$=(S, s_0, F_k)$ with the same constant set of final states as A and for which we have:

$$\text{(i)} \qquad \| L(A) - L(A_k) \| < \frac{1}{k}, \text{ and}$$

$$\text{(ii)} \qquad L(A_k) \text{ is regular.}$$

Let $f_0, f_1, \ldots$ be the sequence which represents F and

$$n = [\log_2 k], \tag{7}$$

where $[m]$ stands for the greatest positive integer less than or equal to m. Let $A_k = (S, s_0, F_k)$ be the t.v.a. with the same constant set of final states as A, in which F_k is obtained from F as follows:

$$\begin{cases} f_j^k = f_j, \ \forall j \leqslant n, \\ f_j^k = f, \ \forall j > n, \end{cases} \tag{8}$$

where $f_0^k, f_1^k, \ldots$ is the sequence which represents F_k, n is given by (7) and f is any fixed function from $\mathcal{F}$. Clearly we have (ii), by theorem 2 [1]. Moreover, by the definition of d in $\mathcal{P}(I^*)$ and by (7) we have:

$$d(L(A) - L(A_k)) > \log_2 k$$

and hence:

$$2^{-d(L(A)-L(A_k))} < \frac{1}{k}.$$

Thus, we have (i).

§ 4. **Final remarks.** 1. In all the theorems we have proved we have considered only t.v.a. with a constant set of final states. But, by theorem 4 [1], this is not a prejudicial restriction.

2. It seems that we can not improve the result of theorem 2 in the same way, that is by enlarging (3) (for example by setting $S^{f_x} \cap S_i^{g_x} \neq \varnothing$, for all i and x with $S_i^{f_x}, S_i^{g_x}$ nonempty) without loosing the equivalence of the languages. But, for improving the above result we must implicate transition functions in non-concomitant moments of time.

3. Theorem 3 states in fact that for every t.v.a.a. language L and every $n \in N$ there is at least one regular language R which has exactly the same words of length less than n as L does. Moreover, R is not trivial (finite) because the choice of f, in (8) may be so that A_k accepts in every moment.

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Received 22. VII. 1980

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Tomul XXVII, s. I a, 1981 f. 2

ON THE UNIQUENESS OF SOLUTION IN SAINT-VENANT PROBLEM

BY

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We consider a beam limited by two planes $x_3 = 0$, $x_3 = h$ $(h > 0)$ and by a cylindrical surface F. We will denote by V the domain occupied by the beam and also the volume of this domain; S will be the domain and also the area of a cross-section of the beam. The boundary of S will be denoted by Γ.

We will take the axes x_1, x_2, x_3 such that

$$\int_S x_1 d\sigma = 0, \quad \int_S x_2 d\sigma = 0, \quad \int_S x_1 x_2 d\sigma = 0,$$

that is the x_3 axis is the central-line of the beam and the axes x_1 and x_2 are the principal axes of inertia of the end $x_3 = 0$.

We consider our beam with the lateral surface free from tractions and loaded only on its ends in such a way that the beam is in equilibrium. Then, the problem is defined by the following equations:

$$\sigma_{ij,j} = 0 \quad \text{in } V^*, \tag{1}$$

$$\sigma_{i\alpha} n_\alpha = 0 \quad \text{on } F; \tag{2}$$

$$\begin{cases} \sigma_{i3} = g_i(x_1, x_2) & \text{on } x_3 = 0, \\ \sigma_{i3} = f_i(x_1, x_2) & \text{on } x_3 = h, \end{cases} \tag{3}$$

where σ_{ij} are the stress components, $g_i(x_1, x_2)$, $f_i(x_1, x_2)$ are given functions such that the beam is in equilibrium, $\mathbf{n}(n_1, n_2, 0)$ is the unit vector along the outer normal to F.

The problem defined by (1), (2), (3) has a unique solution, but it is extremely difficult to find it. If we replace the conditions $(3)_1$ by

$$\begin{cases} R_i = -\int_S \sigma_{i3} d\sigma, \ M_1 = -\int_S x_2 \sigma_{33} d\sigma, \ M_2 = \int_S x_1 \sigma_{33} d\sigma, \\ M_3 = \int_S (x_2 \sigma_{13} - x_1 \sigma_{23}) d\sigma \quad \text{on } x_3 = 0, \end{cases} \tag{4}$$

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*) We use the summation convention over repeated indices. The index j after comma indicates partial differentiation with respect to x_j. All Latin indices take the values 1,2,3 and all Grec indices take the values 1,2.

where R_i are given components of the rezultant tractions and M_i are given components of the rezultant moment of applied tractions.

It is understood that on the end $x_3=h$ there are applied trac i ons so that the beam is in equilibrium. In this way, the uniqueness of so tution is removed out because we have infinitely many traction repartitionls on the end such that their resultant and moment resultant satisfy (4).

The problem defined by (1), (2) and (4) is known as Saint-Venant problem. If, in addition to (1), (2), (4), we require that the displacements u_i are of the form

$$u_i=\sum_{k=0}^{n} U_i^{(k)}(x_1, x_2)x_3^k, \tag{5}$$

then the solution of problem is unique. The function $U_i^{(k)}(x_1, x_2)$ must be determined. We can replace (5) by an analogous supposition for the stress components σ_{ij}.

In the following, we will prove the uniqueness of the solution for the problem (1), (2), (4), (5).

Supposing that the beam is orthotropic, the relations between the stress components σ_{ij} and the strain components ε_{ij} are [1]

$$\begin{aligned} &\sigma_{ii}=c_{1i}\varepsilon_{11}+c_{2i}\varepsilon_{22}+c_{3i}\varepsilon_{33} \text{ (not summed)},\\ &\sigma_{13}=c_{44}\varepsilon_{13}, \quad \sigma_{23}=c_{55}\varepsilon_{23}, \quad \sigma_{12}=c_{66}\varepsilon_{12}, \end{aligned} \tag{6}$$

where $c_{ij}=c_{ji}$ are constants which caracterize the elastic properties of the material of the beam.

The inverse relations of (6) are taken in the form

$$\begin{aligned} E\ \varepsilon_{11}&=\nu_{11}\,\sigma_{11}+\nu_{12}\,\sigma_{22}-\nu_1\,\sigma_{33},\\ E\ \varepsilon_{22}&=\nu_{12}\,\sigma_{11}+\nu_{22}\,\sigma_{22}-\nu_2\,\sigma_{33},\\ E\ \varepsilon_{33}&=-\nu_1\,\sigma_{11}-\nu_2\,\sigma_{22}+\sigma_{33},\\ c_{44}\ \varepsilon_{13}&=\sigma_{13}, \quad c_{55}\ \varepsilon_{23}=\sigma_{23}, \quad c_{66}\ \varepsilon_{12}=\sigma_{12}. \end{aligned} \tag{7}$$

The components ε_{ij} are given in terms of the displacement components u_i by

$$\varepsilon_{ii}=u_{i,i} \text{ (not summed)}, \qquad \varepsilon_{ij}=u_{i,j}+u_{j,i} \ (i\neq j).$$

If we denote by $\varepsilon_{\alpha\beta}^{(k)}$ the strains of the displacements $(U_1^{(k)}, U_2^{(k)}, 0)$ and by $\tau_{\alpha\beta}^{(k)}$ the corresponding stresses, then we have

$$\varepsilon_{\alpha\beta}=\sum_{k=0}^{n}\varepsilon_{\alpha\beta}^{(k)}x_3^k,$$

$$\varepsilon_{i3}=\sum_{k=0}^{n} k\ U_i^{(k)}x_3^{k-1}+\sum_{k=0}^{n}U_{3,i}^{(k)}x_3^k$$

and

$$\sigma_{ii}=\sum_{k=0}^{n}\tau_{ii}^{(k)}\ x_3^k+c_{i3}\sum_{k=0}^{n}kU_3^{(k)}x_3^{k-1} \quad \text{(not summed)},$$

$$\sigma_{12}=\sum_{k=0}^{n}\tau_{12}^{(k)}x_3^k, \qquad \sigma_{\alpha 3}=c_{3+\alpha,3+\alpha}\left[\sum_{k=0}^{n}k\ U_\alpha^{(k)}\ x_3^{k-1}+\sum_{k=0}^{n}U_{3\alpha}^{(k)}\ x_3^k\right].$$

Let us denote by

$$\Delta=c_{44}\frac{\partial^2}{\partial x_1^2}+c_{55}\frac{\partial_2}{\partial x_2^2}, \tag{8}$$

$$D=c_{44}n_1\frac{\partial}{\partial x_1}+c_{55}n_2\frac{\partial}{\partial x_2}. \tag{9}$$

The equilibrium equations (1) and the boundary conditions (2) will be satisfied if and only if, for arbitrary k, we have

$$\begin{aligned} &\Delta U_3^{(k)}+(k+1)[c_{44}U_{1,1}^{(k+1)}+c_{55}U_{2,2}^{(k+1)}+\tau_{33}^{(k+1)}]+\\ &\quad+(k+1)(k+2)c_{33}\ U_3^{(k+2)}=0 \quad \text{in } S, \end{aligned} \tag{10}$$

$$DU_3=-(k+1)[c_{44}U_1^{(k+1)}n_1+c_{55}U_2^{(k+1)}n_2], \text{ on } \Gamma, \tag{11}$$

and

$$\tau_{\alpha\beta,\beta}^{(k)}+(k+1)(c_{\alpha 3}+c_{3+\alpha,3+\alpha})U_{3,\alpha}^{(k+1)}+(k+1)(k+2)c_{3+\alpha,3+\alpha}U_\alpha^{(k+2)}=0 \quad \text{in } S, \tag{12}$$

$$\tau_{\alpha\beta}^{(k)}n_\beta=-(k+1)c_{\alpha 3}U_3^{(k+1)}n_\alpha, \quad \text{on } \Gamma. \tag{13}$$

The problem defined by (12), (13) is a plane problem with body forces.

The necessary and sufficient conditions for the existence of the solutions for the problems (10), (11) and (12), (13) can be found in [2].

Now, let us solve the above problems for $k=n, n-1, n-2,\ldots$. In the following, we will denote by a_s, b_s, c_s, r_s arbitrary constants.

1°. $\underline{k=n}$

We have

$$\Delta U_3^{(n)}=0 \quad \text{in } S, \qquad DU_3^{(n)}=0 \quad \text{on } \Gamma;$$

$$\tau_{\alpha\beta,\beta}^{(n)}=0 \quad \text{in } S, \qquad \tau_{\alpha\beta}^{(n)}n_\beta=0 \quad \text{on } \Gamma.$$

It results

$$\tau_{\alpha\beta}^{(n)}=0,$$

$$U_1^{(n)}=a_n-r_nx_2, \ U_2^{(n)}=b_n+r_nx_1, \ U_3=-c_n, \tag{14}$$

2°. $\underline{k=n-1}$

The corresponding problems are

$$\Delta U_3^{(n-1)}=0 \quad \text{in } S,$$

$$DU_3^{(n-1)}=-n[c_{44}(a_n-r_nx_2)n_1+c_{55}(b_n+r_nx_1)n_2] \text{ on } \Gamma$$

and

$$\tau_{\alpha\beta,\beta}^{(n-1)}=0 \text{ in } S,$$

$$\tau_{\alpha 3}^{(n-1)}n_\beta=n\ c_{\alpha 3}c_n n_\alpha \text{ on } \Gamma \ (\alpha=1, 2).$$

The solutions of these problems are given by

$$(15)\quad \begin{cases} U_1^{(n-1)} = nc_n\nu_1x_1 + a_{n-1} - r_{n-1}x_2, \quad U_3^{(n-1)} = nr_n\varphi - n(a_nx_1 + b_nx_2) - c_{n-1};\\ U_2^{(n-1)} = nc_n\nu_2x_2 + b_{n-1} + r_{n-1}x_1, \end{cases}$$

$$(16)\quad \begin{cases} \tau_{11}^{(n-1)} = n\, c_{13}c_n, \quad \tau_{22}^{(n-1)} = n\, c_{23}c_n, \quad \tau_{12}^{(n-1)} = 0,\\ \tau_{33}^{(n-1)} = nc_n(c_{13}\,\nu_1 + c_{23}\nu_2) = nc_n(c_{33} - E), \end{cases}$$

where φ is the function of torsion defined by

$$\Delta\varphi = 0 \text{ in } S, \quad D\varphi = c_{44}x_2n_1 - c_{55}x_1n_2 \quad \text{on } \Gamma.$$

3°. $\underline{k = n-2}$

It results

$$\Delta U_3^{(n-2)} + n(n-1)(c_{44}\nu_1 + c_{55}\nu_2 - E)c_n = 0 \quad \text{in } S,$$

$$DU_3^{(n-2)} = -(n-1)[c_{44}(nc_n\nu_1x_1 + a_{n-1} - r_{n-1}x_2)n_1 + \\ + c_{55}(nc_n\nu_2x_2 + b_{n-1} + r_{n-1}x_1)n_2] \quad \text{on } \Gamma.$$

This problem has solutions if $c_n = 0$, and then its solution is given by

$$(17)\quad U_3^{(n-2)} = (n-1)r_{n-1}\varphi - (n-1)(a_{n-1}x_1 + b_{n-1}x_2) - c_{n-2}.$$

The problem (12), (13) becomes

$$(18)\quad \begin{cases} \tau_{11,1}^{(n-2)} + \tau_{12,2}^{(n-2)} + n(n-1)[(c_{13} + c_{44})(r_n\varphi_{,1} - a_n) + c_{44}(a_n - r_nx_2)] = 0,\\ \tau_{12,1}^{(n-2)} + \tau_{22,2}^{(n-2)} + n(n-1)[(c_{23} + c_{55})(r_n\varphi_{,2} - b_n) + c_{55}(b_n + r_nx_1)] = 0 \quad \text{in } S. \end{cases}$$

$$(19)\quad \begin{cases} \tau_{11}^{(n-2)}n_1 + \tau_{12}^{(n-2)}n_2 = -(n-1)c_{13}[n(r_n\varphi - a_nx_1 - b_nx_2) - c_{n-1}]n_1,\\ \tau_{12}^{(n-2)}n_1 + \tau_{22}^{(n-2)}n_2 = -(n-1)c_{23}[n(r_n\varphi - a_nx_1 - b_nx_2) - c_{n-1}]n_2 \quad \text{on } \Gamma. \end{cases}$$

The necessary and sufficient conditions for the existence of the solution for (18), (19) require that $r_n = 0$, and we have

$$(20)\quad \begin{cases} \tau_{11}^{(n-2)} = (n-1)c_{13}[n(a_nx_1 + b_nx_2) + c_{n-1}],\\ \tau_{22}^{(n-2)} = (n-1)c_{23}[n(a_nx_1 + b_nx_2) + c_{n-1}],\\ \tau_{33}^{(n-2)} = (n-1)(c_{33} - E)[n(a_nx_1 + b_nx_2) + c_{n-1}], \quad \tau_{12}^{(n-2)} = 0, \end{cases}$$

$$(21)\quad \begin{cases} U_1^{(n-2)} = n(n-1)\left[\dfrac{1}{2}a_n(\nu_1x_1^2 - \nu_2x_2^2) + b_n\nu_1x_1x_2\right] + \\ \qquad + (n-1)c_{n-1}\nu_1x_1 + a_{n-2} - r_{n-2}x_2,\\ U_2^{(n-2)} = n(n-1)\left[a_n\nu_2x_1x_2 + \dfrac{1}{2}b_n(\nu_2x_2^2 - \nu_1x_1^2)\right] + \\ \qquad + (n-1)c_{n-1}\nu_2x_2 + b_{n-2} + r_{n-2}x_1. \end{cases}$$

4°. $\underline{k = n-3}$

The function $U_3^{(n-3)}$ is defined by

$$\Delta U_3^{(n-3)} + (n-1)(n-2)[n(c_{44}\nu_1 + c_{55}\nu_2 - E)(a_nx_1 + b_nx_2) + c_{n-1}] = 0 \quad \text{in } S,$$

$$DU_3^{(n-3)} = -(n-2)\Big\{c_{44}\Big[n(n-1)\Big(\frac{1}{2}a_n(\nu_1x_1^2 - \nu_2x_2^2) + b_n\nu_1x_1x_2\Big) + \\ + (n-1)\nu_1c_{n-1}x_1 + a_{n-2} - r_{n-2}x_2\Big]n_1 + c_{55}\Big[n(n-1)\Big(a_n\nu_2x_1x_2 + \\ + \frac{1}{2}b_n(\nu_2x_2^2 - \nu_1x_1^2)\Big) + (n-1)\nu_2c_{n-1}x_2 + b_{n-2} + r_{n-2}x_1\Big]n_2\Big\} \quad \text{on } \Gamma.$$

and the existence condition of $U_3^{(n-3)}$ require that $c_{n-1} = 0$.
We have

$$U_3^{(n-3)} = n(n-1)(n-2)\,\chi^{(n-3)} + (n-2)[r_{n-2}\varphi - (a_{n-2}x_1 + b_{n-2}x_2)] - c_{n-3},$$

where the function $\chi^{(n-3)}$ is defined by

$$\Delta\chi^{(n-3)} + (c_{44}\nu_1 + c_{55}\nu_2 - E)(a_nx_1 + b_nx_2) = 0 \quad \text{in } S,$$

$$D\chi^{(n-3)} = c_{44}\Big[\frac{1}{2}a_n(\nu_1x_1^2 - \nu_2x_2^2) + b_n\nu_1x_1x_2\Big]n_1 + c_{55}\Big[a_n\nu_2x_1x_2 + \\ + \frac{1}{2}b_n(\nu_2x_2^2 - \nu_1x_1^2)\Big]n_2 \quad \text{on } \Gamma$$

and the function $\chi^{(n-3)}$ exists for arbitrary a_n and b_n.

In consequence, the equations defining $\tau_{\alpha\beta}^{(n-3)}$ are

$$\tau_{11,1}^{(n-3)} + \tau_{12,2}^{(n-3)} + (n-1)(n-2)[(c_{13} + c_{44})(r_{n-1}\varphi_{,1} - a_{n-1}) + \\ + c_{44}(a_{n-1} - r_{n-1}x_2)] = 0,$$

$$\tau_{12,1}^{(n-3)} + \tau_{22,2}^{(n-3)} + (n-1)(n-2)[(c_{23} + c_{55})(r_{n-1}\varphi_{,1} - b_{n-1}) + c_{55}(b_{n-1} + \\ + r_{n-1}x_2)] = 0 \quad \text{in } S,$$

$$\tau_{11}^{(n-3)}n_1 + \tau_{12}^{(n-3)}n_2 = -(n-2)c_{13}[(n-1)(r_{n-1}\varphi - a_{n-1}x_1 - b_{n-1}x_2) - c_{n-2}]n_1,$$
$$\tau_{12}^{(n-3)}n_1 + \tau_{22}^{(n-3)}n_2 = -(n-2)c_{23}[(n-1)(r_{n-1}\varphi - a_{n-1}x_1 - b_{n-1}x_2) - c_{n-2}]n_2$$

and the conditions of the existence of the solution give $r_{n-1} = 0$. At the last, we obtain

$$(22)\quad \begin{cases} \tau_{11}^{(n-3)} = (n-2)c_{13}[(n-1)(a_{n-1}x_1 + b_{n-1}x_2) + c_{n-2}],\\ \tau_{22}^{(n-3)} = (n-2)c_{23}[(n-1)(a_{n-1}x_1 + b_{n-1}x_2) + c_{n-2}],\\ \tau_{33}^{(n-3)} = (n-2)(c_{33} - E)[(n-1)(a_{n-1}x_1 + b_{n-1}x_2) + c_{n-2}], \quad \tau_{12}^{(n-3)} = 0; \end{cases}$$

$$(23)\quad \begin{cases} U_1^{(n-3)} = (n-1)(n-2)\left[\dfrac{1}{2}a_{n-1}(\nu_1x_1^2 - \nu_2x_2^2) + b_{n-1}\nu_1x_1x_2\right] + \\ \qquad + (n-2)c_{n-2}\nu_1x_1 + a_{n-3} - r_{n-3}x_2,\\ U_2^{(n-3)} = (n-1)(n-2)\left[a_{n-1}\nu_2x_1x_2 + \dfrac{1}{2}b_{n-1}(\nu_2x_2^2 - \nu_1x_1^2)\right] + \\ \qquad + (n-2)c_{n-2}\nu_2x_2 + b_{n-3} + r_{n-3}x_1. \end{cases}$$

5°. $k=n-4$

The equations (10), (11) are

$$\Delta U_3^{(n-4)}+(n-2)(n-3)(c_{44}\nu_1+c_{55}\nu_2-E)[(n-1)(a_{n-1}x_1+b_{n-1}x_2)+c_{n-2}]=0 \quad \text{in } S,$$

$$DU_3^{(n-4)}=-(n-3)c_{44}\left\{(n-1)(n-2)\left[\frac{1}{2}a_{n-1}(\nu_1x_1^2-\nu x_2^2)+b_{n-1}\nu_1x_1x_2\right]+(n-2)c_{n-2}\nu_1x_1+a_{n-3}-r_{n-3}x_2\right\}n_1-(n-3)c_{55}\left\{(n-1)(n-2)\left[a_{n-1}\nu_2x_1x_2+\frac{1}{2}b_{n-1}(\nu_2x_2^2-\nu_1x_1^2)\right]+(n-2)c_{n-2}\nu_2x_2+b_{n-3}+r_{n-3}x_1\right\}n_2 \quad \text{on } \Gamma.$$

and the condition for the existence of solution is given by $c_{n-2}=0$.

The equations (12), (13) reduce to the following

$$\tau_{11,1}^{(n-4)}+\tau_{12,2}^{(n-4)}+(n-3)(c_{13}+c_{44})U_{3,1}^{(n-3)}+(n-2)(n-3)c_{44}\left\{n(n-1)\left[\frac{1}{2}a_n(\nu_1x_1^2-\nu_2x_2^2)+b_n\nu_1x_1x_2\right]+a_{n-2}-r_{n-2}x_2\right\}=0,$$

$$\tau_{12,1}^{(n-4)}+\tau_{22,2}^{(n-4)}+(n-3)(c_{23}+c_{55})U_{3,2}^{(n-3)}+(n-2)(n-3)c_{55}\left\{n(n-1)\left[a_n\nu_2x_1x_2+\frac{1}{2}b_n(\nu_2x_2^2-\nu_1x_1^2)\right]+b_{n-2}+r_{n-2}x_1\right\}=0 \quad \text{in } S$$

and

$$\tau_{11}^{(n-4)}n_1+\tau_{12}^{(n-4)}n_2=-(n-3)c_{13}U_3^{(n-3)}n_1,$$

$$\tau_{12}^{(n-4)}n_1+\tau_{22}^{(n-4)}n_2=-(n-3)c_{23}U_3^{(n-3)}n_2 \quad \text{on } \Gamma.$$

From the existence conditions for the solution of this last problem we get

$$a_n=b_n=0,\quad r_{n-2}=0 \quad (n\geqslant 4).$$

Taking into account the above results, we can conclude that in the Saint-Venant problem we have $n\leqslant 3$, and the displacement (5) must be taken in the form

$$u_1=a_3x_3^3+a_2x_3^2+6\left[\frac{1}{2}a_3(\nu_1x_1^2-\nu_2x_2^2)+b_3\nu_1x_1x_2\right]x_3+a_1x_3-r_1x_2x_3+2\left[\frac{1}{2}a_2(\nu_1x_1^2-\nu_2x_2^2)+b_2\nu_1x_1x_2\right]+c_1\nu_1x_1+a_0-r_0x_2,$$

$$u_2=b_3x_3^3+b_2x_3^2+6\left[a_3\nu_2x_1x_2+\frac{1}{2}b_3(\nu_2x_2^2-\nu_1x_1^2)\right]x_3+b_1x_3+r_1x_1x_3+2\left[a_2\nu_2x_1x_2+\frac{1}{2}b_2(\nu_2x_2^2-\nu_1x_1^2)\right]+c_1\nu_2x_2+b_0+r_0x_1,$$

$$u_3=-3(a_3x_1+b_3x_2)x_3^2-2(a_2x_1+b_2x_2)x_3-c_1x_3+U_3^{(0)}(x_1,x_2).$$

Taking $a_1=b_1=a_0=b_0=r_0=0$ and if we put down

$$6a_3=A_3,\ 6b_3=B_3,\ 2a_2=A_2,\ 2b_2=B_2,\ c_1=C_2,$$

$$r_1=\tau,\quad U_3^{(0)}(x_1,x_2)=\Phi(x_1,x_2),$$

the above formulae take the form

$$u_1=-\tau x_2x_3+C_2\nu_1x_1+\left[\frac{1}{2}A_3(\nu_1x_1^2-\nu_2x_2^2)+B_3\nu_1x_1x_2+\frac{1}{6}A_3x_3^2\right]x_3+\frac{1}{2}A_2(\nu_1x_1^2-\nu_2x_2^2)+B_2\nu_1x_1x_2+\frac{1}{2}A_2x_3^2,$$

$$u_2=\tau x_1x_3+C_2\nu_2x_2+\left[A_3\nu_2x_1x_2+\frac{1}{2}B_3(\nu_2x_2^2-\nu_1x_1^2)+\frac{1}{6}B_3x_3^2\right]x_3+A_2\nu_2x_1x_2+\frac{1}{2}B_2(\nu_2x_2^2-\nu_1x_1^2)+\frac{1}{2}B_2x_3^2, \tag{24}$$

$$u_3=\Phi(x_1,x_2)-(A_2x_1+B_2x_2+C_2)x_3-\frac{1}{2}(A_3x_1+B_3x_2)x_3^2,$$

which coincide with that given in [5].

The constans A_3, A_2, B_3, B_2, C_2 and τ are uniquely determined by the conditions (4) (see [2]), and the uniqueness of solution is proved.

The unique solution (24) will be called *standard solution* and the corresponding tractions on the ends *standard repartition*.

In account of the Saint-Venant principle we take this solution as solution for all repartitions which satisfy the conditions (4).

Some remarks: – The treated problem for the orthotropic beams can be extended to the general case.

– The uniqueness of solution, which is a polynomial on x_3, for the general problem of the beams can be proved.

– When the stress tensor is asymmetric (as in [3] *f.i*) the conditions (4) are not sufficient in order to determine an unique solution of the form (5).

It seems that to the conditions (4) we must add also, separately, the following conditions

$$\int_S m_{3i}\,d\sigma=\mu_i,$$

where μ_i are given and m_{3i} are the couple-stress tensor components on the end $x_3=0$. Similarly, in this last case the principle of Saint-Venant must be reexamined.

The same thing is valid for the micropolar media. The remark made long time ago [4], in connection with the rigid body displacements, allows us to believe that the Sanit-Venant problem, for the micropolar case, may be solved if we take the displacements and the microrotations in the form

$$u_1 = \left(U_1 - \frac{1}{2}\tau_1 x_2 x_3 - \frac{1}{6} a_1 x_3^2\right) x_3 + V_1 - \tau_0 x_2 x_3 - \frac{1}{2} b_1 x_3^2 - \tau x_2 x_3 - \frac{1}{2} c_1 x_3^2 ,$$

$$u_2 = \left(U_2 - \frac{1}{6} a_2 x_3^2 + \frac{1}{2}\tau_1 x_1 x_3\right) x_3 + V_2 - \frac{1}{2} b_2 x_3^2 + \tau_0 x_1 x_3 + \tau x_1 x_3 - \frac{1}{2} c_2 x_3^2 ,$$

$$u_3 = U_3 x_3 + V_3 + \frac{1}{2}(a_1 x_1 + a_2 x_2 + a_3) x_3^2 + (b_1 x_1 + b_2 x_2 + b_3 x_3) x_3 +$$
$$+ (c_1 x_1 + c_2 x_2 + c_3) x_3,$$

$$\varphi_1 = \left(\Phi_1 + \frac{1}{2} a_2 x_3\right) x_3 + \psi_1 + b_2 x_3,$$

$$\varphi_2 = \left(\Phi_2 - \frac{1}{2} a_1 x_3\right) x_3 + \psi_2 - b_1 x_3,$$

$$\varphi_3 = \left(\Phi_3 + \frac{1}{2}\tau_1 x_3\right) x_3 + \psi_3 + \tau_0 x_3,$$

where $U_i(x_1, x_2)$, $V_i(x_1, x_2)$ $\Phi_i(x_1, x_2)$, $\psi_i(x_1, x_2)$ are unknown functions and a_i, b_i, c_i, τ_1, τ_0, τ some constants. The existence of this solution is an open problem. It is possible that such a solution do not exist.

It remains to clarify, for example, what is the difference between a lood by tractions (σ_{3i}) only and a lood by couple-stresses (m_{3i}) only, with the same torque in the torsion problem. What are the ways in order to perform such deformations? Can this difference be put in evidence by an experimental way? or the effect of couple-stresses is a secondary one?

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The results of this paper, excepting the last remarks were presented to the „Professor Mendel Haimovici Symposium — in memoriam" (1976).

Received 20. II. 1980

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Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXVII, s.I a, 1981, f.2

RECENZII

R. E. EDWARDS: *A Formal Background to Mathematics* 2a and 2b. A Critical Approach to Elementary Analysis. Universitext. Springer-Verlag, New York, Heidelberg, Berlin, 1980. Part a: XLVII, p. 1—606; Part b: VI, p. 607—1170, DM 72; $ 42.50.

În acest volum autorul tratează fundamentele analizei matematice elementare şi face o critică a expunerilor utilizate în mod curent în facultăţi. Spre deosebire de primul volum, în care se acordă preponderenţă aspectelor formale, aici accentul este pus pe informaţia matematică. Conceptul fundamental este cel de convergenţă a şirurilor reale. Cuprinsul volumului este următorul: convergenţa şirurilor, continuitatea şi limitele funcţiilor, convergenţa seriilor, diferenţierea, integrarea, numerele complexe (exponenţiala complexă şi funcţiile trigonometrice), integrarea aproximativă, coeficienţii diferenţiali, lungimile curbelor, teorema lui Cauchy şi analiza complexă (integrale curbilinii, formula integrală a lui Cauchy, funcţii logaritmice).

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ERIK BALSLEV (editor): *18th Scandinavian Congress of Mathematicians*. Proceedings. 1980. Progress in Mathematics, Vol. 11. Birkhäuser, Boston, Basel, Stuttgart, 1981, IX + 515 p., sFr. 58.

Prima parte a volumului cuprinde următoarele expuneri de sinteză: 1° *L. Carlson*, *BMO — o dezvoltare de 10 ani*; 2° *B. Fuglede*, Topologie fină şi funcţii fin olomorfe; 3° *W. Fulton*, Asupra grupului fundamental al complementului unei curbe plane; 4° *L. Hörmander*, Calcul simbolic şi ecuaţii diferenţiale; 5° *I. Madsen*, Forme spaţiale sferice: o privire generală; 6° *A. Salomaa*, Serii formale de puteri în variabile necomutative; 7° *I. Segal*, Cuantizare, stabilitate şi împrăştiere; 8° *E. Stormer*, Dezvoltări recente asupra C^*-algebrelor. A doua parte a volumului cuprinde 24 comunicări asupra unor probleme matematice variate (topologie, geometrie, analiză funcţională, ecuaţii cu derivate parţiale, mecanică, teoria numerelor, etc.). Autorii acestora sînt: H. H. Anderson, M. C. Blad, J.-E. Jjörk, O. Bratteli, Y. Domar, H. E. Elíasson, G. Ellingsrud & D. Laksov, M. Essen & D. F. Shea, K. Grove, G. Grubb, U. Haagerup, J. P. Hansen, A. Holme, M. Jutila, S. L. Kleiman, E. Laitinen, A. Lima, O. Martio, L. Reichel, S. Rickmann, J.-E. Roos, J. Sjöstrand, S. A. Stromme & J. Mesequer & I. Sols, B. Winzell.

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K. G. BINMORE: *The Foundations of Analysis: A Straightforward Introduction*. Book 1: Logic, Sets and Numbers, Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney, 1980, VII + 131 p.

Cartea este o introducere la nivel informativ în fundamentele algebrice şi logice ale analizei matematice scrisă la un nivel elementar. Sînt discutate noţiunea de convergenţă şi unele rudimente de calcul diferenţial şi integrare. Cartea este foarte frumos scrisă, conţine multe exemple şi ilustraţii fiind o plăcută introducere în analiza matematică.

Sorin Istrail