Conversely, let G be a connected graph such that there is a block B of G and two points $x, y \in G$ which are not adjacent. Because B is 2-connected, there is a cycle:

$$C=[x,\ x_1,\ x_2, \ldots, x_k,\ y,\ y_1,\ y_2, \ldots, y_p,\ x]$$

containing x and y and all points $x_1, \ldots, x_k,\ y_1, \ldots, y_p$ are in B and $k,\ p \geq 1$. There is a path P_1 from x to y such that $S(P_1)$ is a Boolean geodesic and $S(P_1) \subset \{x,\ x_1, \ldots, x_k,\ y\}$ and there is another path P_2 from x to y such that $S(P_2)$ is a Boolean geodesic and $S(P_2) \subset \{x,\ y_1, \ldots, y_p,\ y\}$.

Because x and y are not adjacent in G, it follows that $S(P_1)$ and $S(P_2)$ are two different Boolean geodesics between x and y and the proof is complete.

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A MEASURE FOR BOOLEAN ALGEBRAS OF CENTRAL TENDENCY

By

R. A. MELTER and SERGIU RUDEANU

The values which are to be averaged by a measure of central tendency like the mean or the median are usually real numbers. We define a measure of central tendency β for elements of a Boolean algebra. This measure has properties analogous to those of the classical statistics.

The notation and terminology of [2] are followed herein. We recall that a Boolean function $f: B^n \to B$ is built up from variables and constants of B by superpositions of the basic operations $\vee$ (disjunction), $\cdot$ (conjunction) and $'$ (negation). When constants are not allowed to occur in the construction of f (excepting 0 and 1), f is called a simple Boolean function. A Boolean function f is determined by ist values $f(\alpha_1, \ldots, \alpha_n)$ on the 2^n points $(\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n \subseteq B^n$. Simple Boolean functions are characterized among Boolean functions by the property $f(\alpha_1, \ldots, \alpha_n) \in \{0,1\}$ for all $(\alpha_1, \ldots, \alpha_n) \in \{0,1\}$.

Theorem 1. *Let B be a Boolean algebra. The only isotone symmetric simple Boolean functions from B^n to B are: the constant functions 0,1 and the fundamental symmetric Boolean functions*

$$\beta_k(x_1, \ldots, x_n) = \bigvee_{i_1, \ldots, i_k} x_{i_1} \ldots x_{i_k}, \tag{1}$$

where $1 \leqslant k \leqslant n$ and $\bigvee_{i_1, \ldots, i_k}$ means that the disjunction is extended over all $\binom{n}{k}$ subsets $\{i_1, \ldots, i_k\}$ of distinct indices from the set $\{1, \ldots, n\}$.

Proof. 1. Clearly the constant functions 0,1 and the functions of the form (1) are isotone symmetric simple Boolean functions. Conversely, let $\beta: B^n \to B$ be a simple Boolean function which is also isotone and symmetric. Recall ([2], Theorem 11.3) that a Boolean function β is isotone if and only if it can be written in the form

$$\beta(x_1, \ldots, x_n) = a_0 \vee \bigvee_{p=1}^{n} \bigvee_{i_1, \ldots, i_p} a_{i_1, \ldots, i_p} x_{i_1} \ldots x_{i_p}. \tag{2}$$

Since β is symmetric, the coefficients $a_{i_1, \ldots, i_p}$ are the same for each p-element set, say $a_{i_1, \ldots, i_p} = a_p (p=1, \ldots, n)$. Thus (2) becomes

$$\beta(x_1, \ldots, x_n) = a_0 \vee \bigvee_{p=1}^{n} a_p \bigvee_{i_1, \ldots, i_p} x_{i_1} \ldots x_{i_p}. \tag{3}$$

But $a_0 = \beta(0, \ldots, 0) \in \{0,1\}$ because β is a simple Boolean function. For $a_0 = 1$, we obtain the constant function 1. Now suppose $a_0 = 0$, then

$$\beta(x_1, \ldots, x_n) = \bigvee_{p=1}^{n} a_p \bigvee_{i_1, \ldots, i_p} x_{i_1} \ldots x_{i_p}. \tag{4}$$

We next show (II) that without loss of generality one may suppose that

$$a_p \in \{0,1\} \qquad (p=1, \ldots, n). \tag{5}$$

II. Equation (4) implies

$$\beta(\underset{1}{1}, \ldots, \underset{i}{1}, \underset{i+1}{0}, \ldots, \underset{n}{0}) = \bigvee_{p=1}^{i} a_p \quad (i=1, \ldots, n) \tag{6}$$

and, conversely, it follows from the Verification Theorem ([1]; see also [2], Theorem 2.13) and the symmetry of β that relations (6) imply the identity (4). It therefore remains to prove that equations (6) have a solution $(a_1, \ldots, a_n) \in \{0,1\}^n$. But

$$\beta(\underset{1}{1}, \ldots, \underset{i}{1}, \underset{i+1}{0}, \ldots, \underset{n}{0}) \in \{0,1\} \quad (i=1, \ldots, n) \tag{7}$$

because β is a simple Boolean function. If $\beta(1, \ldots, 1, 0, \ldots)=0$ for all arguments in the left hand sides of (6), then the system (6) has the unique solution $\alpha_i=0 (i=1, \ldots, n)$. Otherwise let k be the first index such that

$$\beta(\underset{1}{1}, \ldots, \underset{k}{1}, \underset{k+1}{0}, \ldots, \underset{n}{0}) = 1;$$

the isotony of the function β implies

$$\beta(\underset{1}{1}, \ldots, \underset{p}{1}, \underset{p+1}{0}, \ldots, \underset{n}{0}) = 1 \quad (p=k, \ldots, n), \text{ and}$$

hence system (6) has the solution

$$a_1 = \ldots = a_{k-1}=0, \; a_{k+1} = \ldots = a_n = 1.$$

III. We have thus proved that β can be written in the form (4) where the a_p's fulfill (5). If $a_p=0 \; (p=1, \ldots, n)$ then β is the constant function 0. Otherwise let k be the first index such that $a_k=1$. Then

$$\beta(x_1, \ldots, x_n) = \bigvee_{i_1, \ldots, i_k} x_{i_1} \ldots x_{i_k} \vee \bigvee_{p=k+1}^{n} a_p \bigvee_{i_1, \ldots, i_p} x_{i_1} \ldots x_{i_p}. \tag{8}$$

But for every $p \geqslant k+1$

$$a_p \bigvee_{i_1, \ldots, i_p} x_{i_1} \ldots x_{i_p} \leqslant \bigvee_{i_1, \ldots, i_k} x_{i_1} \ldots x_{i_k},$$

and thus (8) reduces to (1).

Corollary 1. *Let B be a Boolean algebra. The only isotone symmetric simple Boolean functions β from B^n to B which fulfill*

$$x_1 \ldots x_n \leqslant \beta(x_1, \ldots, x_n) \leqslant x_1 \vee \ldots \vee x_n \tag{9}$$

are the fundamental symmetric Boolean functions $\beta(k=1, \ldots, n)$.

Corollary 2. *Let B be a Boolean algebra. The function*

$$\beta_{\left[\frac{n-1}{2}\right]+1} \tag{10}$$

(where $[x]$ is the greatest integer function) is the only isotone symmetric simple Boolean function β which fulfills the condition: If $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$ then

$$\beta(x_1, \ldots, x_n) = \begin{cases} x_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ x_{\frac{n}{2}+1} & \text{if } n \text{ is even.} \end{cases} \tag{11}$$

Proof. If $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$ then $\beta_k(x_1, \ldots, x_n) = x_{n-k+1}$, and it is easily checked that

$$k = \left[\frac{n-1}{2}\right] + 1 \Leftrightarrow n-k+1 = \begin{cases} \dfrac{n+1}{2} & \text{if } n \text{ is odd} \\ \dfrac{n}{2}+1 & \text{if } n \text{ is even.} \end{cases}$$

Any function of the form $\beta_{\left[\frac{n-1}{2}\right]+1}$ will be called a *measure of central tendency* for B. The first five functions of this kind are:

$$\beta(x_1) = x_1,$$

$$\beta(x_1, x_2) = x_1 \vee x_2,$$

$$\beta(x_1, x_2, x_3) = x_1x_2 \vee x_1x_3 \vee x_2x_3,$$

$$\beta(x_1, x_2, x_3, x_4) = x_1x_2 \vee x_1x_3 \vee x_1x_4 \vee x_2x_3 \vee x_2x_4 \vee x_3x_4,$$

$$\beta(x_1, x_2, x_3, x_4, x_5) = x_1x_2x_3 \vee x_1x_2x_4 \vee x_1x_2x_5 \vee x_1x_3x_4 \vee$$
$$\vee x_1x_3x_5 \vee x_1x_4x_5 \vee x_2x_3x_4 \vee x_2x_3x_5 \vee x_2x_4x_5 \vee x_3x_4x_5.$$

By defining a measure of dispersion δ for elements of B it is also possible to obtain an analogue of Čebyšev's inequality. A Boolean function $\delta: B^n \to B$ will be called a *dispersion function* if $\delta(x_1, x_2, \ldots, x_n)=0$ if and only if $x_1 = x_2 = \ldots = x_n$.

Theorem 2. *A Boolean function $\delta: B^n \to B$ is a dispersion function if and only if it can be expressed as*

$$\begin{aligned} \delta(x_1, x_2, \ldots, x_n) &= x_1 \vee x_2 \vee \ldots \vee x_n - x_1 \ldots x_n = \\ &= (x_1+x_2) \vee (x_1+x_3) \vee \ldots \vee (x_1+x_n) \vee \\ &\vee (x_2+x_3) \vee \ldots \vee (x_2+x_n) \vee \ldots \vee (x_{n-1}+x_n). \end{aligned} \tag{12}$$

Proof. In view of the corollary to Theorem 2.14 in [2], the equation $\delta(x_1, \ldots, x_n)=0$ is equivalent to

$$\begin{aligned} &(x_1+x_2) \vee (x_1+x_3) \vee \ldots \vee (x_1+x_n) \vee (x_2+x_3) \vee \ldots \vee \\ &\vee (x_2+x_{n-1}) \vee \ldots \vee (x_{n-1}+x_n) = 0. \end{aligned} \tag{13}$$

(which expresses the conditions $x_1 = x_2 = \ldots = x_n$) if and only if δ coincides with the function on the left side of (13). It is easy to check that two expressions for δ in (12) can actually be brought to the same disjunctive form.

Remark. A dispersion function also has the following properties:

(i) If $x_1 \leqslant x_2 \ldots \leqslant x_n$ then $\delta(x_1, \ldots, x_n) = x_n - x_1 =$ range,

(ii) δ is symmetric,

(iii) $\delta(x_1, x_2, \ldots, x_j, \ldots, x_n) = 1$ if and only if $\Pi_j x_j = 0$ and $\bigvee_j x_j = 1$,

(iv) $$\delta(x_1 + a,\ x_2 + a, \ldots, x_n + a) = \delta(x_1,\ x_2, \ldots, x_n)$$

for all $a \in B$.

The following is a Boolean analogue of Čebyšev's inequality.

Theorem 3. *The functions β_k in Theorem 1 and the dispersion for B satisfy*

$$x_j + \beta_k(x_1, \ldots, x_n) \leqslant \delta(x_1, \ldots, x_n),\ j = 1,\ 2, \ldots, n. \tag{14}$$

Proof. It follows from (12) and (1) that

$$x_j + \beta_k(x_1, \ldots, x_n) = x_j' \bigvee_{i_1, \ldots, i_k} x_{i_1} \ldots x_{i_k} \vee x_j \cdot \Pi_{i_1, \ldots, i_k} (x_{i_1}' \vee \ldots \vee x_{i_k}') \leqq$$
$$\leqslant x_j' \bigvee_{i=1, i \neq j}^{n} x_j \vee x_j \bigvee_{i=1, i \neq j}^{n} x_i' \leqslant \delta(x_1, \ldots, x_n).$$

In the following, we consider a finite Boolean algebra with 2^m elements. B is represented as $B = \{(x^1,\ x^2, \ldots, x^m) \mid x^i = 0 \text{ or } 1\}$ with the usual component-wise operations. If $z \in B$ then z^j will denote its j-th component and $h(z)$ is the number of atoms which are contained in z. Let $\beta^j = (\beta(x_1,\ x_2, \ldots, x_n))^j$.

Theorem 4. *Let B be a finite Boolean algebra with 2^m elements and let $a \in B$. If $p(a)$ is the probability that $\beta(x_1,\ x_2, \ldots, x_k, \ldots, x_n) = a$, then:*

$$p(a) = \frac{3^{h(a)}}{4^m} \text{ if } n = 2, \tag{15}$$

$$p(a) = \frac{1}{2^m} \text{ (independent of } a) \text{ if } n \text{ is odd.} \tag{16}$$

Proof. We first consider the case of odd m. Here β^j is the union of all products of the x_k^j taken $(n+1)2$ at a time. Thus $\beta^j = 1$ if and only if at least $(n+1)/2$ of the x_k^j are equal to 1. Similarly $\beta^j = 0$ if and only if at least $(n+1)/2$ of the x_k^j are equal to zero. It follows that the number of n-tuples for which $\beta^j = a^j$ is independent of a and since $\Sigma_{a \in B}\ p(a) = 1$ we have $p(a) = 1/2^m$.

If $n = 2$ then $\beta^j = 0$ if and only if $x_1{}^j = x_2{}^j = 0$. There are a total of $3^{h(a)}$ ways in which $x_1,\ x_2$ can be selected with $\beta(x_1,\ x_2) = a$. This last observation yields formula (15).

(The case of n even and greater than two is left open).

Remark. From (15) and $\Sigma_{a \in B}\ p(a) = 1$ it follows that $\Sigma_{r=0}^{m} \binom{m}{r} 3^r = 4^m$, which can be verified independently by use of the binomial theorem.

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