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A NOTE ON KNUTH'S ITERATED POWERS

BY

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In [4] Knuth described occurences of huge numbers in combinatorial theory and suggested the following notation:

$$B(B(\ldots(B))\ldots)) = B\uparrow T,$$

$$B\uparrow B(\uparrow(\ldots\uparrow(B))\ldots)) = B\uparrow\uparrow T,$$

$$B\uparrow\uparrow B(\uparrow\uparrow(\ldots\uparrow\uparrow(B))\ldots)) = B\uparrow\uparrow\uparrow T,$$

where in every equation above there are T copies of a positive integer B, and consequently, there are $T-1$ operations of multiplication or $\uparrow$ or $\uparrow\uparrow$ performed. For instance,

$$B\uparrow T = B^T, \quad B\uparrow\uparrow T = B^{B^{\cdot^{\cdot^{\cdot^{B}}}}}$$

where on the right side of the last equation there are T copies of the symbol B.

The notation above leads to the following inductive definition:

$$B\uparrow T = B\uparrow^1 T = k(B, 1, T), \quad B\uparrow\uparrow\uparrow T = B\uparrow^3 T = k(B, 3, T),$$

$$B\uparrow\uparrow T = B\uparrow^2 T = k(B, 2, T), \quad \vdots$$

$$B\uparrow^d T = k(B, d, T),$$

$$\vdots$$

Hence, following [1], we obtain the equations for a number-theoretic function k (*Knuth's function*):

$$k(B, 1, T) = B^T, \quad k(B, d, 1) = B,$$

$$k(B, d+1, T+1) = k(B, d, k(B, d+1, T)).$$

In order to obtain a total function we add the following two equations, which are compatible with the foregoing definition (the first equation is borrowed from [1]):

$$k(B, d, 0) = 1, \ d \geqslant 1, \quad k(B, 0, T) = B\,.\,T.$$

Let us note that Knuth's function is defined by nested recursion. Consequently, by standard methods [2], one can prove that k is recursive and non primitive recursive.

In [5] R i t c h i e studied an hierarchy of the class of primitive recursive functions (equivalent tio G r z e g o r c z y k's hierarchy) based on the following sequence of primitive recursive functions:

$$r_0(x, y) = x + 1, \quad r_{n+1}(x, 0) = 1$$
$$r_1(x, y) = x + y, \quad r_{n+1}(x, y+1) = r_n(x, r_{n+1}(x, y)) \quad n \geqslant 2.$$
$$r_2(x, y) = xy,$$

For every $n \geqslant 0$ we denote by $\mathcal{R}_n$ the n-th class in Ritchie's hierarchy.

Now let $k_n(x, y) = k(x, n, y)$ and let $\mathcal{K}_n$ be the smallest class of non-negative functions containing the succesor function, the zero function, the projection functions, and $k_n(x, y)$, which is closed under substitution and limited recursion.

The hierarchy $(\mathcal{K}_n)$ *is a proper hierarchy of the class of primitive recursive functions, and for every* $n \geqslant 0$, $\mathcal{K}_n = \mathcal{R}_{n+2}$.

To prove the statement above, we shall state the equality $k_n = r_{n+2}$ for every $n \geqslant 0$. For $n = 0$, $k_0(x, y) = x \cdot y = r_2(x\ y)$; for $n = 1$, $k_1(x, y) = x^y$ and this function satisfies Ritchie's equations: $r_3(x, 0) = 1$, $r_2(x, y+1) = = r_2(x, r_3(x, y))$.

Let us suppose that $k_n = r_{n+2}$. We shall prove by induction (on y) that $k_{n+1} = r_{n+3}$. For $y = 0$, $k_{n+1}(x, 0) = 1 = r_{n+3}(x, 0)$; if $k_{n+1}(x, y) = r_{n+3}(x, y)$, then $k_{n+1}(x, y+1) = k_n(x, k_{n+1}(x, y)) = k_n(x, r_{n+3}(x, y)) = r_{n+2}(x, r_{n+3}(x, y)) = = r_{n+3}(x, y+1)$.

In viev of the third Corrolary of Theorem 3.4 [5] the proof is complete.

Let us note that the equality above motivates – a posteriori – the modification operated by Rithchie in the original definition of Ackermann's function (Ritchie replaced the equation $r_n(x, 0) = x$ by the equation $r_n(x, 0) = = 1$): Ritchie's modified equation agrees with the definition of the power operation.

Finally we display a FORTRAN iterative program (using the method in [3]) for the calculation of Knuth's powers:

```
FUNCTION KNUTH (B, D, T)
INTEGER B, D, T
DIMENSION M(D), ITERATION (D),
JT(D)
IF (D.EQ.O) GO TO 8
IF (T.EQ.O) GO TO 9
KNUTH=B
M(1)=B
IF (D.EQ.1) GO TO 10
DO 7 I=2, D
JT(1)=B
IF (I.EQ.D) JT(1)=T
PLACE=1
IF (PLACE. GT. I-1) GO TO 7
1 ITERATION (PLACE)=3
2 IF (ITERATION (PLACE). GT. JT
(PLACE) GO TO 4
IF (PLACE+1. GT. I-1) GO TO 3
JT (PLACE+1)=KNUTH
KNUTH=M(I-PLACE-1)
PLACE=PLACE+1
GO TO 1
3 KNUT=B**KNUTH
GO TO 5
4 IF (PLACE. EQ. 1) GO TO 6
PLACE=PLACE-1
5 ITERATION (PLACE)=ITERATION
(PLACE)+1
GO TO 2
6 KNUTH=B**KNUTH
7 M(I)=KNUTH
  GO TO 11
8 KNUTH=B.T
  RETURN
9 KNUTH=1
  RETURN
10 KNUTH=B**T
11 RETURN
   END
```


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