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ON THE GLOBAL BEHAVIOUR OF DIFFERENTIABLE FUNCTIONS IN NON ARCHIMEDEAN FIELDS

BY

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1. Introduction. Similarity properties of analytic functions in fields with a non archimedean valuation were first observed by Krasner [1]. The following more general quasilinearity property of functions differentiable in a field K of this type was pointed out by Schramm [2]. Let $f: K \to K$ have a non vanishing derivative at $a \in K$ and denote the valuation of an element $y \in K$ by $|y|$. Then a ball $B(a, \rho)$ of center a and radius $\rho > 0$ exists, such that $x \in B(a, \rho)$ entails

$$f(x) - f(a) = [f'(a) + \varepsilon(a, x)]\,(x - a), \tag{1}$$

$$|\varepsilon(a, x)| < |f'(a)| \tag{2}$$

and thus also

$$|f(x) - f(a)| = |f'(a)|\,|x - a|. \tag{3}$$

Because of (3), we say that f is *quasilinear for the ordered pair* (a, x), and that $B(a, \rho)$ is a *quasilinearity ball of f at a*. The ball of largest radius centered at a, in which (1)–(3) hold, is the *exact quasilinearity ball of f at a*. Its radius is the *radius of quasilinearity $r(f, a)$ of f at a*.

In [3], as an application of quasilinearity, conditions for the term by term differentiability of a function series were formulated in terms of the radii of quasilinearity of the series.

In some applications, the relationships (1)–(3) are of interest even for single pairs of points (a, x). When, e.g., (1)–(3) hold in the field $K = Q_p$ of p-adic numbers, then the first term in the standard p-adic representation of $f(x) - f(a)$ is just the product modulo p of the first terms in the respective respresentations of $f'(a)$ and $x - a$.

The significance of quasilinearity thus calls for the setting up of conditions to determine quasilinearity in practice. Sometimes it is easier to establish the *absence* of quasilinearity, i.e., the inequality

$$|\varepsilon(a, x)| \geqslant |f'(a)| > 0, \tag{4}$$

instead of (2). Formulation of the relevant criteria constitutes the main object of §3. From Theorem 2, e.g., quasilinearity or quasisuperlinearity for a pair of points can be derived from the quasilinearity or quasisuperlinearity of the function concerned for one, two or, at most, three other suitable pairs of points.

In Theorem 3, which will not be invoked later on, conditions are provided by which the quasilinearity of the sum function of a function series for the pair (a, x) can be derived from the quasilinearity of the terms of that series for (a, x).

The theorems of §4 treat of more complex configurations. These generally, involve the radii of quasilinearity of the function in question at various points of the domain. (Ths. 4 – 6).

The central result of this paper is Theorem 4, and most later results are derived from it. It says, generally speaking that in every ball of quasilinearity the radius of quasilinearity and the derivative of the function concerned stand in certain relationships to each other. Some of these relationships display a striking simplicity. When x is a point in the exact quasilinearity ball of f at a, e.g., then $r(f, x)$ never exceeds $r(f, a)$ or, differently stated, the exact quasilinearity ball of f at x is always contained in the exact quasilinearity ball of f at a. (Th, 4 (*i*)). Thus two exact quasilinearity balls of a function coincide when each contains the center of the other (Th. 4 (*iii*)). According to the slightly more complex enunciation of Theorem 4 (*iv*), when x is a point in the exact quasilinearity ball of f at a such that $f'(x)\neq 0$ and $r(f, x)<|x-a|$, then $|f'(x)|\leqslant |f'(a)|\,|x-a|/r(f, x)$.

Theorem 5 treats of the set $\Gamma_\alpha(f)$ of those points in the domain of a given function f at which the radius of quasilinearity does not exceed the number α. It turns out that when f' vanishes nowhere, then $\Gamma_\alpha(f)$ possesses a very simple structure, always being expressible as the intersection of the domain of f with a disjoint union of balls. The latter are exact quasilinearity balls of f when K is discretely valued. Conditions are provided under which the number of the balls is finite.

Theorem 6 is concerned with coverings of an open subset D of the domain of a function f by a set of pairwise disjoint exact quasilinearity balls of f. When K is discretely valued and f' vanishes nowhere, then by Theorem 6 there is exactly one such set τ. It turns out that when τ is finite, the number of its balls is less than the number of balls in any other covering of D by quasilinearity balls of f.

The rest of §4 is concerned with so called *similarity balls*. We shall now recall the meaning of this concept.

When for all x, y in some ball $B(a, \rho)$ a function satisfies the conditions

$$f(x)-f(y)=[f'(a)+\eta(x, y)]\,(x-y), \tag{5}$$

$$|\eta(x, y)|<|f'(a)| \tag{6}$$

and hence

$$|f(x)-f(y)|=|f'(a)|\,|x-y|, \tag{7}$$

then $f|_{B(a,\rho)}$ is a *similarity transformation*, $B(a, \rho)$ is a *similarity ball* of f at a and ρ *a similarity radius*. The largest ρ for which (5) – (7) hold is the *exact similarity radius of f at a*, the corresponding ball being the *exact similarity ball of f at a*. By (7) the restriction of f to a similarity ball is univalent. For K locally compact there is a much stronger result [2] : The function f maps $B(a, \rho)$ homeomorphically onto $B[f(a), |f'(a)|\,r(f, a)]$. For an exact formulation see Theorem C in §4.

The last theorem of this paper, Theorem 7, is particularly important. It provides conditions under which the domain of a function f possesses a unique covering by pairwise disjoint exact similarity balls of f. From the converig 7 so obtained we shall develop an estimate for the number $n(\gamma)$ of times an element $\gamma\in K$ is attained by f or, in other words, for the number of solutions of the equation $f(x)=\gamma$ in D. We shall finally show that when the field K is locally compact, $n(\gamma)$ can be computed exactly, provided the convering can be determined.

§2. **Notation and Review of Earlier Results.** Before presenting our new results, we shall give exact definitions of the concepts employed, introduce some notation, and review some of the results of [3] used herein.

K invariably denotes a field with a non-archimedean valuation, perfect with respect to the topology induced. The valuation of $a\in K$ is $|a|$. Balls in K are denoted by $B(a, \rho)=\{x\in K \mid |x-a|<\rho\}$ or $C(a, \rho)=\{x\in K \mid |x-a|\leqslant\rho\}$, as the case may be. In formulations common to discrete valuations and to continuous valuations, we let the symbol $P(a, \rho)$ stand for $C(a, \rho)$ or $B(a, \rho)$ for K discretely valued or continuously valued, respectively. For functions $f: A\to K$ we define the derivative of f at the point a interior to $A\subset K$ by $f'(a)=\lim_{h\to 0}[f(a+h)-f(a)]/h$. The assumption of interiority is not necessary for our results to be true, but it is helpful in keeping enunciations simple.

Definition 1. *Let a be an interior point of $A\subset K$ and let $f: A\to K$ have a non-vanishing derivative at a. When for some point $x\in A$ (1) – (3) hold, assuming also $\varepsilon(a, a)=0$, then f is quasilinear (ql) for the (ordered) pair (a, x). When (2) does not hold, i.e. when (1) and (4) hold, we shall say that f is quasisuperlinear (qsl) for the (ordered) pair (a, x).*

Definition 2. *Let a be an interior point of $A\subset K$ and let $f: A\to K$ have a non-vanishing derivative at a. A number $\rho>0$ such that f is quasilinear at a with respect to all $x\in P(a, \rho)\subset \mathrm{dom}(f)$ is a radius of quasilinearity (rql) of f at a. Such numbers obviously exist. The ball $P(a, \rho)$ is a quasilinearity ball (qlb) of f at a.*

Let ρ^* be the upper bound of the quasilinearity radii of f at a. Then the exact radius of quasilinearity of f at a (erql), the radius of quasilinearity of f at a for short, $r(f, a)$, is $\sup\{|y| \mid y\in K, |y|<\rho^*\}$. The exact quasilinearity ball (eqlb) of f at a is $P(a, r(f, a)]$. The abbreviations „q l b s“ and „e q l b s“ will stand for „quasilinearity balls“ and „exact quasilinearity balls“, respectively.

Remark 1. It follows from Definition 2 that f need not be defined at all the points of a ball of quasilinearity. This was not pointed out in [3].

In [3] it was shown that in a quasilinearity ball of a function f at a this function has yet a stronger property than the one stated in Definition 1.

Theorem A [3]. *When $\{x, y\}\subset P[a, r(f, a)]\cap \mathrm{dom}(f)$ and $|y-x|=\max(|y-a|, |x-a|)$ then (5) – (7) also hold.*

The function f thus multiplies by the constant factor $|f'(a)|$ the distances of certain points in $P(a, r(f, a)]$, in particular the distance of two points not equidistant from a.

Remark 2. It follows from Theorem A that the equality $f(x)=f(y)$ for $\{x, y\}\subset P[a, r(f, a)]$ entails $|x-a|=|y-a|$. We can easily state conditions under which a given element $\gamma\in K$ is attained by f at *most* once in $P[a, r(f, a)]$. We treat only discrete valuations, and assume that $|\gamma-f(a)|\leqslant\leqslant|f'(a)|r(f, a)$, or γ will not be attained in $C(a, r(f, a)]$ even once, by quasilinearity. For $\gamma\in K$ define $S(f, a, \gamma)=\{x\in K \mid |x-a|=|\gamma-f(a)|/|f'(a)|\}$. Assume now that all $x\in S(f, a, \gamma)$ belonging to $\mathrm{dom}(f)$ are interior points of $\mathrm{dom}(f)$ and satisfy $f'(x)\neq 0$, $r(f, x)\geqslant|\gamma-f(a)|/|f'(a)|$. (We shall see in Theorem 4 (*iii*) that this entails $r(f, x)=r(f, a)$). Then $\{x, y\}\subset S(f, a, \gamma)\cap\mathrm{dom}(f)$, $x\neq y$ entail $|f(x)-f(y)|=|f'(y)||x-y|$ and therefore $f(x)\neq f(y)$. Thus γ is attained in $C(a, r(f, a)]$ at most once.

The following definition of similarity transformations is slightly more general than the one given in [3].

Definition 3. *Let a be an interior point of $A\subset K$, let $f: A\to K$ have a non-vanishing derivative at a, and let $D\subset A$ be a set, such that for $\{x, y\}\subset D$ (5)–(7) hold. Then f is a similarity transformation on D with respect to a. When D is the intersection of $\mathrm{dom}(f)$ with a ball $P(a, \rho)$ of positive radius, then $P(a, \rho)$ is a similarity ball (sb) of f at a, and ρ is a similarity radius (sr) of f at a. Let S be the set of similarity radii of f at a. Then the exact similarity radius (esr) of f at a (esr), $r_s(f, a)$, is $\sup\{|y| \mid y\in K, |y|<\sup S\}$, i.e. $\sup S$ for continuous valuations and $\max\{|y| \mid y\in K, |y|<\sup/S\}$ for discrete valuations. The exact similarity ball (esb) of f at a is $P[a, r_s(f, a)]$. The abbreviations sbs and esbs will stand for „similarity balls" and „exact similarity balls", respectively.*

It follows from Definition 3 that the restriction of a function to any of its similarity balls is univalent.

The following theorem is a very slight generalization of an existence theorem for similarity balls in [3].

Theorem B. *Let $P\subset K$ be a ball of radius $\rho>0$, $A\subset K$ and $A\cap P$ open. With $f: A\to K$, for all $x\in A\cap P$ let $f'(x)$ exist, $f'(x)\neq 0$ and $r(f, x)\geqslant\rho$. Then P is a sb of at every $a\in P\cap A$.*

In [3] additional existence theorems were formulated.

We shall now look at some of the basic properties of similarity transformations which were not dealt with in [3].

Theorem 1. *Let a, b be interior points of $A\subset K$, and let $D\subset A$ contain two points at least.*

(*i*) *Let $f: A\to K$ be a similarity transformation on D with respect to a and with respect to b. Then*

$$|f'(a)-f'(b)|<|f'(a)|=|f'(b)|. \tag{8}$$

(*ii*) *Let f be a similarity transformation on D with respect to a and let (8) hold. Then f is a similarity transformation also with respect to b.*

Proof. (*i*) Let $\{x, y\}\subset D$, $x\neq y$. Then $[f(x)-f(y)]/(x-y)=f'(a)+\eta_1(a, x, y)=f'(b)+\eta_1(b, x, y)$, with $|\eta_1(a, x, y)|<|f'(a)|$, $|\eta_1(b, x, y)|<<|f'(b)|$, and (8) follows. (*ii*) When (5), (6) and (8) hold then $f(x)-f(y)==[f'(b)+\eta_1(b, x, y)](x-y)$, with $|\eta_1(b, x, y)|=|f'(a)-f'(b)+\eta_1(a, x, y)|<<|f'(b)|$.

Corollary to Theorem 1. *Let the sbs $P(a, \rho)$ and $P(b, \sigma)$ of a function $f: E\to K$ intersect. Then $r_s(f, a)=r_s(f, b)$.*

Proof. The esbs $A=P[a, r_s(f, a)]$ and $B=P[b, r_s(f, b)]$ also intersect, whence $A\subset B$ or $B\subset A$. Assume $A\subset B$ first. Since a is an interior point of E, f is a similarity transformation on the infinite set $A\cap E$ with respect to a and with respect to b. By Th. 1(*i*) (8) holds. By Th. 1(*ii*) f is a similarity transformation on $B\cap E$ with respect to a, whence $r_s(f, a)\geqslant r_s(f, b)$, $r_s(f, a)=r_s(f, b)$. The case $B\subset A$ is treated analogously.

§3. Basic Quasilinearity Properties of Functions. In Theorem 2 we shall provide conditions under which the quasilinearity or quasisuperlinearity of a function for a given pair of points can be derived from the quasilinearity or quasisuperlinearity of that function for one, two, or, at most, three other pairs of points. Th. 2 will be widely used in the proofs of the theorems of §4.

Theorem 2. *Let $f: A\to K$ have a non-vanishing derivative at the interior points a, b of $A\subset K$. Also, let $C\in A$, $c\neq a$. Then the following five enunciations are true.*

(*i*) *Defining the function ε by (1), the following equality is an identity.*

$$f(c)-f(a)=\left\{f'(a)+[f'(b)-f'(a)+\varepsilon(b, c)]\frac{c-b}{c-a}+\varepsilon(a, b)\frac{b-a}{c-a}\right\}(c-a). \tag{9}$$

(*ii*) α) *When f is ql for both (a, b) and (b, a) then*

$$|f'(a)|=|f'(b)|. \tag{10}$$

$$|f'(a)-f'(b)|<|f'(a)|. \tag{11}$$

β) *When, instead of α), f is ql for (a, b) and either (11) or*

$$|f'(a)-f'(b)|<|f'(b)|. \tag{12}$$

holds then f is ql for (b, a).

(*iii*) *When*

$$|c-a|\geqslant|b-a| \tag{13}$$

and f is ql for (a, b), (b, a) and (b, c) then f is ql for (a, c).

(*iv*) *Let f be ql for (b, c), and let*

$$|c-b|<\min\{|b-a|, |f'(a)/f'(b)||b-a|\}. \tag{14}$$

When f is ql or qsl for (a, b), respectively, then f has the same property for (a, c). Thus f has the same property for (a, c) whenever c is in a suitable neighbourhood of b.

(*v*) *Let f be ql for (b, a), let*

$$|a-b|<|c-b|, \tag{15}$$

and let (12) hold. When, in addition, f is ql or for (b, c), respectively, then f has the same property for (a, c). Thus f has the same property for (a, c) whenever a is in a suitable neighbourhood of b, provided $c\neq b$ and f' is continuous at b.

Proof. (*i*) We have

$$f(c)-f(a)=[f'(b)+\varepsilon(b,c)]\,(c-b)+[f'(a)+\varepsilon(a,b)]\,(b-a)=$$
$$=\left\{f'(a)+[f'(b)-f'(a)+\varepsilon(b,c)]\,\frac{c-b}{c-a}+\varepsilon(a,b)\,\frac{b-a}{c-a}\right\}(c-a).$$

(*ii*) α) Take absolute values on both sides of the identity

$$f'(a)+\varepsilon(a,b)=f'(b)+\varepsilon(b,a) \tag{16}$$

to obtain (10). Transpose terms in (16) and take absolute values again. β) is also obtained after transposing terms in (16).

(*iii*) Since f is ql for (a,b), (b,a) and (b,c), (10), (11) and $|f'(b)-f'(a)+\varepsilon(b,c)|<|f'(a)|$ follow. Also, $|c-b|\leqslant|c-a|$, by (13). Thus (*iii*) follows from (*i*).

(*iv*) By (14) $|b-a|=|c-a|$. Another application of (14) renders $|f'(b)\,(c-b)/(c-a)|<|f'(a)|$. From this and from the assumptions (*iv*) follows by (*i*).

(*v*) From (15) $|c-a|=|c-b|>0$ follows. From this, from $|\varepsilon(b,a)|<|f'(b)|$ and from (*ii*) we have $|\varepsilon(a,b)\,(b-a)/(c-a)|<|f'(a)|$. From this, from (12) and from the assumptions (*v*) follows by (*i*).

In Theorem 3, which will not be used later on, conditions will be presented, under which the quasilinearity of the sum function of a function series for a pair of points (a,b) can be derived from the quasilinearity of the terms of this series for (a,b).

Theorem 3. (*i*) *Let a be an interior point of $A\subset K$, and $b\in A$. For all $i\in N$, let $f_i\colon A\to K$ be differentiable at a, $f_i'(a)\neq 0$ and f_i ql for (a,b). Assume also that the series $\sum_1^\infty f_i(a)$, $\sum_1^\infty f_i'(a)$ converge. Then $\sum_1^\infty f_i(b)$ converges and*

$$\left|\sum_1^\infty f_i(b)\right|\leqslant\max\{\max[|f_i(a)|,\ |f_i'(a)|\,|b-a|]\mid i\in N\}.$$

(*ii*) *When $\sum_1^\infty f_i$ is differentiable at a term by term and $\max\left\{|f_i'(a)|\,\middle|\, i\in N\right\}=\left|\sum_1^\infty f_i'(a)\right|$ then $\sum_1^\infty f_i$ is ql for (a,b).*

Remark 3. By [3, Th. 5] differentiability of $\sum_1^\infty f_i$ at a term by term is guaranteed when the numbers $r(f_i,a)$ have a positive lower bound.

Proof of Theorem 3. By *ql*, for all $i\in\mathbf{N}$, $|f_i(b)|\leqslant\max[|f_i(a)|,\ |f_i'(a)|\,|b-a|]$. Thus (*i*) follows from the convergence of $\sum_1^\infty f_i(\alpha)$, $\sum_1^\infty f_i'(a)$.

(*ii*) follows from the equality $\left(\sum_1^\infty f_i\right)(b)=\sum_1^\infty\{f_i(a)+[f_i'(a)+\varepsilon_i(a,b)](b-a)\}=$

$$=\sum_1^\infty f_i(a)+\left[\sum_1^\infty f_i'(a)+\sum_1^\infty\varepsilon_i(a,b)\right](b-a),$$ since $|\varepsilon_i(a,b)|<|f_i'(a)|\leqslant$

$$\leqslant\max\{|f_i'(a)|\mid i\in\mathbf{N}\}=\left|\sum_1^\infty f_i'(a)\right|.$$

4. The Global Behaviour of Differentiable Functions. When Q is a quasilinearity ball of the function f at the point a, there exist interrelations between the values of $r(f,a)$ and $f'(a)$ on the one hand the values of $r(f,x)$ and $f'(x)$ with x being in Q, on the other hand. Theorems 4 and 5 will elucidate these interrelations. Thus, knowing more about the values which $r(f,x)$ can attain in a ball of quasilinearity, we have at our disposal a powerful means to determine quasilinearity or quasisuperlinearity of a function for infinitely many pairs of points in the domain of f.

In our formulations we shall conveniently be using the concept of a set-centered ball or set of balls.

Definition 4. *When $S\subset K$ and P is a ball centered at $a\in S$ we shall say that P is S-centered. Analogously, when τ is a set of balls all the centers of which are in S we shall say that τ is S-centered.*

Theorem 4. *Let $f\colon A\to K$ have a non-vanishing derivative at the interior points a,b of $A\subset K$. The following enunciations are then true:*

(*i*) *Every eqlb P of f contains every P-centered qlb of f.*

(*ii*) *When a is in the eqlb of f at b and*

$$r(f,a)\geqslant|b-a| \tag{17}$$

for discrete valuations or

$$r(f,a)>|b-a| \tag{18}$$

for continuous valuations, then $r(f,a)=r(f,b)$.

(*iii*) *When a and b are each in a qlb of f at the other point then the eqlbs of f at a and b coincide.*

(*iv*) *Let $b\in P[a,r(f,a)]$. When $r(f,b)<|b-a|$ and the valuation is discrete, or when $r(f,b)\leqslant|b-a|$ and the valuation is continuous, then $r(f,b)\leqslant|f'(a)|/|f'(b)|\,|b-a|$.*

(*v*) *Let $r(f,a)\geqslant\rho$, $r(f,b)=\sigma>\rho$, with f ql for (a,b). When the valuation is continuous and $|b-a|\leqslant\rho$, or when the valuation is discrete and $|b-a|$ does not exceed the smallest value of elements in K larger than ρ, then $r(f,a)=\sigma$.*

(*vi*) *The union of an infinite strictly increasing sequence of exact quasilinearity balls of a function is not a qlb of that function.*

Proof. (*i*) Let K be discretely valued and $b\in C[a,r(f,a)]$. (*i*) is evident for $r(f,b)\leqslant|b-a|$. For $r(f,b)>|b-a|$ show that f is ql for (a,c) whenever $c\in C[b,r(f,b)]=C[a,r(f,b)]$. For $c\in C[a,|b-a|]=C[b,|b-a|]$ this is evident. For $c\in C[b,r(f,b)]/C(b,|b-a|]$ it follows by Th. 2 (*iii*).

The proof for continuous valuations is analogous. (*ii*) and (*iii*) are immediate consequences of (*i*). (*iv*) For a discretely valued K let $c\in K$ satisfy $|c-b|=r(f,b)$. Then $|c-b|<|b-a|$, $|c-a|=|b-a|$. Suppose $|f'(b)|\,|c-b|>|f'(a)|\,|b-a|$. Then $|f'(a)|\,|c-a|=|f(c)-f(a)|=|[f'(a)+\varepsilon(a,b)]\,(b-a)+[f'(b)+\varepsilon(b,c)]\,(c-b)|=|f'(b)|\,|c-b|>|f'(a)|\,|b-a|=|f'(a)|\,|c-a|$ would follow, since $|\varepsilon(a,b)|<|f'(a)|$, $|\varepsilon(b,c)|<|f'(b)|$. Therefore $|f'(b)|\,|c-b|=|f'(b)|\,r(f,b)\leqslant|f'(a)|\,|b-a|$.

For K continuously valued define $c_i\in B[b,r(f,b)]$ for all $i\in\mathbf{N}$ such that $\lim_{i\to\infty}|c_i-b|=r(f,b)$. Then $|f'(b)|\,|c_i-b|\in|f'(a)|\,|b-a|$ follows, as in the discrete case. Now pass to the limit $i\to\infty$.

(v) Let K be discretely valued. For every $c \in K$ with $\rho < |c-a| \leqslant \sigma$ we have $|c-b| \leqslant \max(|c-a|, |a-b|) \leqslant |c-a| \leqslant \sigma$, whence f is ql for (b, c) and Theorem 2 (iii) applies. Thus $r(f, a) \geqslant \sigma$. But $r(f, a) \leqslant \sigma$, by (i), since $a \in C[b, r(f, b)]$.

The proof for the continuous case is analogous.

(vi) For all $i \in \mathrm{N}$, let S_i be eqbs of f at x_i, with S_i properly contained in S_{i+1} and let $S = \bigcup\limits_{1}^{\infty} S_i$. Then S is a qlb of f at no $x_0 \in S$, since for some $j \in \mathrm{N}$ we must have $x_0 \in S_j$ and since every qlb of f at x_0 is contained in S_j.

Corollary to Theorem 4 (iv). *When K is discretely valued, when everywhere in $C_0 = C[a, r(f, a)]$ the derivative of f vanishes nowhere and $r(f, x)/|x-a|$ is bounded away from zero, then f' is bounded in C_0.*

Proof. When the valuation is continuous and $r(f, x) \geqslant |x-a|$ or when the valuation is discrete and $r(f, x) > |x-a|$ we have $r(f, x) = r(f, a)$, by Th. 4 (ii), and therefore $|f'(x)| = |f'(a)|$, by Th. 2 (ii). In all other cases the corollary immediately follows from Th. 4 (iv).

Corollaries to Theorem 4 (v). *Let $f\colon A \to K$ have a non-vanishing derivative at the interior points a, b of $A \subset K$. The folllowing enunciations then hold.*

(α) *Let ρ, ρ_1 be successive values in the discretely valued field K, let $r(f, a) = \rho < \rho_1 = |b-a|$ and $r(f, b) \geqslant \rho_1$. Then f is qsl for (a, b). Thus, when $r(f, x) \geqslant \rho_1$ everywhere on the „sphere" $S = \{x \in K \mid |x-a| = \rho_1\}$ then f is qsl for (a, x) whenever $x \in S$.*

(β) *Let K be continuously valued, let $|b-a| = r(f, a) = \rho < r(f, b)$. Then f is qsl for (a, b). Again, when $r(f, x) > \rho$ everywhere on the sphere $S = \{x \in K \mid |x-a| = \rho\}$ then f is qsl for every $x \in S$.*

(γ) *Let ρ, ρ_1 be successive values of elements in the discretely valued field K, $\rho < \rho_1$, $C(a, \rho_1) \subset A \subset K$, $r(f, a) = \rho$. Then for at least one element $x \in K$ with $|x-a| = \rho_1$ either $f'(x)$ does not exist or $f'(x) = 0$ or $r(f, x) < \rho_1$.*

(δ) *Let K be continuously valued, $B(a, \rho) \subset A \subset K$, $r(f, a) = \rho$. Then for at least one $x \in K$ with $|x-a| = \rho$ either $f'(x)$ does not exist, or $f'(x) = 0$, or $r(f, x) \leqslant \rho$.*

It is worthwhile mentioning in connection with Theorem 4 (vi) that the union of a strictly increasing sequence of exact quasilinearity balls of a function may sometimes be continued to a ball which is a quasilinearity ball of f.

Example 1. Let K be continuously valued. For $i \in \mathrm{N}$, let $\alpha_i \in K$ satisfy $0 < |\alpha_i| < |\alpha_{i+1}|$ and let $S_i = B\left(0, 1 - \dfrac{1}{i+1}\right)$. With $S_0 = \emptyset$ define $f\colon C(0,1) \to K$ by $f(x) = \alpha_i x$ when $x \in S_i \setminus S_{i-1}$. We shall show that every S_i is an eqlb of f: Let $x_i \in S_i \setminus S_{i-1}$, $x \in S_j \setminus S_{j-1}$, $j \neq i$. Then $f(x) - f(x_i) = \alpha_j x - \alpha_i x = (\alpha_i + (\alpha_j - \alpha_i / x - x_i\, x)\,(x - x_i)$. Also $1 - 1/j \leqslant |x| < 1 - 1/j+1$, $1 - 1/i \leqslant |x_i| < 1 - 1/(i+1)$. Thus, when $j < i$ we have $\left|\dfrac{\alpha_j - \alpha_i}{x - x_i}\, x\right| = \dfrac{|\alpha_i|\,|x|}{|x_i|} <$

$$< \frac{|\alpha_i|\left(1 - \dfrac{1}{j+1}\right)}{1 - \dfrac{1}{i}} \leqslant |\alpha_i| = |f'(x_i)|.$$

On the other hand, when $j > i$ we have $|\alpha_j - \alpha_i / x - x_i\, x| = |\alpha_j| > |\alpha_i| = |f'(x_i)|$.

When $|\alpha_i| < 1$ for all $i \in \mathrm{N}$, then f can be continued to $C(0, 1)$ such that $C(0, 1)$ is a qlb of the continued function, also denoted by f. Just define $f(x) = x$ for $|x| = 1$. For $|x_0| = 1$, $x \in S_j \setminus S_{j-1}$ we then have $f(x) - f(x_0) = \alpha_j x - x_0 = (1 + \alpha_j - 1/x - x_0\, x)\,(x - x_0)$ and $|\alpha_j - 1/x - x_0\, x| = |x|/|x_0| = |x| < 1 = |f'(x_0)|$. When on the other hand $|\alpha_i| \to \infty$, then $C(0, 1)$ is a qlb of f for no definition of f on $|x| = 1$, since $|f(x)| \to \infty$ for $x \uparrow 1$, which is not possible in a qlb.

Quasilinearity pairs (a, x) of a function f, such that $|x-a|$ is the smallest value of elements in K larger than $r(f, a)$ do actually occur.

Example 2. Let $K = Q_5$, $a = 1$, $f(x) = x^3$. Then $r(f, a) = 5^{-1}$, since $|[(1+h)^3 - 1]/h - 3| = |h|\,|3+h|$. For $|h| \leqslant 5^{-1}$, this equals 5^{-1} which is smaller than $|3 \cdot 1^2|$. For $h = 1$ the expression equals 1, which *equals* $|3 \cdot 1^2|$. On the other hand, for $h = 2$ it equals $|2|\,|3+2| = 5^{-1}$ which is *smaller* than $|3.1^2|$.

We shall now see that the set of points in a field K at which a function f has a quasilinearity radius which does not exceed a given number α has a particularly simple structure.

Theorem 5. *Let $f\colon \Gamma \to K$ have a non-vanishing derivative in the open set $\Gamma \subset K$. For every $\alpha > 0$ define $\Gamma_\alpha = \{x \in \Gamma \mid r(f, x) \leqslant \alpha\}$. Then every non empty Γ_α is the intersection of Γ with the disjoint union of a set of balls. When K is discretely valued these balls are exact quasilinearity balls of f.*

Proof. Let $\alpha > 0$ be a number such that $\Gamma_\alpha \neq \emptyset$ and let $P_\alpha = \{P[y, r(f, y)] \mid y \in \Gamma_\alpha\}$. Then $\Gamma_\alpha \subset U(P_\alpha \cap \Gamma) \subset \Gamma_\alpha$, by Theorem 4($i$). Therefore $\Gamma_\alpha = \Gamma \cap (\cup P_\alpha)$. However, for discrete valuations the union of a set of balls of bounded radii which have a non empty intersection is identical with one of the intersecting balls.

From Theorem 4 and Corollary (2) to Theorem 4(v) we learn that in at least one important case, the quasilinearity of a function f for pairs (b, x), with x being in the „sphere" $S = \{z \mid |z-b| = r(f, b)\}$, cannot be extended beyond S. More exactly, let K be discretely valued, let α, β, γ be consecutive values of elements in K, with $\alpha < \beta < \gamma$, and let $b \in \Gamma_\beta \setminus \Gamma_\alpha$. Thus $r(f, b) = \beta$ and f is ql for (b, x) whenever $x \in S$. However, f is qsl for (b, y), $y \in S' = \{z \mid |z-b| = \gamma\}$, whenever S' belongs to $\Gamma \setminus \Gamma_\beta$, by Corollary ($\alpha$) to Theorem 5($v$). The inclusion $S' \subset \Gamma \setminus \Gamma_\beta$ is true, in particular, when each Γ_μ is a ball. In that case $\mu < \nu$ entails $\Gamma_\mu \subset \Gamma_\nu$.

The shape of the sets Γ_α is particularly simple when K is locally compact, when Γ is bounded and every point of Γ is contained in some similarity ball of f. Since Γ is closed as well as open, it is compact and can be covered by finitely many similarity balls of f. On the intersection of each of these balls with Γ, $r(f, x)$ is constant, by the corollary to Th 1. Thus in Γ, $r(f, x)$ takes only finitely many values $r_1 < r_2 \ldots < r_n$. We conclude that for every $i \in \{1, \ldots, n\}$ the set P_{r_i}, defined in the proof of Th. 5, is bounded. Its unboundedness would entail the unboundedness of $\cup P_{r_i}$, since the radius of every ball in P_{r_i} is at least r_1 and K is locally compact. The unboundedness of $\cup P_{r_i}$ would, in turn, entail the unboundedness of $\Gamma_{\gamma_i} \subset \Gamma$, since every $Q \in P_{r_i}$ contains a point of Γ_{r_i}. Thus Γ_α is the intersection of Γ with the union of a finite number of exact quasilinearity balls of f.

The relationship just presented can be reversed. When the number of different quasilinearity radii of f in Γ is finite then every point of Γ lies in a similarity ball of f. In order to see this, let a be any point of Γ and let ρ be the smallest rql of f in Γ. Then $C(a, \rho)$ is obviously a similarity ball of f.

Theorem 6. *Let K be discretely valued, $D \subset A \subset K$, D open. Let $f: A \to K$ have a non-vanishing derivative on D. Assume that the set T of D-centered exact quasilinearity balls of f does not contain an infinite strictly increasing sequence of balls of T.*

Then a unique covering of D by pairwise disjoint, D-centered, exact quasilinearity balls of f exists, and this covering is identical with the set τ of balls in T which are not proper subsets of other balls in T.

In order to see the significance of the condition imposed in Th. 6 on the set of D-centered exact quasilinearity balls of f, consider the following example.

Example 3. Let $K = Qp$ for some fixed p. For all $i \in \mathbf{N}$, let $\alpha_i \in K$, with $0 < |\alpha_i| < |\alpha_{i+1}|$, and $S_i = C(0, p^{i-1})$. With $S_0 = \emptyset$ let $f: K \to K$ be defined by $f(x) = \alpha_i x$ for $x \in S_i \setminus S_{i-1}$. Then every S_i is an eqlb of f with respect to every point $x_i \in S_i \setminus S_{i-1}$. Thus all the exact qlbs of f intersect, but a covering of K by pairwise disjoint eqlbs of f does not exist.

Proof of Theorem 6. Either T contains some ball C^* of infinite radius or all balls in T have finite radius. In the first case let $\tau = \{C^*\}$. In the second case every $C \in T$ is contained in some ball $C_1(C) \in T$ which is *not* a proper subset of another ball in T. Let $\tau = \{C_1(C) \mid C \in T\}$. In both cases τ is not empty, the balls in τ are pairwise disjoint and $\cup \tau = \cup T$, whence $D \subset \cup \tau$. Thus a covering of D by pairwise disjoint, D-centered eqlbs of f exists.

Let ρ also be a covering of D by pairwise disjoint, D-centered, eqlbs of f. Let $U = C[u, r(f, u)]$, with $u \in D$, be any fixed ball in τ and let $S = C[s, r(f, s)]$, with $s \in D$, be that ball in ρ which contains u. Then $S \subset U$, since $U \in \tau$ and since no ball in τ is properly contained in any D-centered eqlb of f. By Theorem 4 (*iii*) $S = U$. Therefore the particular ball in ρ which convers a given $d \in D$ is identical with the ball in τ which covers d, whence $\rho = \tau$. □

As an immediate consequence of Theorem 6 we observe that when τ is finite the number of balls in τ is at most equal to the number of balls in any other covering σ of D by D-centered qlbs of f. One immediately sees this by replacing every ball in σ by a concentric eqlb of f. Either the balls in the new covering are not pairwise disjoint or the new covering is identical with τ.

When for some index set I, $C[x_i, r(f, x_i)] \mid i \in I\}$ is the covering τ of Theorem 6, then $f(D)$ is contained in $\bigcup_I \{C[f(x_i), |f'(x_i)| r(f, x_i)]\}$. Thus, when in a given case τ can be determined, a simple superset for $f(D)$ can be obtained.

From a practical point of view the last theorem of this paper, Th. 7, is of greater portent, as we shall presently see.

Theorem 7. *Let K be discretely valued and $D \subset K$ open. Let $f: D \to K$ have a similarity ball at every point of D. Then a covering of D by pairwise disjoint similarity balls of f exists and this covering is unique.*

Proof. Every $d \in D$ belongs to the sb $Q = P[d, r_s(f, d)]$. Let Q intersect with the sb $S = P[e, r_s(f, e)]$. Then either $Q \subset S$ or $S \subset Q$. Assume first that $Q \subset S$. By the corollary to Theorem 1 we have $r_s(f, d) = r_s(f, e)$, $S = Q$. The case $S \subset Q$ is proved analogously.

When in a given case the covering of Theorem 7 can be determined, it is possible to obtain information on the number of times a given element $\gamma \in K$ is attained by f or, differently expressed, on the number of solutions $n(\gamma)$ of the equation $f(x) = \gamma$. Since f is univalent in each of its similarity balls, $n(\gamma)$ does not exceed the number of balls in the covering σ of D by pairwise disjoint esbs of f. The estimate $n(\gamma) \leq \mathrm{card}(\sigma)$ thus obtained can be improved by disregarding certain balls in the count of the elements of σ. We shall show this for the discrete case. Let $Q = C[a, r_s(f, a)] \in \sigma$. Then $x \in Q$ entails $|f(x) - f(a)| = |f'(a)| \, |x - a|$. Thus, when for some γ we have $|\gamma - f(a)| > |f'(a)| \, r_s(f, a)$ then γ is not attained by f in Q and thus Q can be disregarded. When K is locally compact and σ can be determined we can even compute $n(\gamma)$ exactly, on the basis of the following *Inverse Function Theorem*:

Theorem C [2]. *Let K be locally compact. let the similarity ball $S = C(a, \rho)$ of $f: A \to K$ be contained in A. Then every point γ in $C[f(a), |f'(a)| \rho]$ is attained in S exactly once.*

Thus, $n(\gamma)$ exactly equals the number of those esbs $C[a, r_s(f, a)]$ in σ for which $|\gamma - f(a)| \leq |f'(a)| \, r_s(f, a)$.

We have seen that a function f is not necessarily univalent in its quasilinearity balls. Therefore no estimate of $n(\gamma)$ can be obtained from knowledge of the covering mentioned in Theorem 5. On the basis of Remark 2 an element $\gamma \in K$ is attained by the function f at most once in a given quasilinearity ball of f, when $r(f, x)$ satisfies a certain inequality for all x belonging to some γ-dependent point set. Thus, when this inequality is satisfied in all the balls of τ, an estimate for $n(\gamma)$ can be gained also for this case.

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