property (vii). A trivial computation reveals that (vii) is equivalent, in the case of Hartley's conditional entropy (3), to the statement.

$$\mathcal{H}(\mathcal{P} \times \mathcal{Q} \times \mathcal{R})\,\mathcal{H}(\mathcal{P}) \leqslant \mathcal{H}(\mathcal{P} \times \mathcal{Q})\,\mathcal{H}(\mathcal{P} \times \mathcal{R}). \tag{5}$$

It would not be hard to list cases in which (5) holds. For instance, whenever $\mathcal{H}(\mathcal{P}) = 1$, the inequility (5) is true. However (5) is not true in general, as the following example shows.

Let $\mathcal{P} \times \mathcal{Q} \times \mathcal{R} \in \Gamma_{mnr}$ be such that the only positive entries are π_{111}, π_{121}, π_{112}, π_{122} and π_{211}. Thus $\mathcal{H}(\mathcal{P} \times \mathcal{Q} \times \mathcal{R}) = 5$, $\mathcal{H}(\mathcal{P} \times \mathcal{Q}) = \mathcal{H}(\mathcal{P} \times \mathcal{R}) = 3$ and $\mathcal{H}(\mathcal{P}) = 2$; therefore

$$\mathcal{H}(\mathcal{P} \times \mathcal{Q} \times \mathcal{R})\,\mathcal{H}(\mathcal{P}) = 10 > 9 = \mathcal{H}(\mathcal{P} \times \mathcal{Q})\,\mathcal{H}(\mathcal{P} \times \mathcal{R}).$$

4. **Final remarks**. The properties (vi) and (vii) of Shannon's conditional entropy are not translated into analogous properties of Hartley's conditional entropy, and, as a consequence, of the conditional entropy (2), with $b > 0$ This has an important negative effect on the application of the additive and subadditive entropies (1), at least as long as one adopt the definition of conditional entropy used in this note. In fact the property (vii) is essential (see [3] or [4]) in order to define the entropy of a sequence of discrete random variables. This is no longer possible in view of our result. More important, it is even impossible to speak about an analogue of Mac Millan's theorem in connexion with the additive and subadditive entropies (1), if $b > 0$. Thus one is inevitably lead to the choice $b = 0$, thus justifying what has always been done in the approach to Statistical Mechanics through Information Theory. There it is entirely natural to use the additive and subadditive entropies (see [5] and [6]), but nevertheless one has always chosen $b = 0$.

It is an open question whether a different definition of conditional entropy would yield also the crucial properties (vi) and (vii).

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REMARKS ON DISTANCES IN GRAPHS

By

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It is standard practice to define the distance $d(x, y)$ between two points in a connected graph as the length of the shortest path between x and y. In [2] the Boolean distance $b(x, y)$ is defined to be the set of all points on all paths from x to y, a path from x to y containing, by definition [1], pairwise defferent nodes only.

This distance satisfies the following properties :

1) $b(x, y) = \varnothing$ if and only if $x = y$;

2) $b(x, y) = b(y, x)$ for any points x, y ;

3) $b(x, y) \subset b(x, z) \cup b(z, y)$ for any three points x, y, z of G.

In this paper we first investigate certain connections between the numerical distance and the Boolean distance.

For a connected graph G having point set X with $|X| = n$ let $\rho(x, y) = |b(x, y)|$ for any $x, y \in X$. This function is a numerical metric on X. Indeed, $\rho(x, y) = 0$, if and only if $d(x, y) = \varnothing$ or $x = y$, $\rho(x, y) = \rho(y, x)$ and for any three points $x, y, z \in X$:

$$\rho(x, y) = |b(x, y)| \leqslant |b(x, z) \cup b(z, y)| \leqslant |b(x, z)| + |b(z, y)| = \\ = \rho(x, z) + \rho(z, y).$$

For any point $x \in X$, let $e(x) = \max\limits_{y \in X} \rho(x, y)$.

Definition 1. *A point $x_0 \in X$ will be called an s-center of G if $e(x_0) = \min\limits_{y \in X} e(x)$.*

If T is a tree then it is clear that $\rho(x, y)$ $(x \neq y)$ is greater by one than the numerical distance between x and y. Hence centers and s-centers as defined above coincide for trees. In Figure 1 the number next to each point x is equal to $e(x)$.

We derive $\min e(x) = 8$ and the s-centers are two points which lie in an acyclic block ; they are marked with small circles. The graph has a unique center ; it is adjacent to an s-center and is marked with a small square. From the definition it follows that s-centers are „near" the big blocks of the graph. However, this notion is not useful for 2-connected graphs, since if G is 2-connected, $b(x, y) = X$ for any $x \neq y$ and the s-radius of G equals $n = |X|$ and any point is an s-center. It is well known [1] that the centers of a connected graph lie in the same block. This property is also valid for s-centers.

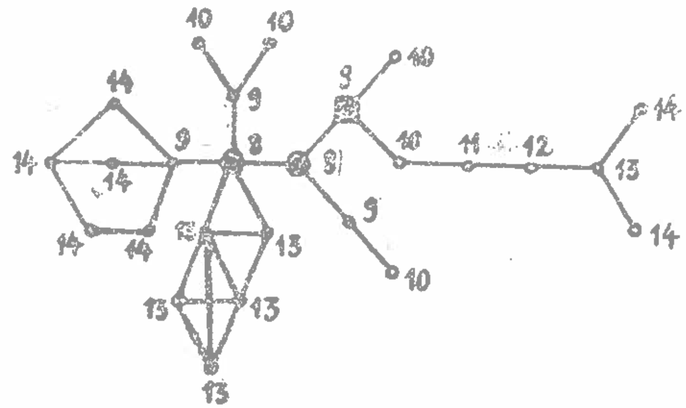

Fig. 1

Theorem 1. *For any connected graph G all s-centers lie in the same block B of G. Either every s-center is a cutpoint of G that belongs to B or every point of B is an s-center.*

Proof. If B is a block of G and $a, b, c \in B$ are three distinct point such that a, b are not cutpoints of G and c is a cutpoint of G, then $b(a,x) = b(b, x)$ and $b(a, x) \supset b(c, x)$ for any point $x \neq a, b$. Hence $e(a) = e(b)$ $e(a) \geq e(c)$. This implies that if $a \in B$, a is not a cutpoint of G and is an s-center, then all points $x \in B$ are s-centers. Suppose that it is not the case that all s-centers of G belong to the same block of G. Suppose that there are two cutpoints c_1, c_2 of G lying in different blocks: $c_1 \in B_1$, $c_2 \in B_2$, $c_1 \notin B_2$ and $c_2 \notin B_1$, such that $e(c_1) = e(c_2) = \min_{x \in X} e(x)$.

Let $d_1, d_2 \in X$ such that $e(c_1) = \rho(c_1, d_1)$ and $e(c_2) = \rho(c_2, d_2)$. Suppose further that $c(c_1, c_2) = \{f_1, f_2, \ldots, f_p\}$ with $p \geq 1$, i. e. all paths $(c_1, \ldots, c_2)$ in G pass through cutpoints $f_1, \ldots, f_p$ in this order, or $(c_1, \ldots, c_2) = (c_1, \ldots, f_1, \ldots, f_2, \ldots, f_p, \ldots, c_2)$.

It follows that $d_2 \notin B_2$ since if $d_2 \in B_2$ we have $b(c_2, d_2)$ is a proper subset of $b(c_1, d_2)$, and hence $\rho(c_1, d_2) > \rho(c_2, d_2) = e(c_2) = e(c_1)$. But $e(c_1) \geq \rho(c_1, d_2)$ by definition.

It can similarly be shown that $d_1 \notin B_1$. It follows that $c(c_1, d_1) = \{h_1, \ldots, h_q\}$ and $c(c_2, d_2) = \{g_1, \ldots, g_r\}$ where $h_1, \ldots, h_q$ and $g_1, \ldots, g_r$ are cut points of G, q, $r \geq 1$.

If $\{f_1, \ldots, f_p\} \cap \{h_1, \ldots, h_q\} = \emptyset$ then $b(c_1, d_1)$ is a proper subset of $b(c_2, d_1)$ and hence $e(c_2) > e(c_1)$, contrary to the hypothesis. Hence $\{f_1, \ldots, f_p\} \cap \{h_1, \ldots, h_q\} \neq \emptyset$ and also $\{f_1, \ldots, f_p\} \cap \{g_1, \ldots, g_r\} \neq \emptyset$. If for example $f_1 = h_s$ with $s \geq 2$, then there exists a cycle passing through different blocks, which contradicts the maximality of the blocks. It follows that $f_1 = h_1$, $f_2 = h_2, \ldots, f_u = h_u$ and $f_p = g_1$, $f_{p-1} = g_2, \ldots f_v = g_{p+1-v}$. Consider two cases: i) $u > v$ and ii) $u \leq v$.

i) If $u > v$, let t be an index satisfying $v < t \leq u$. Suppose that $e(f_t) = \rho(f_t, d_3)$ and $c(f_t, d_3) = \{k_1, k_2, \ldots, k_s\}$.

If $\{f_1, \ldots, f_q\} \cap \{k_1, \ldots = k_s\} = \emptyset$ then $\rho(f_t, d_3) \leq \rho(f_t, d_1)$ since otherwise the equality $e(c_1) = \rho(c_1, d_1)$ does not hold.

But this implies $e(f_t) \leq \rho(f_t, d_1) < e(c_1)$, contrary to the assumption that c_1 is an s-center. Hence it follows that $\{f_1, \ldots, f_p\} \cap \{k_1, \ldots, k_s\} \neq \emptyset$ and if w denotes the number of cutpoints of $\{f_1, \ldots, f_p\} \cap \{k_1, \ldots, k_s\}$, then $k_1, \ldots, k_w$ $(w \geq 1)$ are identical with either:

i1) the first w cutpoints of $f_{t-1}, f_{t-2}, \ldots, f_1, c_1$, or

i2) the first w cutpoints of $f_{t+1}, f_{t+2}, \ldots, c_2$.

In case i1) $\rho(f_t, d_3) \leq \rho(f_t, d_2)$, since otherwise we have

$$e(c_2) = \rho(c_2, d_2) = \rho(c_2, f_t) + \rho(f_t, d_2) - 1 < \rho(c_2, f_t) + \rho(f_t, d_3) - 1 = \rho(c_2, d_3),$$

contrary to the definition of $e(c_2)$.

This implies that $e(f_t) = \rho(f_t, d_3) \leq \rho(f_t, d_2) < \rho(c_2, d_2) = e(c_2)$, and thus contradicts the equality $e(c_2) = \min_{x \in X} e(x)$.

A similar argument shows that i2) does not hold.

ii) If $u \leq v$, in view of the fact that $\rho(c_1, d_2) \leq \rho(c_1, d_1)$, it follows that:

$$\rho(c_1, f_u) + \rho(f_u, f_v) + \rho(f_v, d_2) - 2 \leq \rho(c_1, f_u) + \rho(f_u, d_1) - 1, \text{ or}$$

$$(1) \qquad \rho(f_u, f_v) + \rho(f_v, d_2) \leq \rho(f_v, d_1) + 1.$$

And since $\rho(c_2, d_1) \leq \rho(c_2, d_2)$ one can show that:

$$(2) \qquad \rho(f_u, f_v) + \rho(f_u, d_1) \leq \rho(f_v, d_2) + 1.$$

Addition of (1) and (2) yields $\rho(f_u, f_v) \leq 1$. If $u \neq v$ we have $f_u \neq f_v$ and hence $\rho(f_u, f_v) \geq 2$. Thus $u = v$ and if $t = u = v$ then $1 \leq t \leq p$ and $e(c_1) = e(c_2)$ implies that $\rho(c_1, f_t) = \rho(f_t, c_2)$ and $\rho(f_t, d_1) = \rho(f_t, d_2)$. A repetition of the argument used in case i) can be used to show that $e(f_t) < e(c_1) = e(c_2)$, thus contradicting the hypothesis.

It follows that all s-centers of G belong to the same block of G.

Definition 2. *A Boolean geodesic between two points x and y of a connected graph G is defined to be the set $S(L)$ of all points on a path L from x to y satisfying the property that there is no other path L_1 from x to y, such that $S(L_1)$ is a proper subset of $S(L)$.*

The following result characterizes those connected graphs G such that between any two points of G there is a unique Boolean geodesic, in terms of H u s i m i trees, i. e., of those connected graphs whose all blocks are complete subgraphs.

Theorem 2. *A connected graph G has the property that between any two points of G there is a unique Boolean geodesic, if and only if G is a Husimi tree.*

Proof. It is obvious that between any two points of a H u s i m i tree there exists a unique Boolean geodesic. Indeed, if x and y belong to the same block of a Husimi tree, then the Boolean geodesic between them is $\{x, y\}$; otherwise, this unique Boolean geodesic contains x, y and the cutpoints lying on the shortest path connecting them.

Conversely, let G be a connected graph such that there is a block B of G and two points $x, y \in G$ which are not adjacent. Because B is 2-connected, there is a cycle:

$$C = [x,\ x_1,\ x_2, \dots, x_k,\ y,\ y_1,\ y_2, \dots, y_p,\ x]$$

containing x and y and all points $x_1, \dots, x_k,\ y_1, \dots, y_p$ are in B and $k,\ p \geq 1$. There is a path P_1 from x to y such that $S(P_1)$ is a Boolean geodesic and $S(P_1) \subset \{x,\ x_1, \dots, x_k,\ y\}$ and there is another path P_2 from x to y such that $S(P_2)$ is a Boolean geodesic and $S(P_2) \subset \{x,\ y_1, \dots, y_p,\ y\}$.

Because x and y are not adjacent in G, it follows that $S(P_1)$ and $S(P_2)$ are two different Boolean geodesics between x and y and the proof is complete.

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A MEASURE FOR BOOLEAN ALGEBRAS OF CENTRAL TENDENCY

By

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The values which are to be averaged by a measure of central tendency like the mean or the median are usually real numbers. We define a measure of central tendency β for elements of a Boolean algebra. This measure has properties analogous to those of the classical statistics.

The notation and terminology of [2] are followed herein. We recall that a Boolean function $f: B^n \to B$ is built up from variables and constants of B by superpositions of the basic operations $\vee$ (disjunction), $\cdot$(conjunction) and $'$ (negation). When constants are not allowed to occur in the construction of f (excepting 0 and 1), f is called a simple Boolean function. A Boolean function f is determined by ist values $f(\alpha_1, \dots, \alpha_n)$ on the 2^n points $(\alpha_1, \dots, \alpha_n) \in \{0,1\}^n \subseteq B^n$. Simple Boolean functions are characterized among Boolean functions by the property $f(\alpha_1, \dots, \alpha_n) \in \{0,1\}$ for all $(\alpha_1, \dots, \alpha_n) \in \{0,1\}$.

Theorem 1. *Let B be a Boolean algebra. The only isotone symmetric simple Boolean functions from B^n to B are: the constant functions 0,1 and the fundamental symmetric Boolean functions*

$$\beta_k(x_1, \dots, x_n) = \bigvee_{i_1, \dots, i_k} x_{i_1} \dots x_{i_k}, \tag{1}$$

where $1 \leq k \leq n$ and $\bigvee_{i_1, \dots, i_k}$ means that the disjunction is extended over all $\binom{n}{k}$subsets $\{i_1, \dots, i_k\}$ of distinct indices from the set $\{1, \dots, n\}$.

Proof. I. Clearly the constant functions 0,1 and the functions of the form (1) are isotone symmetric simple Boolean functions. Conversely, let $\beta: B^n \to B$ be a simple Boolean function which is also isotone and symmetric. Recall ([2], Theorem 11.3) that a Boolean function β is isotone if and only if it can be written in the form

$$\beta(x_1, \dots, x_n) = a_0 \vee \bigvee_{p=1}^{n} \bigvee_{i_1, \dots, i_p} a_{i_1, \dots, i_p}\, x_{i_1} \dots x_{i_p}. \tag{2}$$

Since β is symmetric, the coefficients $a_{i_1, \dots, i_p}$ are the same for each p-element set, say $a_{i_1, \dots, i_p} = a_p (p = 1, \dots, n)$. Thus (2) becomes

$$\beta(x_1, \dots, x_n) = a_0 \vee \bigvee_{p=1}^{n} a_p \bigvee_{i_1, \dots, i_p} x_{i_1} \dots x_{i_p}. \tag{3}$$

But $a_0 = \beta(0, \dots, 0) \in \{0,1\}$ because β is a simple Boolean function. For $a_0 = 1$, we obtain the constant function 1. Now suppose $a_0 = 0$, then

$$\beta(x_1, \dots, x_n) = \bigvee_{p=1}^{n} a_p \bigvee_{i_1, \dots, i_p} x_{i_1} \dots x_{i_p}. \tag{4}$$