(8) \qquad Q_n(x) = \int\limits_a^x F[x, t; y_n(t)]\ dt\ ,$$

and α_k are parameters which will be determined as it follows :

In the first approximation, we consider

$$(9) \qquad z_1(x) = P_n(x) + \alpha_1 Q_n(x)\ .$$

If we replace $z_1(x)$ in (6), this equation will be verified with some error, which depends on α_1 :

$$(10) \qquad \varepsilon_1(x, \alpha_1) = z_1(x) - P_n(x) - \int\limits_a^x F[x, t;\ y_n(t)]\ z_1(t)\ dt,$$

or, taking into account (9),

$$\varepsilon_1(x, \alpha_1) = \alpha_1 Q_n(x) - \int\limits_a^x F[x, t;\ y_n(t)]\ P_n(t)\ dt - \alpha_1 \int\limits_a^x F[x, t;\ y_n(t)]\ Q_n(t)\ dt.$$

If we denote

$$(11) \qquad R_n(x) = Q_n(x) - \int\limits_a^x F[x, t;\ y_n(t)]\ Q_n(t) dt\ ,$$

then

$$\varepsilon_1(x, \alpha_1) = \alpha_1\ R_n(x) - \int\limits_a^x F[x, t;\ y_n(t)]\ P_n(t)\ dt.$$

To determine α_1 we consider the minimum related to α_1 of the functional

$$(12) \qquad I_1 = \int\limits_a^b \varepsilon_1^2(x, \alpha_1)\ dx = \int\limits_a^b \left\{ \alpha_1\ R_n(x) - \int\limits_a^x F[x, t; y_n(t)]\ P_n(t) dt \right\}^2 dx\ ,$$

which leads to the equation.

$$\int\limits_a^b R_n(x) \left\{ \alpha_1\ R_n(x) - \int\limits_a^x F[x, t;\ y_n(t)]\ P_n(t)\ dt \right\} dx = 0,$$

where from

$$(13) \qquad \alpha_1 = \int\limits_a^b R_n(x) \left(\int\limits_a^x F[x, t;\ y_n(t)]\ P_n(t)\ dt \right) dx \Bigg/ \int\limits_a^b R_n^2(x)\ dx.$$

In the second approximation, according to (7), we have

$$(14) \qquad z_2(x) = P_n(x) + \alpha_2 Q_n(x) + \int\limits_a^x F[x, t;\ y_n(t)]\ z_1(t)\ dt.$$

Proceeding, as before, we obtain the error

$$\varepsilon_2(x, \alpha_2) = z_2(x) - P_n(x) - \int\limits_a^x F[x, t;\ y_n(t)]\ z_2(t)\ dt,$$

and if we replace $z_2(x)$ from (14), we can write

$$\varepsilon_2(x, \alpha_2) = \alpha_2 Q_n(x) + \int\limits_a^x F[x, t;\ y_n(t)]\ z_1(t)\ dt - \int\limits_a^x F[x, t; y_n(t)] \Bigg\{ P_n(t) +$$

$$+ \alpha_2 Q_n(t) + \int\limits_a^t F[t, \tau;\ y_n(\tau)]\ z_1(\tau)\ d\tau \Bigg\} dt = \alpha_2\ R_n(x) + \int\limits_a^x F[x, t;\ y_n(t)] \Bigg\{ z_1(t) -$$

$$- P_n(t) - \int\limits_a^t F[t, \tau;\ y_n(\tau)]\ z_1(\tau)\ d\tau \Bigg\} dt\ ,$$

and according to (10), with α_1 already determined, it results

$$\varepsilon_2(x, \alpha_2) = \alpha_2 R_n(x) + \int_a^x F[x, t;\ y_n(t)]\ \varepsilon_1(t)\, dt. \tag{15}$$

α_2 will be determined so that the functional

$$I_2 = \int_a^b \varepsilon_2^2(x, \alpha_2)\, dx = \int_a^b \left\{ \alpha_2 R_n(x) + \int_a^x F[x, t;\ y_n(t)]\ \varepsilon_1(t)\, dt \right\}^2 dx \tag{16}$$

be minim relativ to α_2. From this, it follows that

$$\int_a^b R_n(x) \left\{ \alpha_2 R_n(x) + \int_a^x F[x, t;\ y_n(t)]\ \varepsilon_1(t)\, dt \right\} dx = 0,$$

where from

$$\alpha_2 = -\int_a^b R_n(x) \left(\int_a^x F[x, t;\ y_n(t)]\ \varepsilon_1(t)\, dt \right) dx \Big/ \int_a^b R_n^2(x)\, dx. \tag{17}$$

In the same way, we request that $z_k(s)$ satisfy equation (6) with the error $\varepsilon_k(x, \alpha_k)$, given by

$$\varepsilon_k(x, \alpha_k) = z_k(x) - P_n(x) - \int_a^x F[x, t;\ y_n(t)]\ z_k(t)\, dt,$$

where from, introducing $z_k(x)$ given by (7), it follows that

$$\varepsilon_k(x, \alpha_k) = \alpha_k Q_n(x) + \int_a^x F[x, t:\ y_n(t)]\ z_{k-1}(t)\, dt - \int_a^x F[x, t;\ y_n(t)] \Big\{ P_n(t) +$$

$$+ \alpha_k Q_n(t) + \int_a^t F[t, \tau;\ y_n(\tau)]\ z_{k-1}(\tau)\, d\tau \Big\} dt,$$

or

$$\varepsilon_k(x, \alpha_k) = \alpha_k R_n(x) + \int_a^x F[x, t;\ y_n(t)] \Big\{ z_{k-1}(t) - P_n(t) -$$

$$- \int_a^t F[t, \tau:\ y_n(\tau)]\ z_{k-1}(\tau)\, d\tau \Big\} dt,$$

and finally

$$\varepsilon_k(x, \alpha_k) = \alpha_k R_n(x) + \int_a^x F[x, t;\ y_n(t)]\ \varepsilon_{k-1}(t)\, dt. \tag{18}$$

We determine, as we did before, α_k from the condition of minim of the functional

$$I_k = \int_a^b \varepsilon_k^2(x, \alpha_k)\, dx = \int_a^b \left\{ \alpha_k R_n(x) + \int_a^x F[x, t;\ y_n(t)]\ \varepsilon_{k-1}(t)\, dt \right\}^2 dx, \tag{19}$$

and its minim relative to α_k is accomplished for

$$\alpha_k = -\int_a^b R_n(x) \left(\int_a^x F[x, t;\ y_n(t)]\ \varepsilon_{k-1}(t)\, dt \right) dx \Big/ \int_a^b R_n^2(x)\, dx. \tag{20}$$

II. Study of the convergence. We will study the convergence of the sequences $\{z_k(x)\}$, $\{y_n(x)\}$, and finally we will give an evaluation of the global error by using the previous method.

In this sense, we shall denote :

$$B_n^2 = 4(b-a) \int_a^b \left(\int_a^x |F[x, t;\ y_n(t)]|^2\, dt \right) dx\ ;\ A_n^2 = \sup_{x \in [a, b]} \int_a^x |P_n(t)|^2 dt. \tag{21}$$

We have

Theorem 1. *If besides the hypotheses on f and K, we suppose also that the integrals of the quadratic absolute values of the functions P_n, R_n for any y_n on $[a, b]$ and $B_n^2 < 1$ make sense, then the sequences $\{\varepsilon_k\}$ and $\{\alpha_k\}$ converge to zero, the sequence $\{z_k(x)\}$ converges uniformly to the solution of the equation (6) and we have the evaluation.*

$$\{D_k\}^{\frac{1}{2}} \leqslant \frac{2(b-a)^{\frac{1}{2}}}{2(b-a)^{\frac{1}{2}} - B_n} \cdot A_n B_n^k, \tag{22}$$

where $\quad D_k = \int_a^b \delta_k^2(x)\, dx\ ;\ \delta_k(x) = z(x) - z_k(x)$.

Proof. 1) Using in (19) the inequality of Minkowski, we have

$$\{I_k\}^{\frac{1}{2}} \leqslant |\alpha_k| \cdot \left\{ \int_a^b R_n^2(x) dx \right\}^{\frac{1}{2}} + \left\{ \int_a^b \left(\int_a^x F[x, t;\ y_n(t)]\ \varepsilon_{k-1}(t)\, dt \right)^2 dx \right\}^{\frac{1}{2}},$$

where, if we replace $|\alpha_k|$, we obtain,

$$\{I_k\}^{\frac{1}{2}} \leqslant \left|\int_a^b R_n(x)\left(\int_a^x F[x,\ t;\ y_n(t)]\ \varepsilon_{k-1}(t)\,dt\right)dx\right| \Big/ \left\{\int_a^b R_n^2(x)\,dx\right\}^{\frac{1}{2}} +$$

$$+\left\{\int_a^b\left(\int_a^x F[x\ t;\ y_n(t)]\ \varepsilon_{k-1}(t)\,dt\right)^2 dx\right\}^{\frac{1}{2}}.$$

Now, applying the inequality of Cauchy-Buniakovski and after the necessary transformations, we have

$$\{I_k\}^{\frac{1}{2}} \leqslant 2\left\{\int_a^b\left(\int_a^x |F[x,t;\ y_n(t)]\ |^2\,dt\right)dx\right\}^{\frac{1}{2}}\cdot\left\{\int_a^b\left(\int_a^x |\varepsilon_{k-1}(t)\ |^2\,dt\right)dx\right\}^{\frac{1}{2}}.$$

Since $\int_a^x |\ \varepsilon_{k-1}(t)\ |^2\,dt \leqslant I_{k-1}$, then $\{I_k\}^{\frac{1}{2}} \leqslant 2.\left\{\int_a^b\left(\int_a^x |\ F[x,t;\ y_n(t)]\ |^2\,dt\right)dx\right\}^{\frac{1}{2}}.$

$.\ \{I_{k-1}\}^{\frac{1}{2}}\cdot(b-a)^{\frac{1}{2}}$ and taking into account (21), it follows the recurrent relation $I_k \leqslant B_n^2\cdot I_{k-1}$, for each n, where from it is easy to deduce

$$I_k \leqslant (B_n^2)^{k-1}\cdot I_1. \tag{23}$$

For the evaluation of I_1, we start from its expression and we obtain

$$\{I_1\}^{\frac{1}{2}} \leqslant 2\left\{\int_a^b\left(\int_a^x |\ F[x,t;\ \ y_n(t)\ ]\ |^2\,dt\right)dx\right\}^{\frac{1}{2}}\cdot\left\{\int_a^b\left(\int_a^x |\ P_n(t)\ |^2\,dt\right)dx\right\}^{\frac{1}{2}} \leqslant A_n.B_n.$$

With this, (23) can be written

$$\{I_k\}^{\frac{1}{2}} \leqslant A_n\cdot B_n^k. \tag{23'}$$

As a result of the fact that $B_n<1$, when $k\to\infty$, it follows that $I_k\to 0$, $\varepsilon_k\to 0$ and $\alpha_k\to 0$.

2) We prove now that the sequence $\{z_k(x)\}$ converges uniformly on $[a,b]$ to $z(x)$ which is the solution of equation (6) and we obtain the evaluation (22).

Having in mind that $z_k(x)=z_{k-1}(x)+\alpha_k$ for $k=2,3,\ldots$ it is easy to prove, we have

$$|\ z_k(x)-z_{k-1}(x)\ |=|\ \alpha_k\ | \leqslant \frac{B_n}{2\,(b-a)}\{I_{k-1}\}^{\frac{1}{2}}.\ (b-a)^{\frac{1}{2}}\frac{1}{\left\{\int_a^b R_n^2(x)\,dx\right\}^{\frac{1}{2}}} \leqslant \tag{24}$$

$$\leqslant \frac{B_n^{k-1}}{2\,(b-a)^{\frac{1}{2}}\cdot R_n},$$

where $R_n^2=\int_a^b R_n^2(x)\,dx$.

Since $B_n<1$, it comes that the series $z_1(x)+\sum_{k=2}^{\infty}[z_k(x)-z_{k-1}(x)]$ is absolutely and uniformly convergent on $[a,b]$ to $z(x)$. As it can be seen in (7), the continuous function z with the values $z(x)=\lim_{k\to\infty} z_k(x)$ is a solution of the equation (6).

For the error $\delta_k(x)$ we have

$$z_k(x)=P_n(x)+\int_a^x F[x,t;\ y_n(t)]\ z_k(t)\,dt+\varepsilon_k(x) \tag{25}$$

and by substraction of (25) from (6), we obtain

$$\delta_k(x)=z(x)-z_k(x)=\int_a^x F[x,t;\ y_n(t)]\ \delta_k(t)\,dt-\varepsilon_k(x). \tag{26}$$

It follows that

$$D_k=\int_a^b \delta_k^2(x)\,dx=\int_a^b\left\{\int_a^x F\ [x,\ t\ ;y_n(t)]\ \delta_k(t)\ dt-\varepsilon_k(x)\right\}^2 dx, \tag{27}$$

where, it we apply Minkowski's inequality, we obtain

$$\{D_k\}^{\frac{1}{2}} \leqslant \left\{\int_a^b\left(\int_a^x F[x,t;y_n(t)]\ \delta_k(t)\,dt\right)^2 dx\right\}^{\frac{1}{2}}+\left\{\int_a^b \varepsilon_k^2(x)\,dx\right\}^{\frac{1}{2}}.$$

With the help of the Cauchy-Buniakovski inequality, (19) and (21), we obtain the evaluation

$$\{D_k\}^{\frac{1}{2}} \leqslant \frac{B_n}{2\,(b-a)^{\frac{1}{2}}}\cdot\{D_k\}^{\frac{1}{2}}+\{I_k\}^{\frac{1}{2}} \text{ or } \{D_k\}^{\frac{1}{2}} \leqslant \frac{2\,(b-a)^{\frac{1}{2}}\cdot\{I_k\}^{\frac{1}{2}}}{2\,(b-a)^{\frac{1}{2}}-B_n}.$$

Since $\{I_k\}^{\frac{1}{2}} \leqslant B_n^{k-1}\{I_1\}^{\frac{1}{2}}$, $\{I_1\}^{\frac{1}{2}} \leqslant A_n\,B_n$, it follows that

$$\{D_n\}^{\frac{1}{2}} \leqslant \frac{2(b-a)^{\frac{1}{2}}}{2(b-a)^{\frac{1}{2}}-B_n}\,A_n\cdot B_n^k.$$

Knowing that $B_n<1$, it follows that $D_k\to 0$ when $k\to\infty$.

The relation (22), gives in a certain sense, the evaluation of the error for the solution of the equation (6).

3) In what follows, we shall denote

$$\delta_{n+1}(x)=y_{n+1}(x)-y_n(x)\ ;\ \Delta_{n+1}=\int_a^b|\delta_{n+1}(x)|^2dx\ ;\ M=\max_D|K'_y(x,t;y)|\ \text{and}$$

(28) $$K=\max_D|K(x,t;y)|.$$

We have

Theorem 2. *If the function f is continuous on $[a,b]$, the function K is continuous and is derivable with respect to y, continuous on D, then the sequence of approximations $\{y_{n+1}(x)\}$ converges uniformly towards the solution of the equation (1) and we have the evaluation:*

(29) $$\{\Delta_{n+1}\}^{\frac12}\leqslant\frac{2K(b-a)^{3/2}}{\sqrt3}\cdot\frac{[2M(b-a)]^n}{n!}e^{M(b-a)}.$$

Proof. We must first notice that from the continuity of K and its derivative with respect to y it follows that it satisfies also a condition of Lipschitz type.

We will first evaluate $|y_{n+1}(x)-y_n(x)|$. For this we have that

$$y_{n+1}(x)-y_n(x)=\int_a^x\{(K[x,t;y_n(t)]-K[x,t;y_{n-1}(t)])+$$

$$+K'_y[x,t;y_{n-1}(t)](y_{n-1}(t)-y_n(t))+K'_y[x,t;y_n(t)](y_{n+1}(t)-y_n(t))\}dt.$$

Taking the absolute values and (28), we obtain

(30) $$|y_{n+1}(x)-y_n(x)|\leqslant2M\int_a^x|y_n(t)-y_{n-1}(t)|dt+M\int_a^x|y_{n+1}(t)-y_n(t)|dt.$$

We evaluate now $|y_1(x)-y_0(x)|$ and we have

$$|y_1(x)-y_0(x)|\leqslant\int_a^x|K[x,t;f(t)]|dt+\int_a^x|K'_y[x,t;f(t)]|\cdot|y_1(t)-y_0(t)|dt,$$

or

$$|y_1(x)-y_0(x)|\leqslant K(x-a)+M\int_a^x|y_1(t)-y_0(t)|dt.$$

We may apply Gronwall's lemma (see [5] p. 13) with
$x(t)=|y_1(t)-y_0(t)|$; $h(t)=K(t-a)$; $K(t)=M$, and obtain

$$|y_1(x)-y_0(x)|\leqslant K(x-a)+KM\int_a^x(t-a)\exp\int_t^xM\,dt=$$

$$=\frac KMe^{M(x-a)}-\frac KM\leqslant\frac KMe^{M(x-a)}\leqslant\frac KMe^{M(b-a)}.$$

Further on, according to (30), we have

$$|y_2(x)-y_1(x)|\leqslant2M\int_a^x|y_1(t)-y_0(t)|dt+M\int_a^x|y_2(t)-y_1(t)|dt\leqslant$$

$$\leqslant2K\int_a^xe^{M(t-a)}dt+M\int_a^x|y_2(t)-y_1(t)|dt=\frac{2K}{M}[e^{M(x-a)}-1]+M\int_a^x|y_2(t)-y_1(t)|dt.$$

After Gronwall's lemma, it follows that

$$|y_2(x)-y_1(x)|\leqslant\frac{2K}{M}[e^{M(x-a)}-1]+2K\int_a^x[e^{M(t-a)}-1]e^{M(x-t)}dt=$$

$$=2K(x-a)e^{M(x-a)}\leqslant2K(b-a)e^{M(b-a)}.$$

Proceeding analogously, we find the evaluations

$$|y_3(x)-y_2(x)|\leqslant4KM\frac{(x-a)^2}{2!}e^{M(x-a)}\ ;\ |y_4(x)-y_3(x)|\leqslant8KM^2\frac{(x-a)^3}{3!}e^{M(x-a)}$$

and so on, by induction

(31) $$|y_{n+1}(x)-y_n(x)|\leqslant\frac KM(2M)^n\frac{(x-a)^n}{n!}e^{M(x-a)}\leqslant\frac KM(2M)^n\frac{(b-a)^n}{n!}e^{M(b-a)}.$$

Since the series $\frac KMe^{M(b-a)}\sum_0^\infty\frac{[2M(b-a)]^n}{n!}$ is convergent and has the sum $\frac KMe^{3M(b-a)}$, the series $y_0(x)+\sum_0^\infty[y_{n+1}(x)-y_n(x)]$ is uniformly convergent on $[a,b]$, it follows that the sequence $\{y_{n+1}(x)\}$ is uniformly convergent on $[a,b]$ towards a continuous function y which will be the solution of the equation (1), by passing to the limit in (3), as it is obvious.

If we replace $y_{n+1}(x)$ in the equation (1), it is verified with some error $\eta_{n+1}(x)$, i.e. we have

(32) $$y_{n+1}(x)=f(x)+\int_a^xK[x,t;y_{n+1}(t)]dt+\eta_{n+1}(x).$$

By subtraction of (32) from (1) and if we denote $\delta_{n+1}(x)=y_{n+1}(x)-y(x)$, we have

$$\delta_{n+1}(x)=y_{n+1}(x)-y(x)=\int_a^x\{K[x,t;y_{n+1}(t)]-K[x,t;y(t)]\}dt+\eta_{n+1}(x).$$

If we consider the absolute values, we obtain

$$|\delta_{n+1}(x)| \leqslant M \int_a^x |\delta_{n+1}(t)|\,dt + |\eta_{n+1}(x)|.$$

and

$$\Delta_{n+1} = \int_a^b |\delta_{n+1}(x)|^2 dx \leqslant \int_a^b \Big[M \int_a^x |\delta_{n+1}(t)|\,dt + |\eta_{n+1}(x)|\Big]^2 dx.$$

With the Minkowski inequality, it follows that

$$\{\Delta_{n+1}\}^{\frac{1}{2}} \leqslant M\{\Delta_{n+1}\}^{\frac{1}{2}} \cdot \frac{b-a}{\sqrt{2}} + \left\{\int_a^b |\eta_{n+1}(x)|^2 dx\right\}^{\frac{1}{2}}.$$

From (32) and (3), we can write that

$$\eta_{n+1}(x) = y_{n+1}(x) - f(x) - \int_a^x K[x, t\,; y_{n+1}(t)]\,dt = \int_a^x \{K[x, t\,; y_n(t)] -$$

$$- K_y'[x, t\,; y_n(t)]\,y_n(t) + K_y'[x, t\,; y_n(t)]\,y_{n+1}(t) - K[x, t\,; y_{n+1}(t)]\}\,dt,$$

where from it comes that

$$|\eta_{n+1}(x)| \leqslant 2M \int_a^x |y_{n+1}(t) - y_n(t)|\,dt.$$

Taking into consideration the evaluation of (31), it follows that

$$|\eta_{n+1}(x)| \leqslant 2M \frac{K}{M} \frac{[2M(b-a)]^n}{n!} e^{M(b-a)} (x-a) = 2K \frac{[2M(b-a)]^n}{n!} e^{M(b-a)} (x-a)$$

and then

$$\int_a^b |\eta_{n+1}(x)|^2 dx \leqslant \frac{4K^2}{3} \frac{[2M(b-a)]^{2n}}{(n!)^2} e^{2M(b-a)} (b-a)^3.$$

From (33), if $M < \dfrac{b-a}{\sqrt{2}}$, it follows that

$$\{\Delta_{n+1}\}^{\frac{1}{2}} \leqslant \frac{2K(b-a)^{3/2}}{\sqrt{3}} \cdot \frac{[2M(b-a)]^n}{n!} e^{M(b-a)}$$

and the theorem is proved.

3) Given the conditions of the theorems 1 and 2, we have the following evaluation

$$(34)\quad \{D_{n+1}^k\}^{\frac{1}{2}} \leqslant \frac{2(b-a)^{\frac{1}{2}}}{2(b-a)^{\frac{1}{2}} - B_n} \cdot A_n B_n^k + \frac{2K(b-a)^{3/2}}{\sqrt{3}} \cdot \frac{[2M(b-a)]^n}{n!} e^{M(b-a)},$$

where $D_{n+1}^k = \int_a^b |z_k(x) - y(x)|^2 dx.$

Indeed, if we write $|z_k(x) - y(x)| = |z_k(x) - y_{n+1}(x) + y_{n+1}(x) - y(x)|$ with the inequality of Minkowski and the notations made in the previous theorems, we have $\{D_{n+1}^k\}^{\frac{1}{2}} \leqslant \{\Delta_{n+1}\}^{\frac{1}{2}} + \{D_K\}^{\frac{1}{2}}$, where from (34) follows.

In the same way as in the given theorems, this is an evaluation of the global error of the method.

As we have seen, the approximations $y_n(x)$ are solutions of some linear integral equations for which they are assured and the conditions of existence and uniqueness may be precised, while for the approximation of these a variant of the Sokolov method has been used, more advantageous than the one given in [1], where the smallest solution α_k of nonlinear algebraic equation must be determined all the time, if there exists. If we compare the method suggested here with the one from [2], they are approached by the simplicity of the determination of the corrections α_k, only that in the quoted work, α_k are determined as solutions of some linear differential ordinary equations of first order and nonhomogenous, while here they are determined with the help of the relation (20).

By means of the Sokolov method and the suggested method, the structure of the approximations $z_k(x)$ is the same, but, sometimes, the parameters α_k in the method of functional correction can be found as average value among the approximations of k and $k-1$ orders, while here α_k are determined from the condition of minim of the errors of average quadrature ε_k, which is the result of replacing the approximation $z_k(x)$ in the integral equation (6).

Consequently, the approximation of order k, obtained by Sokolov's method replaced in equation (6) leads to an error superior to ε_k. As far as the method suggested by us is concerned, more calculations are necessary. That means that for each approximation $y_n(x)$ of the solution of the equation (1), the approximations $z_k(x)$ of the solutions of the integral equation (6) should be found; the parameters α_k must be determined every time.

Nevertheless, the simple way of determining the α_k parameters gives a certain advantage to the method.

Finally, the method may be used for the Fredholm equation as well, and analogous results are obtained.

REFERENCES

1. Sokolov Ju. D. — *O primenenii metoda osrednenia functzional'nih popravoc c nelinejnah uravneniiam.* Ukrainskii. Matem. Jurn., Matematica, T. IX, No. 4, 1957.
2. Svaricevskaia, M. A. Yagubov N. A., — *Priblijennoe reșenie nelineinogo integral'nogo uravneniia s parametrom s pomoșciu socetaniia cvazilinearizațiia i osredneniia, functional'nih popravoc.* Ma'ruzalari Az. Elmler. Akad. S.S.R. Tom. 31, No. 4 1975.

3. Mamedov Ya, D., Aşirov S. A. — *Nelineinîe uravnenia Volterra*. Ashabad, 1977.

4. Miller R. K. — *Nonlinear Volterra Integral Equations*. Mathematical Lecture Note, Series Benjamin (1971).

5. Corduneanu C. — *Ecuaţii diferenţiale şi integrale*. Iaşi, 1977.

Received 15. VIII. 1980

Department of Mathematics
University of Galaţi
R. S. Romania

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXVII, s. I a, 1981 f. 2

ON THE CONVERGENCE OF SOME MULTISTEP METHODS

BY

N. CALISTRU

1. Introduction. In [1] we have established a certain family of multistep algorithms for the numerical solution of the problem

$$\ddot{x} = f(t, x, \dot{x}),\ t \in J \subset R, \tag{1.1}$$

$$x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0, \quad t_0 \in J, \tag{1.2}$$

where $x(t) = (x^1(t), x^2(t), \ldots, x^n(t))^T$ is an unknown n-vector function, $x_0 = (x_0^1, x_0^2, \ldots, x_0^n)^T$, $\dot{x} = (\dot{x}_0^1, \dot{x}_0^2, \ldots, \dot{x}_0^n)^T$ are prescribed n-vectors and $f(t, x, \dot{x}) = (f^1(t, x, \dot{x}), f^2(t, x, \dot{x}), \ldots, f^n(t, x, \dot{x}))^T$ is a n-vector function defined on $J \times R^{2n}$.

We supposed that the system (1.1), (1.2) has a unique continuous and sufficiently smooth solution on J. We have considered the following sets of points from $[t_0, T]$

$$\Sigma_h = \{t_j |\ t_j = t_0 + jh\ ;\ j = \overline{0, N_1}\ ;\ N_1 = [(T - t_0)/h]\},$$

$$\Sigma_h(\alpha) = \{t_{j-\alpha} | t_{j-\alpha} = t_0 + (j - \alpha)h\ ;\ j = -[\alpha], -[\alpha] + 1, \ldots, [\alpha + N_1]\},$$

where $T \in J$, $T > t_0$, α is a real noninteger parameter and $h \in (0, T - t_0)$.

Then we obtained the family of multistep methods for the numerical solution of the problem (1.1), (1.2)

$$(\alpha A A^*) = (\mathscr{M}_{AA^*}^{(k)}(\alpha), (1.3), (A), (A^*)),$$

where

$$\mathscr{M}_{AA^*}^{(k)}(\alpha) = \{u_{k-1}(\alpha), \dot{u}_{k-1}(\alpha), u_{k-j-\alpha}(\alpha), \dot{u}_{k-j-\alpha}(\alpha)\ ;\ j = \overline{0,\ p+1}\ ;\ k \geqslant 1\}$$

$$\ddot{u}_{j-\alpha}(\alpha) = f(t_{j-\alpha}, u_{j-\alpha}(\alpha), \dot{u}_{j-\alpha}(\alpha))\ ; \tag{1.3}$$

$$\text{(A)} \quad \begin{cases} \dot{u}_k(\alpha) = \dot{u}_{k-1}(\alpha) + h \sum\limits_{i=0}^{p} A_i(\alpha) \ddot{u}_{k-i-\alpha}(\alpha), \\[2ex] u_k(\alpha) = u_{k-1}(\alpha) + h\dot{u}_{k-1}(\alpha) + h^2 \sum\limits_{i=0}^{p} \bar{A}_i(\alpha) \ddot{u}_{k-i-\alpha}(\alpha), \end{cases}$$

$$\text{(A}^*\text{)} \quad \begin{cases} \dot{u}_{k-\alpha}(\alpha) = \dot{u}_{k-1}(\alpha) + h \sum\limits_{i=0}^{p} A_i^*(\alpha) \ddot{u}_{k-i-1-\alpha}(\alpha), \\[2ex] u_{k-\alpha}(\alpha) = u_{k-1}(\alpha) + h\dot{u}_{k-1}(\alpha) + h^2 \sum\limits_{i=0}^{p} \bar{A}_i^*(\alpha) \ddot{u}_{k-i-1-\alpha}(\alpha),\ k = \overline{1, N_1}. \end{cases}$$