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ON THE CONVERGENCE OF SOME MULTISTEP METHODS

BY

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1. Introduction. In [1] we have established a certain family of multistep algorithms for the numerical solution of the problem

$$\ddot{x} = f(t, x, \dot{x}),\ t \in J \subset R, \tag{1.1}$$

$$x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0, \quad t_0 \in J, \tag{1.2}$$

where $x(t) = (x^1(t), x^2(t), \ldots, x^n(t))^T$ is an unknown n-vector function, $x_0 = (x_0^1, x_0^2, \ldots, x_0^n)^T$, $\dot{x} = (\dot{x}_0^1, \dot{x}_0^2, \ldots, \dot{x}_0^n)^T$ are prescribed n-vectors and $f(t, x, \dot{x}) = (f^1(t, x, \dot{x}), f^2(t, x, \dot{x}), \ldots, f^n(t, x, \dot{x}))^T$ is a n-vector function defined on $J \times R^{2n}$.

We supposed that the system (1.1), (1.2) has a unique continuous and sufficiently smooth solution on J. We have considered the following sets of points from $[t_0, T]$

$$\Sigma_h = \{t_j \mid t_j = t_0 + jh\ ;\ j = \overline{0, N_1}\ ;\ N_1 = [(T - t_0)/h]\},$$

$$\Sigma_h(\alpha) = \{t_{j-\alpha} \mid t_{j-\alpha} = t_0 + (j - \alpha)h\ ;\ j = -[\alpha], -[\alpha] + 1, \ldots, [\alpha + N_1]\},$$

where $T \in J$, $T > t_0$, α is a real noninteger parameter and $h \in (0, T - t_0)$.

Then we obtained the family of multistep methods for the numerical solution of the problem (1.1), (1.2)

$$(\alpha A A^*) = (\mathcal{M}_{AA^*}^{(k)}(\alpha), (1.3), (A), (A^*)),$$

where

$$\mathcal{M}_{AA^*}^{(k)}(\alpha) = \{u_{k-1}(\alpha), \dot{u}_{k-1}(\alpha), u_{k-j-\alpha}(\alpha), \dot{u}_{k-j-\alpha}(\alpha)\ ;\ j = \overline{0, p+1}\ ;\ k \geqslant 1\}$$

$$\ddot{u}_{j-\alpha}(\alpha) = f(t_{j-\alpha}, u_{j-\alpha}(\alpha), \dot{u}_{j-\alpha}(\alpha))\ ; \tag{1.3}$$

$$\text{(A)} \quad \begin{cases} \dot{u}_k(\alpha) = \dot{u}_{k-1}(\alpha) + h \sum\limits_{i=0}^{p} A_i(\alpha) \ddot{u}_{k-i-\alpha}(\alpha), \\ u_k(\alpha) = u_{k-1}(\alpha) + h\dot{u}_{k-1}(\alpha) + h^2 \sum\limits_{i=0}^{p} \bar{A}_i(\alpha) \ddot{u}_{k-i-\alpha}(\alpha), \end{cases}$$

$$\text{(A}^*\text{)} \quad \begin{cases} \dot{u}_{k-\alpha}(\alpha) = \dot{u}_{k-1}(\alpha) + h \sum\limits_{i=0}^{p} A_i^*(\alpha) \ddot{u}_{k-i-1-\alpha}(\alpha), \\ u_{k-\alpha}(\alpha) = u_{k-1}(\alpha) + h\dot{u}_{k-1}(\alpha) + h^2 \sum\limits_{i=0}^{p} \bar{A}_i^*(\alpha) \ddot{u}_{k-i-1-\alpha}(\alpha),\ k = \overline{1, N_1}. \end{cases}$$

The set $\mathcal{M}^{(1)}_{AA^*}(\alpha)$ is called the set of start values of the algorithms (αAA^*). The coefficients $A_i(\alpha)$, $\bar{A}_i(\alpha)$, $A_i^*(\alpha)$, $\bar{A}_i^*(\alpha)$, $i=\overline{0, p}$, are algebraic polynomials of parameter α.

Our aim is to prove here the convergence and numerical stability of the above method.

2. **Preliminaries.** We shall give here some results for later use. The following lemma is the main tool in the proof of convergence of the methods (αAA^*).

Lemma 2.1. *Let* $\rho_k(\alpha)$, $\tau_k(\alpha)$, $\bar{\rho}_k(\alpha)$, $\bar{\tau}_k(\alpha)$, $b_k(\alpha)$, $c_k(\alpha)$, $b_{j-\alpha}(\alpha)$, $c_{j-\alpha}(\alpha)$ $e_{j-\alpha}(\alpha)$ $k=1,2,\dots$; $j=-p-1, -p, -p+1,\dots$, *be sequences of real n-dimensional vector functions defined on R.*

Let $\beta_s(\alpha)$, $\gamma_s(\alpha)$, $\bar{\beta}_s(\alpha)$, $\bar{\gamma}_s(\alpha)$ $s=\overline{0, p}$, $p\in N$, *be real functions defined on R. We assume that the following hypotheses are satisfied :*

(I). *There are positive real functions* $R(\alpha)$, $G(\alpha)$ *and* $E(\alpha)$ *defined on R, and nonnegative constants L, K such that*

$$\max(\|\rho_k(\alpha)\|, \|\tau_k(\alpha)\|, \|\bar{\rho}_k(\alpha)\|, \|\bar{\tau}_k(\alpha)\|) \leqslant R(\alpha),\ k=1, 2,\dots; \tag{2.1}$$

$$\max\left(\sum_{s=0}^{p} |\beta_s(\alpha)|, \sum_{s=0}^{p} |\gamma_s(\alpha)|, \sum_{s=0}^{p} |\bar{\beta}_s(\alpha)|, \sum_{s=0}^{p} |\bar{\gamma}_s(\alpha)|\right) \leqslant G(\alpha); \tag{2.2}$$

$$\|e_{j-\alpha}(\alpha)\| \leqslant L(\|b_{j-\alpha}(\alpha)\| + \|c_{j-\alpha}(\alpha)\|),\ j=-p-1, -p,\dots; \tag{2.3}$$

$$\|b_{s-\alpha}(\alpha)\| + \|c_{s-\alpha}(\alpha)\| \leqslant E(\alpha),\ s=\overline{-p, 0}; \tag{2.4}$$

(II). *For* $k=1, 2, 3,\dots$, *the following inequalities are satisfied :*

$$\|b_k(\alpha)\| \leqslant \sum_{s=1}^{k} \|Z_s(\alpha)\|, \tag{2.5}$$

$$\|c_k(\alpha)\| \leqslant h\sum_{s=1}^{k-1} \|b_s(\alpha)\| + \sum_{s=1}^{k} \|V_s(\alpha)\|, \tag{2.6}$$

$$\|b_{k-\alpha}(\alpha)\| \leqslant \|b_{k-1}(\alpha) + \|\bar{Z}_k(\alpha)\|, \tag{2.7}$$

$$\|c_{k-\alpha}(\alpha)\| \leqslant \|c_{k-1}(\alpha)\| + |1-\alpha| h\|b_{k-1}(\alpha)\| + \|\bar{V}_k(\alpha)\|, \tag{2.8}$$

where

$$\left\{\begin{aligned} & Z_k(\alpha) = h\sum_{s=0}^{p} \beta_s(\alpha)e_{k-s-\alpha}(\alpha) + \rho_k(\alpha),\quad V_k(\alpha) = h^2\sum_{s=0}^{p} \gamma_s(\alpha)e_{k-s-\alpha}(\alpha) + \tau_k(\alpha),\\ & \bar{Z}_k(\alpha) = h\sum_{s=0} \bar{\beta}_s(\alpha)e_{k-s-1-\alpha}(\alpha) + \bar{\rho}_k(\alpha),\\ & \bar{V}_k(\alpha) = h^2\sum_{s=0}^{p} \bar{\gamma}_s(\alpha)e_{k-s-1-\alpha}(\alpha) + \bar{\tau}_k(\alpha), \end{aligned}\right. \tag{2.9}$$

and $h>0$ *is such that*

$$kh \leqslant K. \tag{2.10}$$

Then, for every $k\in N$, *we have*

$$E_k(\alpha) \leqslant (1+hM(\alpha))^{k+1}E(\alpha) + (1+hM(\alpha))^k \frac{\bar{R}(\alpha)}{h}, \tag{2.11}$$

where

$$E_k(\alpha) = \max(\|b_s(\alpha)\| + \|c_s(\alpha)\|;\ s=\overline{1, k}), \tag{2.12}$$

$$\begin{gathered} M(\alpha) = q(\alpha)LG(\alpha)(K+2p+1),\ \bar{R}(\alpha) = 2q(\alpha)K(K+2)R(\alpha),\\ q(\alpha) = \max(1, |1-\alpha|). \end{gathered} \tag{2.13}$$

(We use the norm $\|x\| = \max(|x^i|,\ i=\overline{1, n})$).

Proof. We consider the functions

$$E_{k-\alpha}(\alpha) = \max(\|b_{s-\alpha}(\alpha)\| + \|c_{s-\alpha}(\alpha)\|;\ s=\overline{1, k}),\ \alpha\in R,\ k=1, 2,\dots, \tag{2.14}$$

by using the hypotheses of Lemma 2.1 we obtain the following estimates :

$$\sum_{s=1}^{k} \|Z_s(\alpha)\| \leqslant h\sum_{s=1}^{k}\sum_{m=0}^{p} \|\beta_m(\alpha)e_{s-m-\alpha}(\alpha)\| + \sum_{s=1}^{k} \|\rho_s(\alpha)\| \leqslant$$

$$h\sum_{s=1}^{k}\sum_{m=0}^{p} |\beta_m(\alpha)|\|e_{s-m-\alpha}(\alpha)\| + \frac{R(\alpha)K}{h} = h\sum_{s=-p}^{-1} \|e_{s+1-\alpha}(\alpha)\| \sum_{m=-s}^{r} |\beta_m(\alpha)| +$$

$$h\sum_{s=0}^{k-1} \|e_{s+1-\alpha}(\alpha)\| \sum_{m=0}^{r} |\beta_m(\alpha)| + \frac{KR(\alpha)}{h} \leqslant hpLG(\alpha)E(\alpha) + hLG(\alpha)\sum_{s=0}^{k-1} E_{s+1-\alpha}(\alpha) + \frac{KR(\alpha)}{h},$$

where $r=\min(p,\ k-s-1)$. Thus

$$\sum_{s=1}^{k} \|Z_s(\alpha)\| \leqslant phLG(\alpha)E(\alpha) + hLG(\alpha)\sum_{s=1}^{k} E_{s-\alpha}(\alpha) + \frac{KR(\alpha)}{h}.\ k=1, 2,\dots \tag{2.15}$$

Similarly one can find the estimations

$$\sum_{s=1}^{k} \|V_s(\alpha)\| \leqslant ph^2LG(\alpha)E(\alpha) + h^2LG(\alpha)\sum_{s=1}^{k} E_{s-\alpha}(\alpha) + \frac{KR(\alpha)}{h}, \tag{2.16}$$

$$\|\bar{Z}_k(\alpha)\| \leqslant (p+1)hLG(\alpha)(E(\alpha) + E_{k-1-\alpha}(\alpha)) + R(\alpha), \tag{2.17}$$

$$\|\bar{V}_k(\alpha)\| \leqslant (p+1)h^2LG(\alpha)(E(\alpha) + E_{k-1-\alpha}(\alpha)) + R(\alpha),\ k=1, 2,\dots \tag{2.18}$$

Using (2.5) and (2.15) we can write

$$\|b_k(\alpha)\| \leqslant phLG(\alpha)E(\alpha) + hLG(\alpha)\sum_{s=1}^{k} E_{s-\alpha}(\alpha), + \frac{KR(\alpha)}{h}, \tag{2.19}$$

and from here we obtain

$$\sum_{s=1}^{k-1} \|b_s(\alpha)\| \leqslant p(k-1)hLG(\alpha)E(\alpha) + hLG(\alpha)\sum_{s=1}^{k-1} (k-s)E_{s-\alpha}(\alpha) + \frac{K^2R(\alpha)}{h}. \tag{2.20}$$

According to (2.6), 2.16), (2.20) ; (2.7), (2.17), (2.19) and (2.8), (2.13), (2.18), (2.20) we get respectively

$$\|c_k(\alpha)\| \leqslant p(k-1)h^2LG(\alpha)E(\alpha) + h^2LG(\alpha)\sum_{s=1}^{k}(k+1-s)E_{s-\alpha}(\alpha) + \tag{2.21}$$
$$+ \frac{K(K+1)R(\alpha)}{h},$$

$$\|b_{k-\alpha}(\alpha)\| \leqslant (2p+1)hLG(\alpha)E(\alpha) + (p+2)hLG(\alpha)\sum_{s=1}^{k-1}E_{s-\alpha}(\alpha) + \frac{2KR(\alpha)}{h}. \tag{2.22}$$

$$\|c_{k-\alpha}(\alpha)\| \leqslant (1 + h^2q(\alpha)(kp+p+1)LG(\alpha)E(\alpha) + q(\alpha)\frac{2K(K+1)R(\alpha)}{h} + \tag{2.23}$$
$$+ (p+1)q(\alpha)h^2LG(\alpha)\sum_{s=1}^{k-1}(k+1-s)E_{s-\alpha}(\alpha), \quad k=1,2,\ldots$$

Summing (2.19), (2.21) ; (2.22), (2.23), using (2.13) and suitable estimates, we have :

$$\|b_k(\alpha)\| + \|c_k(\alpha)\| \leqslant (1 + hM(\alpha))E(\alpha) + hM(\alpha)\sum_{s=1}^{k}E_{s-\alpha}(\alpha) + \frac{\bar{R}(\alpha)}{h}, \tag{2.24}$$

$$\|b_{k-\alpha}(\alpha)\| + c_{k-\alpha}(\alpha)\| \leqslant (1 + hM(\alpha))E(\alpha) + hM(\alpha)\sum_{s=1}^{k-1}E_{s-\alpha}(\alpha) + \tag{2.25}$$
$$+ \frac{\bar{R}(\alpha)}{h}, \quad k = 1, 2, \ldots$$

In view of (2.12) and (2.14), we can assert that for each fixed k, there are two natural numbers n_k and m_k, $1 \leqslant n_k$, $m_k \leqslant k$, such that

$$E_k(\alpha) \leqslant \|b_{n_k}(\alpha)\| + \|c_{n_k}(\alpha)\|, \quad E_{k-\alpha}(\alpha) \leqslant \|b_{m_k}(\alpha)\| + \|c_{m_k}(\alpha)\|.$$

Thus the following inequalities hold :

$$E_{k-\alpha}(\alpha) \leqslant (1 + hM(\alpha))E(\alpha) + hM(\alpha)\sum_{s=1}^{m_k-1}E_{s-\alpha}(\alpha) + \frac{\bar{R}(\alpha)}{h},$$

$$E_k(\alpha) \leqslant (1 + hM(\alpha))E(\alpha) + hM(\alpha)\sum_{s=1}^{n_k}E_{s-\alpha}(\alpha) + \frac{\bar{R}(\alpha)}{h}, \quad k = 1, 2, \ldots$$

Because m_k, $n_k \leqslant k$, so much the more we have

$$E_{k-\alpha}(\alpha) \leqslant (1 + hM(\alpha))E(\alpha) + hM(\alpha)\sum_{s=1}^{k-1}E_{s-\alpha}(\alpha) + \frac{\bar{R}(\alpha)}{h}, \tag{2.26}$$

$$E_k(\alpha) \leqslant (1 + hM(\alpha))E(\alpha) + hM(\alpha)\sum_{s=1}^{k}E_{s-\alpha}(\alpha) + \frac{\bar{R}(\alpha)}{h}, \quad k = 1, 2, \ldots \tag{2.27}$$

By induction with respect to k, from (2.26) it results

$$E_{k-\alpha}(\alpha) \leqslant (1 + hM(\alpha))^kE(\alpha) + (1 + hM(\alpha))^{k-1}\frac{\bar{R}(\alpha)}{h}, \quad k = 1, 2, \ldots \tag{2.28}$$

By (2.27) and (2.28) it follows (2.11) and thus Lemma 2.1 is proved.

Let now $x(t)$ be the solution of the problem (1.1), (1.2) and let $x(t_k)$, $\dot{x}(t_k)$, $x(t_{k-\alpha})$, $\dot{x}(t_{k-\alpha})$ be the accurate values of the $x(t)$, $\dot{x}(t)$ in the knots of the network $\Sigma_h \cup \Sigma_h(\alpha)$. We suppose that the following conditions are satisfied :

(CA). The elements of the starting set $\mathcal{M}^{(1)}_{AA^*}(\alpha)$ verify the relations

$$u_0(\alpha) = x(t_0),\ \dot{u}_0(\alpha) = \dot{x}(t_0),\ u_{j-\alpha}(\alpha) = x(t_{j-\alpha}),\ \dot{u}_{j-\alpha}(\alpha) = \dot{x}(t_{j-\alpha}),\ j = -\overline{(p+1), 0}\ ;$$

(CA^*). In the condition (CA) the values $u_k(\alpha)$, $\dot{u}_k(\alpha)$, $u_{k-\alpha}(\alpha)$, $\dot{u}_{k-\alpha}(\alpha)$, $k = \overline{1, N_1}$, furnished by the algorithms (αAA^*) are accurate determined.

We consider the differences

$$\varepsilon_k(\alpha) = x(t_k) - u_k(\alpha),\quad \dot{\varepsilon}_k(\alpha) = \dot{x}(t_k) - \dot{u}_k(\alpha),\quad k = \overline{1, N_1},$$

and define the functions $E_k(\alpha)$ by the relations

$$E_k(\alpha) = \max\left(\|\varepsilon_s(\alpha)\|, \|\dot{\varepsilon}_s(\alpha)\|\ ;\ s = \overline{1, k}\right),\quad k = \overline{1, N_1}. \tag{2.29}$$

Definition 2.1. *The α-error up to the step k of the family (αAA^*) is the function $E_k(\alpha)$ defined by (2.29). For $\alpha = \bar{\alpha}$, the value $E_k(\bar{\alpha})$ is the error up to the step k of the method $(\bar{\alpha}AA^*)$.*

Remark 2.1. In practice the elements of the starting set $\bar{\mathcal{M}}^{(1)}_{AA^*}(\alpha)$, do not precisely satisfy the condtion (CA). Generally, we use starting set of the form

$$\bar{\mathcal{M}}^{(1)}_{AA^*}(\alpha) = \{\bar{u}_0(\alpha), \dot{\bar{u}}_0(\alpha), \bar{u}_{j-\alpha}(\alpha), u_{j-\alpha}(\alpha)\ ;\ j = 0, -1, \ldots -p-1\},$$

whose elements are approximations of the solutions at the corresponding knots. The approximation errors are given by the norm of the vectors

$$\delta_0(\alpha) = x(t_0) - \bar{u}_0(\alpha),\ \dot{\delta}_0(\alpha) = \dot{x}(t_0) - \dot{\bar{u}}_0(\alpha), \tag{2.30}$$
$$\delta_{j-\alpha}(\alpha) = x(t_{j-\alpha}) - \bar{u}_{j-\alpha}(\alpha),\ \dot{\delta}_{j-\alpha}(\alpha) = \dot{x}(t_{j-\alpha}) - \dot{\bar{u}}_{j-\alpha}(\alpha),\ j = -\overline{(p+1), 0}.$$

Definition 2.2. *The initial α-error of the family (αAA^*) with respect to $\bar{\mathcal{M}}^{(1)}_{AA^*}(\alpha)$ is the function $\mathcal{J}(\alpha)$ defined by*

$$\mathcal{J}(\alpha) = \max_{-p-1 \leqslant s \leqslant 0}\left(\|\delta_{s-\alpha}(\alpha)\| + \|\dot{\delta}_{s-\alpha}(\alpha)\|, \|\delta_0(\alpha)\| + \|\dot{\delta}_0(\alpha)\|\right). \tag{2.31}$$

For $\alpha = \bar{\alpha}$, the value $\mathcal{J}(\bar{\alpha})$ is the initial error of the method $(\bar{\alpha}AA^*)$ with respect to the starting set $\bar{\mathcal{M}}^{(1)}_{AA^*}(\bar{\alpha})$.

Remark 2.2. As a consequence of the rounding errors, the values given by (αAA^*) do not verify the condition (CA^*). Generally, these values are not precisely obtained and they do not verify exactly the formulae of (αAA^*). In fact, if $\tilde{u}_k(\alpha)$, $\dot{\tilde{u}}_k(\alpha)$, $\tilde{u}_{k-\alpha}(\alpha)$, $\dot{\tilde{u}}_{k-\alpha}(\alpha)$ are the values given by the method (αAA^*) with respect to the starting set $\tilde{\mathcal{M}}^{(1)}_{AA^*}(\alpha)$, they verify the following relations

$$(2.32)\qquad \begin{aligned} \dot{\tilde{u}}_k(\alpha) &= \dot{\tilde{u}}_{k-1}(\alpha) + h\sum_{s=0}^{p} A_s(\alpha)\ddot{\tilde{u}}_{k-s-\alpha}(\alpha) + \tilde{\rho}_{p,k}(\alpha),\\ \tilde{u}_k(\alpha) &= \tilde{u}_{k-1}(\alpha) + h\dot{\tilde{u}}_{k-1}(\alpha) + h^2\sum_{s=0}^{p} \overline{A}_s(\alpha)\ddot{\tilde{u}}_{k-s-\alpha}(\alpha), + \tilde{\tau}_{p,k}(\alpha), \end{aligned}$$

$$(2.33)\qquad \begin{aligned} \dot{\tilde{u}}_{k-\alpha}(\alpha) &= \dot{\tilde{u}}_{k-1}(\alpha) + h\sum_{s=0}^{p} A^*_s(\alpha)\ddot{\tilde{u}}_{k-s-1-\alpha}(\alpha) + \tilde{\rho}^*_{p,k}(\alpha)\\ \tilde{u}_{k-\alpha}(\alpha) &= \tilde{u}_{k-1}(\alpha) + h(1-\alpha)\dot{\tilde{u}}_{k-1}(\alpha) + h^2\sum_{s=0}^{p} \overline{A}^*_s(\alpha)\ddot{\tilde{u}}_{k-s-1-\alpha}(\alpha) + \tilde{\tau}^*_{p,k}(\alpha), \end{aligned}$$

$k=\overline{1, N_1}$, where

$$(2.34)\qquad \ddot{\tilde{u}}_{k-j-\alpha}(\alpha) = f(t_{k-j-\alpha}, \tilde{u}_{k-j-\alpha}(\alpha), \dot{\tilde{u}}_{k-j-\alpha}(\alpha)),\ j = -(\overline{p+1}), 0.$$

The formulae (2.32), (2.33) differ from the formulae (A), (A^*) by the terms $\tilde{\rho}_{p,k}(\alpha)$, $\tilde{\tau}_{p,k}(\alpha)$, $\tilde{\rho}^*_{p,k}(\alpha)$, $\tilde{\tau}^*_{p,k}(\alpha)$, which appear as a consequence of the rounding errors in the process the computing operations.

The norm of the vectors

$$(2.35)\qquad \begin{cases} \delta_{k-\alpha}(\alpha) \equiv u_{k-\alpha}(\alpha) - \tilde{u}_{k-\alpha}(\alpha),\ \dot{\delta}_{k-\alpha}(\alpha) \equiv \dot{u}_{k-\alpha}(\alpha) - \dot{\tilde{u}}_{k-\alpha}(\alpha),\\ \delta_k(\alpha) \equiv u_k(\alpha) - \tilde{u}_k(\alpha),\ \dot{\delta}_k(\alpha) \equiv \dot{u}_k(\alpha) - \dot{\tilde{u}}_k(\alpha),\ k = \overline{1, N_1}, \end{cases}$$

where $u_k(\alpha)$, $\dot{u}_k(\alpha)$, $u_{k-\alpha}(\alpha)$, $\dot{u}_{k-\alpha}(\alpha)$ satisfy the condition (CA^*), are measuring the deviation of the values $\tilde{u}_k(\alpha)$, $\dot{\tilde{u}}_k(\alpha)$, $\tilde{u}_{k-\alpha}(\alpha)$, $\dot{\tilde{u}}_{k-\alpha}(\alpha)$, from the values $u_k(\alpha)$, $\dot{u}_k(\alpha)$, $u_{k-\alpha}(\alpha)$, $\dot{u}_{k-\alpha}(\alpha)$.

Definition 2.3. *The rounding α-erorr up to the step k of the family (αAA^*) is the function $\mathcal{R}_k(\alpha)$ defined by*

$$(2.36)\qquad \mathcal{R}_k(\alpha) = \max\,(\|\delta_s(\alpha)\| + \|\dot{\delta}_s(\alpha)\|\ ;\ s = \overline{1, k}).$$

For $\alpha=\overline{\alpha}$ the value $\mathcal{R}_k(\overline{\alpha})$ is the rounding error up to the step k of the method $(\overline{\alpha}AA^*)$.

Remark 2.3. Taking into account Remarks 2.1 and 2.2 we shall prove that the values of $x(t)$ and $\dot{x}(t)$ at knots Σ_h, are approximated by $\tilde{u}_k(\alpha)$, $\dot{\tilde{u}}_k(\alpha)$ respectively. The accuracy of approximation defined with

$$(2.37)\qquad x(t_k) - \tilde{u}_k(\alpha) = \varepsilon_k(\alpha) + \delta_k(\alpha)\ ;\ \dot{x}(t_k) - \dot{\tilde{u}}_k(\alpha) = \dot{\varepsilon}_k(\alpha) + \dot{\delta}_k(\alpha),\ k = \overline{1, N_1},$$

will depend on the initial and rounding α-errors.

Definition 2.4. *The total α-error up to the step k of the family (αAA^*) is the function $\mathcal{T}_k(\alpha)$ defined by*

$$(2.38)\qquad \mathcal{T}_k(\alpha) = E_k(\alpha) + \mathcal{R}_k(\alpha).$$

For $\alpha=\overline{\alpha}$, the value $\mathcal{T}_k(\overline{\alpha})$ is the total error up to the step k of the method $(\overline{\alpha}AA^*)$.

Now, we are able to establish in the following section the main results of the paper.

3. Convergence and numerical stability of the methods (αAA^*).

Theorem 3.1. *If:*

(AH_1). *The problem* (1.1), (1.2) *has a unique continuous solution with continuous derivatives up to the order $p+4$ for $t\in[t_0, T]$;*

(AH_2). *The function $f(t, x, y)$ is continuous together with its first order partial derivatives with respect to x and y, and there exists a positive constant L such that*

$$\left\|\frac{\partial f^i}{\partial x^j}(t, x, y)\right\| \leq \frac{L}{n},\quad \left\|\frac{\partial f^i}{\partial y^j}(t, x, y)\right\| \leq \frac{L}{n},\quad i, j = \overline{1, n},$$

for every $(t, x, y) \in J\times R^{2n}$, and

(AH_3). *The conditions (CA) and (CA^*) are satisfied,*

then the methods (αAA^*) are convergent. In addition, the α-error up to the step k of the family (αAA^*) is of the order $O(h^{p+1})$.

Proof. The solution $x(t)$ of the problem (1.1), (1.2) verifies the relations.

$$(3.1)\qquad \begin{cases} \dot{x}(t_k) = \dot{x}(t_{k-1}) + h\displaystyle\sum_{s=0}^{p} A_s(\alpha)\ddot{x}(t_{k-s-\alpha}) + \rho_{p,k}(\alpha)\\ x(t_k) = x(t_{k-1}) + h\dot{x}(t_{k-1}) + h^2\displaystyle\sum_{s=0}^{p} \overline{A}_s(\alpha)\ddot{x}(t_{k-s-\alpha}) + \tau_{p,k}(\alpha), \end{cases}$$

$$(3.2)\qquad \begin{cases} \dot{x}(t_{k-\alpha}) = \dot{x}(t_{k-1}) + h\displaystyle\sum_{s=0}^{p} A^*_s(\alpha)\ddot{x}(t_{k-s-1-\alpha}) + \rho^*_{p,k}(\alpha)\\ x(t_{k-\alpha}) = x(t_{k-1}) + h(1-\alpha)\dot{x}(t_{k-1}) + h^2\displaystyle\sum_{s=0}^{p} \overline{A}^*_s(\alpha)\ddot{x}(t_{k-s-1-\alpha}) + \tau^*_{p,k}(\alpha), \end{cases}$$

$k=\overline{1, N_1}$, where (see [1])

$$(3.3)\qquad \begin{cases} \rho^{(\alpha)}_{p,k} = h^{p+2}P_{p+1}(\alpha)x^{(p+3)}(t_{k-\alpha}) + O(h^{p+3})\\ P^{(\alpha)}_{p+1} = \displaystyle\int_{\alpha-1}^{\alpha}\frac{t(t+1)\dots(t+p)}{(p+1)!}\,dt, \end{cases}$$

$$(3.4)\qquad \begin{cases} \tau_{p,k}(\alpha) = h^{p+3}Q_{p+1}(\alpha)x^{(p+3)}(t_{k-\alpha}) + O(h^{p+4})\\ Q_{p+1}(\alpha) = \displaystyle\int_{\alpha-1}^{\alpha}\frac{(\alpha-t)t(t+1)\dots(t+p)}{(p+1)!}\,dt. \end{cases}$$

$$
(3.5)\qquad \begin{cases} \rho^*_{p,k}(\alpha) = h^{p+2} P^*_{p+1}(\alpha) x^{(p+3)}(t_{k-1-\alpha}) + O(h^{p+3}) \\ P^*_{p+1}(\alpha) = \displaystyle\int_{\alpha-1}^{0} \frac{(t+1)(t+2)\dots(t+p+1)}{(p+1)!}\, dt, \end{cases}
$$

$$
(3.6)\qquad \begin{cases} \tau^*_{p,k}(\alpha) = h^{p+3} Q^*_{p+1}(\alpha) x^{(p+3)}(t_{k-1-\alpha}) + O(h^{p+4}) \\ Q^*_{p+1}(\alpha) = \displaystyle\int_{0}^{\alpha-1} \frac{t(t+1)\dots(t+p+1)}{(p+1)!}\, dt. \end{cases}
$$

In view of the hypothesis (AH_3) it follows that the functions

$$
\varepsilon_k(\alpha) = x(t_k) - u_k(\alpha), \quad \dot\varepsilon_k(\alpha) = \dot x(t_k) - \dot u_k(\alpha)
$$

$$
\varepsilon_{k-\alpha}(\alpha) = x(t_{k-\alpha}) - u_{k-\alpha}(\alpha), \quad \dot\varepsilon_{k-\alpha}(\alpha) = \dot x(t_{k-\alpha}) - \dot u_{k-\alpha}(\alpha),\ k = \overline{1, N_1},
$$

verify the relations

$$
(3.7)\qquad \begin{aligned} &\dot\varepsilon_s(\alpha) = \dot\varepsilon_{s-1}(\alpha) + F_{p,s}(\alpha)\,; \quad \varepsilon_s(\alpha) = \varepsilon_{s-1}(\alpha) + h\,\dot\varepsilon_{s-1}(\alpha) + G_{p,s}(\alpha)\,; \\ &\dot\varepsilon_{s-\alpha}(\alpha) = \dot\varepsilon_{s-1}(\alpha) + \overline{F}_{p,s}(\alpha)\,; \quad \varepsilon_{s-\alpha}(\alpha) = \varepsilon_{s-1}(\alpha) + (1-\alpha)h\,\dot\varepsilon_{s-1}(\alpha) + \\ &\qquad + \overline{G}_{p,s}(\alpha),\ k = \overline{1, N_1}, \end{aligned}
$$

where

$$
(3.8)\qquad \begin{aligned} F_{p,k}(\alpha) &= h \sum_{s=0}^{p} A_s(\alpha)\ddot\varepsilon_{k-s-\alpha}(\alpha) + \rho_{p,k}(\alpha), \\ G_{p,k}(\alpha) &= h^2 \sum_{s=0}^{p} \overline{A}_s(\alpha)\ddot\varepsilon_{k-s-\alpha}(\alpha) + \tau_{p,k}(\alpha), \\ \overline{F}_{p,k}(\alpha) &= h \sum_{s=0}^{p} A^*_s(\alpha)\ddot\varepsilon_{k-s-1-\alpha}(\alpha) + \rho^*_{p,k}(\alpha), \\ \overline{G}_{p,k}(\alpha) &= h^2 \sum_{s=0}^{p} \overline{A}^*_s(\alpha)\ddot\varepsilon_{k-s-1-\alpha}(\alpha) + \tau^*_{p,k}(\alpha),\ k = \overline{1, N_1}, \end{aligned}
$$

and

$$
(3.9)\quad \ddot\varepsilon_{j-\alpha}(\alpha) = f(t_{j-\alpha}, x(t_{j-\alpha}), \dot x(t_{j-\alpha})) - f(t_{j-\alpha}, u_{j-\alpha}(\alpha), \dot u_{j-\alpha}(\alpha)),\ j = \overline{-(p+1), N_1}.
$$

Now, we shall show that the systems of functions $\rho_{p,k}(\alpha)$, $\tau_{p,k}(\alpha)$, $\rho^*_{p,k}(\alpha)$ $\tau^*_{p,k}(\alpha)$, $\varepsilon_k(\alpha)$, $\dot\varepsilon_k(\alpha)$, $\varepsilon_{j-\alpha}(\alpha)$, $\dot\varepsilon_{j-\alpha}(\alpha)$, $\ddot\varepsilon_{j-\alpha}(\alpha)$, $A_s(\alpha)$, $\overline{A}_s(\alpha)$, $A^*_s(\alpha)$, $\overline{A}^*_s(\alpha)$, $k = \overline{1, N_1}$, $j = \overline{-(p+1), N_1}$, $s = \overline{0, p}$ satisfy the conditions of Lemma 2.1.

Indeed, in accordance with the hypothesis (Ah_1), it results that there are real positive functions $M_\rho(\alpha)$, $M_\tau(\alpha)$ and a positive constant C such, that

$$
(3.10)\qquad \begin{aligned} \rho_{p,k}(\alpha) &= h^{p+2} P_{p+1}(\alpha) x^{(p+3)}(t_{k-\alpha}) + h^{p+3} M_\rho(\alpha), \\ \tau_{p,k}(\alpha) &= h^{p+3} Q_{p+1}(\alpha) x^{(p+3)}(t_{k-\alpha}) + h^{p+4} M_\tau(\alpha), \\ \rho^*_{p,k}(\alpha) &= h^{p+2} P^*_{p+1}(\alpha) x^{(p+3)}(t_{k-1-\alpha}) + h^{p+3} M_\rho(\alpha), \\ \tau^*_{p,k}(\alpha) &= h^{p+3} Q^*_{p+1}(\alpha) x^{(p+3)}(t_{k-1-\alpha}) + h^{p+4} M_\tau(\alpha),\ k = \overline{1, N_1}, \\ \| x^{(p+3)}(t) \| &\leqslant C,\ t \in [t_0, T]. \end{aligned}
$$

From these relations, we obtain the estimates

$$
(3.11)\qquad \max(\|\rho_{p,k}(\alpha)\|, \|\tau_{p,k}(\alpha)\|, \|\rho^*_{p,k}(\alpha)\|, \|\tau^*_{p,k}(\alpha)\|) \leqslant h^{p+2} R_p(\alpha),\ k = \overline{1, N_1}
$$

where

$$
(3.12)\qquad \begin{aligned} R_p(\alpha) = \max\,(& C|P_{p+1}(\alpha)| + (T-t_0)M_\rho(\alpha),\ C|P^*_{p+1}(\alpha)| + (T-t_0)M_\rho(\alpha), \\ & (T-t_0)(C|Q_{p+1}(\alpha)| + (T-t_0)M_\tau(\alpha)),\ (T-t_0)(C|Q^*_{p+1}(\alpha)| + (T-t_0)M_\tau(\alpha))). \end{aligned}
$$

Let us consider the scalar positive function

$$
(3.13)\qquad G_p(\alpha) = \max\left(\sum_{s=0}^{p} |A_s(\alpha)|,\ \sum_{s=0}^{p} |\overline{A}_s(\alpha)|,\ \sum_{s=0}^{p} |A^*_s(\alpha)|,\ \sum_{s=0}^{p} |\overline{A}^*_s(\alpha)|\right),
$$

defined for $\alpha \in R$.

The hypothesis (AH_2) and the relation (3.9) lead to the estimates

$$
(3.14)\qquad \| \ddot\varepsilon_{j-\alpha}(\alpha) \| \leqslant L(\| \varepsilon_{j-\alpha}(\alpha) \| + \| \dot\varepsilon_{j-\alpha}(\alpha) \|),\ j = \overline{-(p+1), N_1}.
$$

By hypothesis (AH_3), we have $\varepsilon_{s-\alpha}(\alpha) = 0$, $\dot\varepsilon_{s-\alpha}(\alpha) = 0$, $s = \overline{-(p+1), 0}$. Hence we obtain

$$
(3.15)\qquad \| \varepsilon_{s-\alpha}(\alpha) \| + \| \dot\varepsilon_{s-\alpha}(\alpha) \| = 0,\ s = \overline{-(p+1), 0}.
$$

The relations (3.14), (3.11), (3.13) and (3.15) show that the hypothesis (1) of Lemma 2.1 is satisfied with $R(\alpha) = h^{p+2}R_p(\alpha)$, $G(\alpha) = G_p(\alpha)$, $E(\alpha) = 0$, L defined by (AH_2) and $K = T - t_0$.

Next, by summing from $s = 1$ to $s = k$ each of the relations (3.7) and taking into account that $\varepsilon_0(\alpha) = \dot\varepsilon_0(\alpha) = 0$, we get estimations

$$
(3.16)\qquad \begin{aligned} &\| \dot\varepsilon_k(\alpha) \| \leqslant \sum_{k=1}^{k} \| F_{p,s}(\alpha) \|,\ \| \varepsilon_k(\alpha) \| \leqslant h \sum_{s=1}^{k-1} \| \dot\varepsilon_s(\alpha) \| + \sum_{s=1}^{k} \| G_{p,s}(\alpha) \|, \\ &\| \dot\varepsilon_{k-\alpha}(\alpha) \| \leqslant \| \dot\varepsilon_{k-1}(\alpha) \| + \| \overline{F}_{p,k}(\alpha) \|, \\ &\| \varepsilon_{k-\alpha}(\alpha) \| \leqslant |1-\alpha|\, h \sum_{s=1}^{k-1} \| \dot\varepsilon_s(\alpha) \| + \sum_{s=1}^{k} \| \overline{G}_{p,s}(\alpha) \|,\ k = \overline{1, N_1}. \end{aligned}
$$

It is clear that

$$
(3.17)\qquad kh \leqslant T - t_0.
$$

These relations show that the last hypothesis of the Lemma 2.1 is also satisfied. Therefore we have

$$
(3.18)\qquad E_k(\alpha) \leqslant (1 + hM(\alpha))^k \overline{R}_p(\alpha) h^{p+1} \leqslant e^{(T-t_0)M(\alpha)}\, \overline{R}_p(\alpha) h^{p+1},\ k = \overline{1, N_1},
$$

where

$$
M(\alpha) = q(\alpha) L G_p(\alpha)(2p + 1 + T - t_0),\ q(\alpha) = \max\,(1, |1-\alpha|),
$$

$$
\overline{R}_p(\alpha) = 2q(\alpha)(T - t_0)(T - t_0 + 2)\, R_p(\alpha).
$$

The inequality (3.18) completes the proof.

Taking into account the Remarks 2.1 and 2.2, we have to discuss the numerical stability of the methods (αAA^*) with respect to initial and rounding errors. An answer to this problem is given by the following.

Theorem 3.1. *Let* $\tilde{u}_k(\alpha)$, $\tilde{\dot{u}}_k(\alpha)$, $\tilde{u}_{k-\alpha}(\alpha)$, $\tilde{\dot{u}}_{k-\alpha}(\alpha)$ $k=\overline{1, N_1}$, *be the values given by the method* (αAA^*) *with the starting set* $\mathcal{M}^{(1)}_{AA^*}(\alpha)$. *If the hypotheses of the Theorem 3.1. hold and if there is a positive function* $R^*_p(\alpha)$ *defined on* R *such that,*

$$\max(\|\tilde{\rho}_{p,k}(\alpha)\|, \|\tilde{\tau}_{p,k}(\alpha)\|, \|\tilde{\rho}^*_{p,k}(\alpha)\|, \|\tilde{\tau}^*_{p,k}(\alpha)\| ; \ k=1, N_1) \leqslant R^*_p(\alpha), \tag{3.19}$$

where $\tilde{\rho}_{p,k}(\alpha)$, $\tilde{\tau}_{p,k}(\alpha)$ and $\tilde{\rho}^*_{p,k}(\alpha)$, $\tilde{\tau}^*_{p,k}(\alpha)$ are the terms appearing in (2.32) and (2.33) respectively, then for every $\alpha \in R/Z$ the method (αAA^*) is stable with respect to the initial and rounding errors.

Proof. The proof, is similar to that of the Theorem 3.1. One shows that the sequences of vectors $(\tilde{\rho}_{p,k}(\alpha))$, $(\tilde{\tau}_{p,k}(\alpha))$, $(\tilde{\rho}^*_{p,k}(\alpha))$, $(\tilde{\tau}_{p,k}(\alpha))$, $(\dot{u}_k(\alpha) - \tilde{\dot{u}}_k(\alpha))$, $(u_k(\alpha) - \tilde{u}_k(\alpha))$, $(\dot{u}_{k-\alpha}(\alpha) - \tilde{\dot{u}}_{k-\alpha}(\alpha))$, $(u_{k-\alpha}(\alpha) - \tilde{u}_{k-\alpha}(\alpha))$ $k=\overline{1, N_1}$; $(\dot{u}_{j-\alpha}(\alpha) - \tilde{\dot{u}}_{j-\alpha}(\alpha))$, $(u_{j-\alpha}(\alpha) - \tilde{u}_{j-\alpha}(\alpha))$, $(f(t_{j-\alpha}, u_{j-\alpha}(\alpha), \dot{u}_{j-\alpha}(\alpha)) - f(t_{j-\alpha}, \tilde{u}_{j-\alpha}(\alpha), \tilde{\dot{u}}_{j-\alpha}(\alpha)))$ $j = \overline{-(p+1), N_1}$; $A_s(\alpha)$, $\overline{A}_s(\alpha)$, $A^*_s(\alpha)$, $\overline{A}^*_s(\alpha)$ $s=0, p$. satisfy the conditions of the Lemma 2.1 with $R(\alpha) = R^*_p(\alpha)$, $E(\alpha) = \mathcal{J}(\alpha)$ and $L, K, G(\alpha)$ entering in the Theorem 3.1. In accordance with (2.11) we get

$$\mathcal{R}_k(\alpha) \leqslant (1 + hM(\alpha))^{k+1}\mathcal{J}(\alpha) + (1 + hM(\alpha))\frac{\overline{R}^*_p(\alpha)}{h}, \ k = \overline{1, N_1}, \tag{3.20}$$

where

$$M(\alpha) = q(\alpha) LG_p(\alpha)(2p+1+T-t_0), \quad \overline{R}^{**}_p(\alpha) = 2q(\alpha)(T-t_0)(T-t_0+2)R^*_p(\alpha).$$

Remark 3.4. By Theorem 3.2 we obtain the next estimation for the total α-error up the step k of the family (αAA^*).

$$\mathcal{T}_k(\alpha) \leqslant (\mathcal{J}(\alpha) + h^{p+1}(\overline{R}_p(\alpha) + \overline{R}^*_p(\alpha)h^{-p-2}))\, e^{(T-t_0)M(\alpha)}, \ k=\overline{1, N_1}. \tag{3.21}$$

We note that if $\mathcal{J}(\alpha)$ is of the order $O(h^{p+1})$ and $\overline{R}^*_p(\alpha)$ is of the order $O(h^{p+2})$, then $\mathcal{T}_k(\alpha)$ has the order $O(h^{p+1})$.

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WHY SHANNON'S ENTROPY: A FURTHER REASON

BY

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1. The definition of conditional entropy. Additive and subadditive entropies were characterized in [1] (here the notation of [2] will be adopted). For any finite discrete complete probability distribution $p=(p_1, p_2, \dots, p_n) \in \Gamma_n := \{(q_1, q_2, \dots, q_n) : q_i > 0, \sum_{i=1}^{n} q_i = 1\}$ $(n=2, 3, \dots)$ these entropies have the representation

$$\mathcal{H}(\mathcal{P}) := H_n(p_1, p_2, \dots, p_n) = a\, H^s_n(p_1, p_2, \dots, p_n) + b \log \mathcal{N}(\mathcal{P}), \tag{1}$$

where a and b are non-negative constants, $\mathcal{H}(\mathcal{P})$ denotes the number of non-zero componets in the distribution $\mathcal{P}$ and H^s_n is the Shannon's entropy

$$H^s_n(p_1, p_2, \dots, p_n) = -\sum_{i=1}^{n} p_i \log p_i.$$

In order to define the conditional entropy for the additive and subadditive entropies (1), we shall consider the joint distribution $\mathcal{P} \times Q = \{\pi_{ij} ; i = 1, 2, \dots, m ; j = 1, 2, \dots, n\} \in \Gamma_{mn}$ together with its marginals $\pi = \left\{\sum_{j=1}^{n} \pi_{ij} p_i\right\} \in \Gamma_m$ and $Q = \left\{\sum_{i=1}^{m} \pi_{ij} = q_j\right\} \in \Gamma_n$. The conditional distributions $(q_{i1}, q_{i2}, \dots, q_{in}) \in \Gamma_n$ $(i=1, 2, \dots, m)$, where $\pi_{ij} = p_i q_{ij}$ will also be useful.

As is well known, Shannon's entropy satisfies the relationship

$$\mathcal{H}(\mathcal{P} \times Q) = \mathcal{H}(\mathcal{P}) + \mathcal{H}(Q/\mathcal{P}),$$

which can also be used to define the conditional entropy (see [2]). This we *shall* do with the entropy (1).

A simple calculation shows that for (1), one has

$$\mathcal{H}(Q/\mathcal{P}) = \mathcal{H}(\mathcal{P} \times Q) - \mathcal{H}(\mathcal{P}) =$$

$$= -a \sum_{i=1}^{m} p_i \sum_{j=1}^{n} q_{ij} \log q_{ij} + b \log \mathcal{N}(\mathcal{P} \times Q)/\mathcal{P}. \tag{2}$$

This relations supports the following

Definition. *The conditional entropy* $\mathcal{H}(Q/\mathcal{P})$ *for the additive and subadditive entropies (1) is defined as being equal to the r. h. s. of (2) for every distribution* $\mathcal{P} \times Q \in \Gamma_{mn}$ $(m, n=2, 3, \dots)$.