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ON REGULARITY OF TIME-VARIANT LANGUAGES

BY

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§ 1. **Introduction.** The notion of time-varying automaton has been introduced in automata theory by A g a s a n d j a n (1967) and intensively studied by A. S a l o m a a [1]. Like probabilistic automata, it represents a quite natural generalisation of the notion of finite automaton. Definitions 1 – 3 are from [1], while the facts concerning the metric space $\mathcal{P}(I^*)$ are from [2].

Definition 1. *A time-varying automaton (t.v.a.) over an alphabet I is a triple $A=(S, s_0, F)$, where S is the finite, nonempty set of states, s_0 is the initial state, $F: N \to S^{S\times I} \times 2^S$ is a mapping given by $F(i)=(f_i, S_i)$, $f_i: S\times I \to S$ is the transition function at the i-th moment and $S_i \subseteq S$ is the set of final states at the same time.* If for a t.v.a. A all f_i (or S_i) are equal, we are dealing with a t.v.a. which has a constant transition (respectively a constant set of final states).

Definition 2. *The response of a t.v.a. A for a word $P\in I^*$ (denoted by $\mathrm{resp}_A(P)$) is recursively defined by:*

$$\mathrm{resp}_A(\lambda)=s_0,$$
$$\mathrm{resp}_A(Qx)=f_{\lg Q+1}(\mathrm{resp}_A(Q), x),\ \forall Q\in I^*,\ \forall x\in I,$$

where $\lambda\in I^*$ is the empty word and $\lg Q$ is the lenght of the word Q.

Definition 3. *The word P is accepted by A iff $\mathrm{resp}_A(P)\in S_{\lg P}$. $L(A)$, the language accepted by A is $L(A)=\{P\mid P\in I^*, \mathrm{resp}_A(P)\in S_{\lg P}\}$.*

Considering the binary operation „+" on $\mathcal{P}(I^*)$, where $E_1+E_2=(E_1\cup E_2)\setminus(E_1\cap E_2)$, $\forall E_1, E_2\in\mathcal{P}(I^*)$, we obtain the linear space $(\mathcal{P}(I^*), +, \cdot)$ over $GF(2)$.

Let $d:\mathcal{P}(I^*)\to\bar{R}$ be the mapping given by

$$d(E)=\begin{cases} +\infty, \text{ if } E=\varnothing \\ \min\{\lg P\mid P\in E\}, \text{ if } E\neq\varnothing. \end{cases}$$

The norm of the language E is given by $\|E\|=2^{-d(E)}$.
Thus, we obtain a metric in the standard way, by taking $\rho(E_1, E_2)=\|E_1-E_2\|$.

2. **Sufficient conditions for regularity of time-variant languages.**

Theorem 1. *Let $A=(S, s_0, F)$ be a t.v.a. over I, with a constant set of final states, S_1, and suppose that the following condition holds:*

$$\forall i, j\in N((i<j \wedge f_i=f_j)\Rightarrow f_{i+1}=f_{j+1}). \tag{1}$$

Then $L(A)$ *is regular.*

Proof. Denote

$$j_0-i_0=\min\{j-i \mid i,j\in N,\ i<j,\ f_i=f_j\}.$$

It can easily be proved by induction on t, that the following condition holds :

(2) $$\forall t\in N(t\geqslant i_0\Rightarrow f_t=f_{t+j_0-i_0}),$$

that is F is almost periodic, of period j_0-i_0, begining with the moment i_0. Apply now theorem 7 [1].

Remark 1. The condition (1) is quite strong when we compare it with that of theorem 7 [1]. Indeed, for a t.v.a. A in which F is given by :

$$f,\ g,\ g,\ f,\ g,\ g,\ \ldots$$

(1) is not satisfied but F is periodic of period 3, and thus $L(A)$ is regular.

We will give a better condition, even compared with that of theorem 7 [1]. Let $A=(S,\ s_0,\ F)$ be any t.v.a. with a constant set of final states, S_1, and $S^{S\times I}=\mathcal{F}$. For every $f\in\mathcal{F}$ and every $x\in I$, the function $f^x\colon S\to S$ defined by :

$$f^x(s)=f(s,\ x)$$

produces a partition of the set S, by the following sets :

$$S_1^{f^x}=\{s\in S-S_1 \mid f^x(s)\in S-S_1\},$$
$$S_2^{f^x}=\{s\in S-S_1 \mid f^x(s)\in S_1\},$$
$$S_3^{f^x}=\{s\in S_1 \mid f^x(s)\in S-S_1\},$$
$$S_4^{f^x}=\{s\in S_1 \mid f^x(s)\in S_1\},$$

Some of these sets may be empty. We can now define on $\mathcal{F}$ a relation „$\sim$" as follows :

(3) $$f\sim g \text{ if } S_i^{f^x}=S_i^{g^x},\ \forall i=\overline{1,4},\ \forall x\in I.$$

It is obvious that $\sim$ is an equivalence on $\mathcal{F}$. Let :

$$\mathcal{F}/\sim=\{C_1,\ldots,C_m\}$$

and fix a representative g_i in each equivalence class C_i. Let $\varphi:\mathcal{F}\to\mathcal{F}$ be the function given by :

(4) $$\varphi(f)=g_i \text{ if } f\in C_i.$$

Theorem 2. *Let* $A=(S,\ s_0,\ F)$ *be a t.v.a. over* I, *with a constant set of final states,* S_1. *If the function* $\varphi\circ F$ *(where „$\circ$" stands for the usual superposition of functions) is almost periodic then* $L(A)$ *is regular.*

Proof. Let $A_\varphi=(S,\ s_0,\ \varphi\circ F)$ be the t.v.a. with the same constant set of final states as A. We shall prove that $L(A)=L(A_\varphi)$. First, since S, s_0 and S_1 are the same in A and A_φ, if $P=\lambda$ it follows $P\in L(A)\Leftrightarrow P\in L(A_\varphi)$. Consider now any $P\in I^*$, lg $P=n\geqslant 1$. We will show that :

$$P\in L(A)\Leftrightarrow P\in L(A_\varphi)$$

by induction on n. Let $G=\varphi\circ F$ and G be the sequence $g_0, g_1,\ldots$ By hypothesis we note that

$$f_i\sim g_i,\ \forall i\in N.$$

Initial step. Let $p=x\in I$. Then :

$$\mathrm{resp}_A x=f_0\ P(\mathrm{resp}_A(\lambda),\ x)=f_0(s_0,\ x),$$
$$\mathrm{resp}_{A_\varphi} x=g_0(\mathrm{resp}_{A_\varphi}(\lambda),\ x)=g_0(s_0,\ x).$$

Because $f_0\sim g_0$ and $s_0\equiv s_0$, we have :

$$f_0(s_0,\ x)\in S_1\Leftrightarrow g_0(s_0,\ x)\in S_1,$$

and then $P\in L(A)\Leftrightarrow P\in L(A_\varphi)$.

Inductive step. Suppose that the assertion holds for every $Q\in I^*$ with lg$Q=n-1$ and let P be any word of length n :

$$P=Q\ x,\ x\in I.$$

We have :

$$\mathrm{resp}_A(P)=\mathrm{resp}_A\,(Qx)=f_{\lg P}(\mathrm{resp}_A(Q),\ x),$$
$$\mathrm{resp}_{A_\varphi}(P)=\mathrm{resp}_{A_\varphi}(Qx)=g_{\lg P}(\mathrm{resp}_{A_\varphi}(Q),\ x).$$

Because $f_{\lg Q}\sim g_{\lg Q}$ and, by induction hypothesis, $\mathrm{resp}_A\,(Q)\in S_1$ iff $\mathrm{resp}_{A_\varphi}(Q)\in S_1$, we have :

$$f_{\lg P}(\mathrm{resp}_A(Q),\ x)\in S_1\Leftrightarrow g_{\lg}P(\mathrm{resp}_{A_\varphi}P(Q),\ x)\in S_1$$

Thus $P\in L(A)\Leftrightarrow P\in L(A_\varphi)$. Apply now theorem 7 [1] for A_φ.

Remark 2. The condition of theorem 2 is weaker than that of theorem 7 [1]. Indeed, if $A=(S,\ s_0,\ F)$ is the t.v.a. with :

(5) $$S=\{s_0,\ s_1,\ s_2,\ s_3\},\ S_1=\{s_3\},\ I=\{i_1,\ i_2,\ i_3\},$$
$$F: f,g,f,f,g,f,f,f,g,f,f,f,f,g,\ldots$$

where

(6) $$\begin{cases} f(s_0,\ i_1)=s_0,\ g(s_0,\ i_1)=s_1,\\ f(s_1,\ i_1)=s_1,\ g(s_1,\ i_1)=s_0,\end{cases}$$

and $f(s,\ i)=g(s,\ i)$ for all $s\in S$ and $i\in I$ not in (6), then F is not periodic, but $\varphi\circ F$ is. In fact $\varphi\circ F$ is given by the sequence :

$$g,\ g,\ g,\ldots$$

(or $f,\ f,\ f,\ldots$) since, clearly, $f\sim g$.

§. 3. **T.v.a.a. languages and regular languages in the metric space** $\mathcal{P}(I^*)$. **Theorem 3.** *Every t.v.a.a. language which is accepted by a t.v.a. with a constant set of final states can ve arbitrarily well approximated by a regular language within the metric of Bodnarchuk.*

Proof. Let $A=(S,\ s_0,\ F)$ be a t.v.a. over I with a constant set of final states, S_1. We shall prove that for every $k>0$, $k\in N$, there is a t.v.a. $A_k=$

$=(S, s_0, F_k)$ with the same constant set of final states as A and for which we have:

$$\text{(i)} \qquad \| L(A) - L(A_k) \| < \frac{1}{k}, \text{ and}$$

$$\text{(ii)} \qquad L(A_k) \text{ is regular.}$$

Let $f_0, f_1, \ldots$ be the sequence which represents F and

$$n = [\log_2 k], \tag{7}$$

where $[m]$ stands for the greatest positive integer less than or equal to m. Let $A_k = (S, s_0, F_k)$ be the t.v.a. with the same constant set of final states as A, in which F_k is obtained from F as follows:

$$\begin{cases} f_j^k = f_j, \ \forall j \leqslant n, \\ f_j^k = f, \ \forall j > n, \end{cases} \tag{8}$$

where $f_0^k, f_1^k, \ldots$ is the sequence which represents F_k, n is given by (7) and f is any fixed function from $\mathcal{F}$. Clearly we have (ii), by theorem 2 [1]. Moreover, by the definition of d in $\mathcal{P}(I^*)$ and by (7) we have:

$$d(L(A) - L(A_k)) > \log_2 k$$

and hence:

$$2^{-d(L(A) - L(A_k))} < \frac{1}{k}.$$

Thus, we have (i).

§ **4. Final remarks.** 1. In all the theorems we have proved we have considered only t.v.a. with a constant set of final states. But, by theorem 4 [1], this is not a prejudicial restriction.

2. It seems that we can not improve the result of theorem 2 in the same way, that is by enlarging (3) (for example by setting $S^{f_x} \cap S_i^{g_x} \neq \emptyset$, for all i and x with $S_i^{f_x}$, $S_i^{g_x}$ nonempty) without loosing the equivalence of the languages. But, for improving the above result we must implicate transition functions in non-concomitant moments of time.

3. Theorem 3 states in fact that for every t.v.a.a. language L and every $n \in N$ there is at least one regular language R which has exactly the same words of length less than n as L does. Moreover, R is not trivial (finite) because the choice of f, in (8) may be so that A_k accepts in every moment.

REFERENCES

1. Salomaa A. — *On Finite Automata with a Time-Variant Structure*. Information and Control, 13, nr. 2, 1968.
2. Gécseg F. and I. Peák — *Algebraic Theory of Automata*, Akadémiai Kiadó-Budapest 1972, 114—121.

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ON THE UNIQUENESS OF SOLUTION IN SAINT-VENANT PROBLEM

BY

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We consider a beam limited by two planes $x_3 = 0$, $x_3 = h$ $(h > 0)$ and by a cylindrical surface F. We will denote by V the domain occupied by the beam and also the volume of this domain; S will be the domain and also the area of a cross-section of the beam. The boundary of S will be denoted by Γ.

We will take the axes x_1, x_2, x_3 such that

$$\int_S x_1 d\sigma = 0, \quad \int_S x_2 d\sigma = 0, \quad \int_S x_1 x_2 d\sigma = 0,$$

that is the x_3 axis is the central-line of the beam and the axes x_1 and x_2 are the principal axes of inertia of the end $x_3 = 0$.

We consider our beam with the lateral surface free from tractions and loaded only on its ends in such a way that the beam is in equilibrium. Then, the problem is defined by the following equations:

$$\sigma_{ij,j} = 0 \quad \text{in } V^*, \tag{1}$$

$$\sigma_{i\alpha} n_\alpha = 0 \quad \text{on } F; \tag{2}$$

$$\begin{cases} \sigma_{i3} = g_i(x_1, x_2) & \text{on } x_3 = 0, \\ \sigma_{i3} = f_i(x_1, x_2) & \text{on } x_3 = h, \end{cases} \tag{3}$$

where σ_{ij} are the stress components, $g_i(x_1, x_2)$, $f_i(x_1, x_2)$ are given functions such that the beam is in equilibrium, $\mathbf{n}(n_1, n_2, 0)$ is the unit vector along the outer normal to F.

The problem defined by (1), (2), (3) has a unique solution, but it is extremely difficult to find it. If we replace the conditions $(3)_1$ by

$$\begin{cases} R_i = -\int_S \sigma_{i3} d\sigma, \ M_1 = -\int_S x_2 \sigma_{33} d\sigma, \ M_2 = \int_S x_1 \sigma_{33} d\sigma, \\ M_3 = \int_S (x_2 \sigma_{13} - x_1 \sigma_{23}) d\sigma \quad \text{on } x_3 = 0, \end{cases} \tag{4}$$

*) We use the summation convention over repeated indices. The index j after comma indicates partial differentiation with respect to x_j. All Latin indices take the values 1,2,3 and all Grec indices take the values 1,2.