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SOME NEW DISCRETE INEQUALITIES IN TWO INDEPENDENT VARIABLES

BY

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1. **Introduction.** During the last few years the role played by the discrete inequalities in the theory of difference equations and numerical analysis is well known. One of the most used discrete inequalities is the analogue of the celebrated Gronwall-Bellman-Reid inequality established by Jones [4] and Sugiyama [11] and its subsequent generalizations (see [3], [5], [6], [7], [8]). An interesting but apparently neglected generalization of GBR inequality in two independent variables is due to Wendroff [1, p. 154] and this has been generalized for various motivations in different directions, (see [2], [10]). The discrete version of the Wendroff inequality and its generalizations are given in [9]. Like the inequality of Jones and Sugiyama these inequalities are very useful in studying the qualitative and quantitative properties of solutions of partial difference equations.

In this paper we present several new discrete inequalities in two independent variables which can be used as handy tools in the analysis of various problems in the theory of finite difference equations involving two independent variables. Section 2 deals with linear and in section 3 we present some nonlinear inequalities. Finally in section 4 some applications are given.

Before giving the main results of this paper, we shall first introduce some notations which we shall use throughout the paper. N — denotes the et $\{0, 1, \ldots\}$. The expression $\sum_{s=0}^{t-1} b(s)$ represents a solution of the linear difference equation $\Delta Z(t) = b(t)$ for all $t \in N$ under the initial condition $Z(0) = 0$, where Δ is the operator defined by $\Delta Z(t) = Z(t+1) - Z(t)$. It is supposed that $\sum_{s=0}^{-1} b(s) = 0$. The expression $\prod_{s=0}^{t-1} c(s)$ represents a solution of the linear difference equation $Z(t+1) = C(t)\, Z(t)$ for all $t \in N$ under the initial condition $Z(0) = 1$. It is supposed that $\prod_{s=0}^{-1} c(s) = 1$. Also we define

$$\Delta Z_x(x, y) = Z(x+1, y) - Z(x, y), \quad \Delta Z_y(x, y) = Z(x, y+1) - Z(x, y),$$
$$\Delta Z_{xy}(x, y) = \Delta Z_x(x, y+1) - \Delta Z_x(x, y)$$

where the letters x and y are used to denote the two independent variables which are members of N.

In what follows we shall assume that the functions appearing in the inequalities are real-valued, nonnegative and defined for all $x\geq 0$, $y\geq 0$.

2. Linear Inequalities. **Theorem 1.** *Let the following inequality be satisfied for all* $x\geq 0$, $y\geq 0$

$$u(x,y)\leq a(x)+b(y)+c(x,y)\sum_{r=1}^{n}E^r(x,y,u) \tag{1}$$

where

$$E^r(x,y,u)=\sum_{s_1=0}^{x-1}\sum_{t_1=0}^{y-1}f_{r1}(s_1,t_1)\sum_{s_2=0}^{s_1-1}\sum_{t_2=0}^{t_1-1}f_{n2}(s_2,t_2)\ldots$$

$$\sum_{s_r=0}^{s_{r-1}-1}\sum_{t_r=0}^{t_{r-1}-1}f_{rr}(s_r,t_r)u(s_r,t_r)$$

with $a(x)$ *and* $b(y)>0$ *also* $\Delta a(x)$, $\Delta b(y)\geq 0$; $c(x,y)\geq 1$. *Then for all* $x\geq 0$, $y\geq 0$

$$u(x,y)\leq c(x,y)[a(0)+b(y)]\prod_{s=0}^{x-1}\left[1+\frac{\Delta a(s)}{a(s)+b(0)}+\sum_{t=0}^{y-1}\sum_{r=1}^{n}\Delta E^r_{xy}(s,t,c)\right].$$

Proof. Inequality (1) can be rewritten as

$$m(x,y)\leq a(x)+b(y)+\sum_{r=1}^{n}E(x,y,cm) \tag{2}$$

where $m(x,y)=u(x,y)/c(x,y)$. Let $v(x,y)$ be the right members of (2), then

$$\Delta v_{xy}(x,y)=\sum_{r=1}^{n}\Delta E^r_{xy}(x,y,cm)\leq\sum_{r=1}^{n}\Delta E^r_{xy}(x,y,c)\,V(x,y), \tag{3}$$

since $v(x,y)$ is nondecreasing in x and y. From (3)

$$\frac{\Delta v_x(x,y+1)}{v(x,y+1)}-\frac{\Delta v_x(x,y)}{v(x,y)}\leq\sum_{r=1}^{n}E^r_{xy}(x,y,c). \tag{4}$$

Keeping x fixed in (4) and summing over y from 0 to $y-1$, we obtain

$$\frac{\Delta v_x(x,y)}{v(x,y)}\leq\frac{\Delta a(x)}{a(x)+b(0)}+\sum_{t=0}^{y-1}\sum_{r=1}^{n}\Delta E^r_{xy}(x,t,c),$$

or

$$\frac{v(x+1,y)}{v(x,y)}\leq 1+\frac{\Delta a(x)}{a(x)+b(0)}+\sum_{t=0}^{y-1}\sum_{r=1}^{n}\Delta E^r_{xy}(x,t,c). \tag{5}$$

Keeping y fixed in (5) and taking the product over x from 0 to $x-1$, we obtain

$$v(x,y)\leq[a(0)+b(y)]\prod_{s=0}^{x-1}\left[1+\frac{\Delta a(s)}{a(s)+b(0)}+\sum_{t=0}^{y-1}\sum_{r=1}^{n}\Delta E^r_{xy}(s,t,c)\right]$$

and now the result follows from $u/c\leq v$.

Remark 1. For $n=1$ and $c(x,y)\equiv 1$ the result is the same as given in [9].

Remark 2. Let $f_{11}(x,y)=f_{21}(x,y)$, $c(x,y)\equiv 1$ and $n=2$ then the result is not comparable with the result obtained in theorem 2 of [9].

Corollary 2. *In inequality* (*y*) *replace* $a(x)+b(y)$ *by* $p(x,y)$ *where* $p(x,y)$ *is positive and nondecreasing; then for* $x\geq 0$, $y\geq 0$

$$u(x,y)\leq p(x,y)\,c(x,y)\prod_{s=0}^{x-1}\left[1+\sum_{t=0}^{y-1}\sum_{r=1}^{n}\Delta E^r_{xy}(s,t,c)\right].$$

Next result is about the following inequality

$$u(x,y)\leq a(x,y)+\sum_{i=1}^{n}g_i(x,y)\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}h_i(s,t)\,u(s,t). \tag{6}$$

Theorem 3. *Let the inequality* (6) *be satisfied, where (i)* $a(x,y)>0$ *and nondecreasing (ii)* $g_i(x,y)\geq 1$ $(i=1,2,\ldots,n)$ *and nondecreasing for all* $i\geq 2$. *Then*

$$u(x,y)\leq E_n\,a(x,y) \tag{7}$$

where E_k *is defined as follows*

$$E_0w=w,\quad E_kw=w\,(E_{k-1}g_k)\prod_{s=0}^{x-1}\left[1+\sum_{t=0}^{y-1}h_k\,E_{k-1}g_k\right],\quad k=1,2,\ldots,n.$$

Proof. The proof is by finite induction. For $n=1$ the result follows from corollary 2. Let us assume that (7) is true for given integer k, $1<k\leq n-1$. Now we are given

$$u(x,y)\leq\left[a(x,y)+g_{k+1}(x,y)\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}h_{k+1}(s,t)\,u(s,t)\right]+$$

$$+\sum_{i=1}^{k}g_i(x,y)\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}h_i(s,t)\,u(s,t);$$

since the part inside the bracket is nondecreasing, we have

$$u(x,y)\leq E_k\left[a(x,y)+g_{k+1}(x,y)\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}h_{k+1}(s,t)\,u(s,t)\right]\leq$$

$$\leq E_ka(x,y)+g_{k+1}(x,y)\,E_k\,a(x,y)\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}h_{k+1}(s,t)\frac{u(s,t)}{a(s,t)}.$$

Thus we find

$$\frac{u(x,y)}{a(x,y)}\leq E_k\,g_{k+1}(x,y)\left[1+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}h_{k+1}(s,t)\frac{u(s,t)}{a(s,t)}\right].$$

Now applying corollary 2, we obtain

$$u(x,y)\leq a(x,y)E_k\,g_{k+1}(x,y)\prod_{s=0}^{x-1}\left[1+\sum_{t=0}^{y-1}h_{k+1}(s,t)\,E_k\,g_{k+1}(s,t)\right]=E_{k+1}a(x,y).$$

This proves that (7) holds for $k+1$, hence the result is true for all k.

Theorem 4. *Let the inequality (6) be satisfied with $a(x, y)$ nondecreasing and positive also $g_i(x, y)\geq 1$ for all i then for all $x\geq 0$, $y\geq 0$*

$$u(x, y)\leq a(x, y)\prod_{i=1}^{n} g_i(x, y)\prod_{s=0}^{x-1}\left[1+\sum_{t=0}^{y-1}\sum_{r=1}^{n} h_r(s,t)\prod_{i=1}^{n} g_i(s,t)\right].$$

Proof. The proof is similar to previous theorems.

3. Nonlinear Inequalities. Theorem 5. *Let the following inequality be satisfied for all $x\geq 0$, $y\geq 0$*

$$u(x, y)\leq p(x, y)\left[u_0+\sum_{r=1}^{n}\dot{E}^r(x, y, u)\right]. \tag{8}$$

where $u_0>0$ and

$$\dot{E}^r(x, y, u)=\sum_{s_1=0}^{x-1}\sum_{t_1=0}^{y-1} f_{r1}(s_1, t_1)\, u^{k_{r1}}(s_1, t_1)\dots\sum_{s_r=0}^{s_{r-1}-1}\sum_{t_r=0}^{t_{r-1}-1} f_{rr}(s_r, t_r)\, u^{k_{rr}}(s_r, t_r)$$

k_{rj}, $1\leq r\leq n$, $1\leq j\leq r$ are nonnegative real numbers. Then for $\alpha<1$, $x\geq 0$, $y\geq 0$

$$u(x, y)\leq p(x, y)\left[u_0^{1-\alpha}+(1-\alpha)\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=1}^{n}\dot{E}'_{xy}(s, t, p)\, u_0^{\alpha_r-\alpha}\right]^{1/(1-\alpha)}$$

where

$$\alpha_r=\sum_{j=1}^{r} k_{rj}\ \text{and}\ \alpha=\max\alpha_r\quad 1\leq r\leq n.$$

Proof. The inequality (8) can be written as $u(x, y)\leq p(x, y)\, m(x, y)$:

$$\text{where } m(x, y)=u_0+\sum_{r=1}^{n}\dot{E}^r(x, y, u).$$

Thus,

$$\Delta m_{xy}(x, y)\leq\sum_{r=1}^{n}\Delta\dot{E}'_{xy}(x, y, p)\,[m(x, y)]^{\alpha_r}.$$

Since $m(x, y)\geq u_0$, we have

$$\Delta m_{xy}(x, y)\leq m^{\alpha}(x, y)\sum_{r=1}^{n}\Delta\dot{E}'_{xy}(x, y, p)\, u_0^{\alpha_r-\alpha}.$$

By using the nondecreasing nature of $m(x, y)$, we find

$$\frac{\Delta m_x(x, y+1)}{m^{\alpha}(x, y+1)}-\frac{\Delta m_x(x, y)}{m^{\alpha}(x, y)}\leq\sum_{r=1}^{n}\Delta\dot{E}'_{xy}(x, y, p)\, u_0^{\alpha_r-\alpha}. \tag{9}$$

In (9) keeping x fixed and summing over y from 0 to $y-1$, we get on using $\Delta m_x(x, 0)=0$

$$\frac{\Delta m_x(x, y)}{m^{\alpha}(x, y)}\leq\sum_{t=0}^{y-1}\sum_{r=1}^{n}\Delta\dot{E}'_{xy}(x, t, p)\, u_0^{\alpha_r-\alpha}. \tag{10}$$

For $x\geq 0$, $y\geq 0$, we have

$$\frac{m^{1-\alpha}(x+1, y)-m^{1-\alpha}(x, y)}{1-\alpha}=\int_{x}^{x+1}\frac{dm(s, y)}{m^{\alpha}(s, y)}\leq\frac{\Delta m_x(x, y)}{m^{\alpha}(x, y)}$$

and from (10) we obtain

$$\Delta m_x^{1-\alpha}(x, y)\leq(1-\alpha)\sum_{t=0}^{y-1}\sum_{r=1}^{n}\Delta\dot{E}'_{xy}(x, t, p)\, u_0^{\alpha_r-\alpha}. \tag{11}$$

Now keeping y fixed in (11) and summing over x from 0 to $x-1$, we obtain on using $m(0, y)=u_0$

$$m(x, y)\leq\left[u_0^{1-\alpha}+(1-\alpha)\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=1}^{n}\Delta\dot{E}'_{xy}(s, t, p)\, u_0^{\alpha_r-\alpha}\right]^{1/(1-\alpha)}$$

and hence the result follows.

Definition. *A function $W:[0, \infty)\to[0, \infty)$ is said to belong to class S if*

(i) $W(u)$ is positive, nondecreasing and continuous for $u\geq 0$;

(ii) $\frac{1}{v}W(u)\leq W(u/v)$, for all $u\geq 0$, $v>0$.

Theorem 6. *Let the following inequality be satisfied for all $x\geq 0$, $y\geq 0$*

$$u(x, y)\leq a(x, y)+\sum_{r=1}^{n}E^r(x, y, u)+\sum_{i=1}^{m}g_i(x, y)\times\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}h_i(s, t)W_i(u(s, t)) \tag{12}$$

where

(i) $a(x, y)$ is positive and nondecreasing

(ii) $g_i(x, y)\geq 1$ for all $i=1, 2, \dots, m$

(iii) $W_i\in S$ for all $i=1, 2, \dots, m$.

Then,

$$u(x, y)\leq a(x, y)\,\Phi(x, y)\prod_{i=1}^{m}g_i(x, y)\prod_{r=1}^{m}E_r(x, y) \tag{13}$$

where

$$\Phi(x, y)=\prod_{s=0}^{x-1}\left[1+\sum_{t=0}^{y-1}\sum_{r=1}^{n}\Delta E'_{xy}\left(s, t, \prod_{i=1}^{m}g_i\right)\right],$$

$$E_1(x, y)=G_1^{-1}\left[G_1(1)+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}h_1(s, t)\,\Phi(s, t)\prod_{r=1}^{m}g_r(s, t)\right],$$

$$E_k(x, y)=G_k^{-1}\left[G_k(1)+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}h_k(s, t)\,\Phi(s, t)\prod_{r=1}^{m}g_r(s, t)\times\prod_{i=1}^{k-1}E_i(s, t)\right]$$

$$k=2, 3, \dots, m,$$

$$G_k(u)=\int_{u_0}^{u}\frac{ds}{W_k(s)}, u\geq u_0>0$$

as long as

$$G_k(1) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} h_k(s,t)\,\varphi(s,t)\prod_{r=1}^{m} g_r(s,t)\prod_{i=1}^{k-1} E_i(s,t) \in \mathrm{Dom}\,(G_k^{-1})$$

$$k = 1, 2, \ldots, m.$$

Proof. From (ii) inequality (12) can be written as

$$v(x,y) \leq a^*(x,y) + \sum_{r=1}^{n} E^r\Big(x, y, \prod_{i=1}^{m} g_i\, v\Big),$$

where

$$a^*(x,y) = a(x,y) + \sum_{r=1}^{m}\sum_{s=0}^{x-1}\sum_{t=0}^{y-1} h_r(s,t)\prod_{i=1}^{m} g_i(s,t)\, W_r(v(s,t)v(x,y) = \frac{u(x,y)}{\prod_{i=1}^{m} g_i(x,y)}.$$

From corollary 2, we find

$$v(x,y) \leq a^*(x,y)\,\varphi(x,y). \tag{14}$$

Now from (14) we have

$$m(x,y) \leq 1 + \sum_{n=1}^{m}\sum_{s=0}^{x-1}\sum_{t=0}^{y-1} h_r(s,t)\sum_{i=1}^{m} g_i(s,t)\,\varphi(s,t)\times W_r(m(s,t)), \tag{15}$$

where

$$m(x,y) = v(x,y)/(a(x,y)\,\varphi(x,y)).$$

Rest of the proof is by induction and we shall represent

$$p(x,y) = \prod_{i=1}^{m} g_i(x,y), \text{ for } m = 1 \text{ we have}$$

$$m(x,y) \leq 1 + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} h_1(s,t)\,p(s,t)\,\varphi(s,t)\,W_1(m(s,t)). \tag{16}$$

Let $n(x,y)$ be the right member of (16), then

$$\Delta n_{xy}(x,y) \leq h_1(x,y)\,p(x,y)\,\varphi(x,y)\,W_1(n(x,y))$$

or

$$\frac{\Delta n_x(x,y+1)}{W_1(n(x,y+1))} - \frac{\Delta n_x(x,y)}{W_1(n(x,y))} \leq h_1(x,y)\,p(x,y)\,\varphi(x,y).$$

Now keeping x fixed and summing over y from 0 to $y-1$, we get on using. $\Delta n_x(x,0) = 0$

$$\frac{\Delta n_x(x,y)}{W(n(x,y))} \leq \sum_{t=0}^{y-1} h_1(x,t)\,p(x,t)\,\varphi(x,t).$$

From the definition of G_1 we have

$$G_1(n(x+1,y)) - G_1(n(x,y)) = \int_{n(x,y)}^{n(x+1,y)} \frac{ds}{W_1(s)} \leq \frac{\Delta n_x(x,y)}{W_1(n(x,y))} \leq$$

$$\leq \sum_{t=0}^{y-1} h_1(x,t)\,p(x,t)\,\varphi(x,t). \tag{17}$$

Now keeping y fixed in (17) and summing over x from 0 to $x-1$, we obtain

$$n(x,y) \leq G_1^{-1}\Big[G_1(1) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} h_1(s,t)\,\varphi(s,t)\,p(s,t)\Big],$$

and hence the result follows for $m=1$. Assume the theorem is true for a given integer k, $1 < k \leq m-1$; then we are given

$$m(x,y) \leq \Big[1 + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} h_{k+1}(s,t)\varphi(s,t)\,p(s,t)\,W_{k+1}(m(s,t))\Big] +$$

$$+ \sum_{i=1}^{k}\sum_{s=1}^{x-1}\sum_{t=0}^{y-1} h_i(s,t)\,\varphi(s,t)\,p(s,t)\,W_i(m(s,t)),$$

since the part inside the bracket is nondecreasing, we obtain as earlier

$$m(x,y) \leq \Big[1 + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} h_{k+1}(s,t)\,\varphi(s,t)\,p(s,t)\,W_{k+1}(m(s,t))\Big] \times \prod_{i=1}^{k} E_i(x,y);$$

or

$$n(x,y) \leq 1 + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} h_{k+1}(s,t)\,\varphi(s,t)\,p(s,t)\prod_{i=1}^{k} E_i(s,t)\,W_{k+1}(n(s,t)),$$

where

$$n(x,y) = m(x,y)/\Big(\prod_{i=1}^{k} E_i(x,y)\Big).$$

Repeating the same arguments as before, we get

$$n(x,y) \leq G_{k+1}^{-1}\Big[G_{k+1}(1) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} h_{k+1}(s,t)\,\varphi(s,t)\,p(s,t) \times$$

$$\prod_{i=1}^{k} E_i(s,t)\Big] = E_{k+1}(x,y)$$

hence the result is true for $k+1$ and thus the theorem follows.

Corollary 7. *In theorem 6, let $g_i(x,y)$ for all $1 \leq i \leq m$ be nondecreasing. Then*

$$u(x,y) \leq a(x,y)\,\psi(x,y)\prod_{i=1}^{m} E_i(x,y).$$

where

$$\psi(x, y) = \prod_{s=0}^{x-1}\left[1 + \sum_{t=0}^{y-1}\sum_{r=1}^{m} \Delta E'_{xy}(s, t, 1)\right],$$

and

$$E_k(x, y) = g_k(x, y)\, G_k^{-1}\left[G_k(1) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} h_k(s, t)\, g_k(s, t)\, \psi(s, t)\right] k = 1, 2, \ldots, m.$$

The next result deals with the following inequality :

(18) $$u(x, y) \leqslant a(x, y) + \sum_{r=1}^{n} E^r(x, y, u) + \sum_{r=1}^{m} E^r(x, y, W(u)).$$

Theorem 8. *Let the inequality (18) be satisfied with* (i) $a(x, y)$ *nondecreasing and positive* (*ii*) $W \in S$.

Then

(19) $$u(x, y) \leqslant a(x, y)\psi(x, y)\, G^{-1}\left[G(1) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=1}^{m} \Delta E'_{xy}(s, t, \psi)\right]$$

where

$$G_1(u) = \int_{u_0}^{u} \frac{ds}{W(s)},\ 0 < u_0 \leqslant u,$$

and $\psi(x, y)$ *is the same as defined in corollary 7, also the term inside the bracket of (19)* $\in \mathrm{Dom}(G^{-1})$.

The proof of this theorem is similar to previous ones.

Theorem 9. *Let the inequality (18) be satisfied with* (i) $a(x, y)$ *is nondecreasing and positive* (ii) W *is positive, nondecreasing, continuous and submultiplicative. Then*

$$u(x, y) \leqslant a(x, y)\, \psi(x, y)\, G^{-1}\left[G(1) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=1}^{m} \Delta E'_{xy}\left(s, t, \frac{W(\psi a)}{a}\right)\right],$$

where ψ *and* G *are the same as defined in theorem 7, also the term inside the bracket* $\in \mathrm{Dom}(G^{-1})$.

Proof. Inequality (18) can be written as

$$u(x, y) \leqslant a^*(x, y) + \sum_{r=1}^{n} E^r(x, y, u),$$

where

$$a^*(x, y) = a(x, y) + \sum_{r=1}^{m} E^r(x, y, W(u)),$$

since $a^*(x, y)$ is nondecreasing from corollary 2, we obtain

(20) $$u(x, y) \leqslant a^*(x, y)\, \psi(x, y)$$

or

(21) $$m(x, y) \leqslant 1 + \sum_{r=1}^{m} E^r\left(x, y, \frac{W(\psi a m)}{a}\right),$$

where $m(x, y) = u(x, y)/a(x, y)\, \psi(x, y)$.

Let $n(x, y)$ be the right member of (21). Then since W is submultiplicative

$$\Delta n_{xy}(x, y) \leqslant \sum_{r=1}^{m} \Delta E'_{xy}\left(x, y, \frac{W(\psi a)}{a}\right) W(n(x, y)).$$

Now using the similar arguments as in theorem 6, we obtain

$$n(x, y) \leqslant G^{-1}\left[G(1) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=1}^{m} \Delta E'_{xy}\left(s, t, \frac{W(\psi a)}{a}\right)\right]$$

and from this the result follows.

4. Some Applications. There are many possible applications of the inequalities obtained in sections 2 and 3. Here we shall present only two which are sufficient to convey the importance of our results.

Consider the system of sum-difference equations

(22) $$u_1(x, y) = a_1(x) + b_1(y) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} f_1(s, t, u_1(s, t), u_2(s, t))$$

(23) $$u_2(x, y) = a_2(x) + b_2(y) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} f_2(s, t, u_1(s, t))$$

where $a_1(x),\ b_1(y) > 0$; $\Delta a_1(x),\ \Delta b_1(y) \geqslant 0$ for all $x, y \in N$ and

(24) $$|f_1(x, y, u_1, u_2)| \leqslant h_1(x, y)\,|u_1| + h_2(x, y)\,|u_2|$$

(25) $$|f_2(x, y, u_1)| \leqslant h_3(x, y)\,|u_1|.$$

Using (25) in (23) we obtain

(26) $$|u_2(x, y)| \leqslant |a_2(x)| + |b_2(y)| + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} h_3(s, t)\,|u_1(s, t)|.$$

Using (24) and (26) in (22) we find

(27) $$\begin{aligned} |u_1(x, y)| \leqslant a_1(x) + b_1(y) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} h_1(s, t)\,|u_1(s, t)| + \\ + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\left\{h_2(s, t)\,[\,|a_2(s)| + |b_2(t)|\,] + \sum_{\tau=0}^{s-1}\sum_{\eta=0}^{t-1} h_3(\tau, \eta)\,|u_1(\tau, \eta)|\right\}. \end{aligned}$$

For the inequality (27) theorem 1 is applicable and hence we can find upper bound for $|u_1(x, y)|$ and from (26) the upper bound for $|u_2(x, y)|$ in terms of known functions.

In the next example we apply our theorem 8 to obtain a bound on the solution of a partial difference equation of the form

$$\Delta u_{xy} = f\left(x,\ y,\ u(x,\ y),\ \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} k(x, y, s, t, u(s, t)\right) \tag{28}$$

With the given boundary conditions $u(x, 0) = g(x)$, $u(0, y) = h(y)$ $u(0, 0) = 0$ where all the functions are real-valued and defined on N, also

$$|g(x) + h(y)| \leqslant a(x, y), \tag{29}$$

$$|k(x, y, s, t, u)| \leqslant b_3(s, t)\ W(|u(s, t)|), \tag{30}$$

$$|f(x, y, u, \varphi)| \leqslant b_1(x, y)\ |u(x, y)| + b_2(x, y)\ |\varphi| \tag{31}$$

where a and W are same as in theorem 8.

Any solution of (28) satisfying the boundary conditions can be written as

$$u(x, y) = g(x) + h(y) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} f(s, t, u(s, t), \sum_{\tau=0}^{s-1}\sum_{\eta=0}^{t-1} k(s, t, \tau, \eta, u(\tau, \eta)).$$

Using (29 – (31), we find

$$|u(x, y)| \leqslant a(x, y) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b_1(s, t)\ |u(s, t)| + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b_2(s, t) \times$$
$$\times \sum_{\tau=0}^{s-1}\sum_{\eta=0}^{t-1} b_3(\tau, \eta)\ W(|u(\tau, \eta)|).$$

This is the same form discussed in theorem 8 and hence upper estimate for $|u(x, y)|$ can be obtained.

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A SHAPE THEORY FOR CLOSURE ALGEBRAS

BY

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0. Introduction. The closure algebras (i.e. σ-complete Boolean algebras with a closure operation) were studied, from the topological point of view, by many authors, among which we cite: J. Mc Kinsey and A. Tarski, [1], R. Sikorski, [2], [3], G. Nöbeling, [4], [5], M. R. Bunjatov, [6]. Some problems of homology and homotopy theory for closure algebras was investigated in the last years by M. R. Bunjatov, [7], [8], [9], S. A. Bairamov [10], E. A. Raga Mohamed Baha, [11], [12], N. I. Ahmed Hilmi Muhamed, [13], etc.

In this note we construct a shape theory for closure algebras. This is possible by the results in [16] where the notion of ANR-algebra is defined and it is proved that for each σ-complete closure algebra A, there exists a **H**-ANR-**F** Boolean system, which is **H**-ANR-**B** associated with A [16, Theorem 6.5]. Let us denote by **H** the σ-homotopy category of the σ-complete closure algebras and by **A** the σ-homotopy category of the σ-ANR algebras.

We shall use the definitions and results from [16].

1. The Boolean shape category BS.

1.1. Following S. Mardešić, [14], and K. Morita, [15], we shall define the Boolean shape category **BS**: The objects of **BS** are all σ-complete closure algebras. The morphisms of **BS** are called Boolean shapings (or Boolean shape homomorphisms) and are defined as follows: let A and B be σ-complete closure algebras. A Boolean shaping $f \in$ **BS**(A, B) is a function which assigns to every $[h] \in \mathbf{H}(Q, A)$, $Q \in \mathrm{Ob}\,\mathbf{A}$, a σ-homotopy class $f([h]) \in \mathbf{H}(Q, B)$, in such a way that for $Q' \in \mathrm{Ob}\,\mathbf{A}$, $[h'] \in \mathbf{H}(Q', A)$ and $[k] \in \mathbf{H}(Q, Q')$, the equality $[h']\ [k] = [h]$ implies $f([h'])\ [k] = f([h])$:

$$\begin{tikzcd} & A & \\ Q \arrow[ur, "{[h]}"] \arrow[rr, "{[k]}"'] & & Q' \arrow[ul, "{[h']}"'] \end{tikzcd} \Longrightarrow \begin{tikzcd} & B & \\ Q \arrow[ur, "{f([h])}"] \arrow[rr, "{[k]}"'] & & Q' \arrow[ul, "{f([h'])}"'] \end{tikzcd}$$

If A, B, C are σ-complete closure algebras and $f \in \mathbf{BS}(A, B)$, $g \in \mathbf{BS}(B, C)$ are Boolean shapings, then the composite Boolean shaping $g \circ f$ is defined as follows: if $[h] \in \mathbf{H}(Q, A)$, $Q \in \mathrm{Ob}\mathbf{A}$, then $(g \circ f)([h]) = g(f([h]))$. If $[h'] \in \mathbf{H}(Q', A)$, $Q' \in \mathrm{Ob}\mathbf{A}$, $[k] \in \mathbf{H}(Q, Q')$, and $[h']\ [k] = [h]$, then $f([h'])[k] = f([h])$ and therefore $g(f([h']))\ [k] = g(f([h]))$, that is $(g \circ f)([h'])\ [k] = (g \circ f)([h])$, which proves that $g \circ f$ is indeed a Boolean shaping.

The identical Boolean shaping $1_A \in \mathbf{BS}(A, A)$ is defined by $1_A([h]) = [h]$, for $[h] \in \mathbf{H}(Q, A)$, $Q \in \mathrm{Ob}\mathbf{A}$. It easily results that for every Boolean shaping $f \in \mathbf{BS}(A, B)$, $1_B \circ f = f$ and $f \circ 1_A = f$.