$$\Delta u_{xy} = f\left(x,\ y,\ u(x,\ y),\ \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} k(x, y, s, t, u(s, t))\right) \tag{28}$$

With the given boundary conditions $u(x, 0) = g(x)$, $u(0, y) = h(y)$ $u(0, 0) = 0$ where all the functions are real-valued and defined on N, also

$$| g(x) + h(y) | \leqslant a(x, y), \tag{29}$$

$$| k(x, y, s, t, u) | \leqslant b_3(s, t)\ W(\ | u(s, t) |), \tag{30}$$

$$| f(x, y, u, \varphi) | \leqslant b_1(x, y) \ | u(x, y) | + b_2(x, y) \ | \varphi | \tag{31}$$

where a and W are same as in theorem 8.

Any solution of (28) satisfying the boundary conditions can be written as

$$u(x, y) = g(x) + h(y) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} f\left(s,\ t,\ u(s, t),\ \sum_{\tau=0}^{s-1}\sum_{\eta=0}^{t-1} k(s, t, \tau, \eta, u(\tau, \eta))\right).$$

Using (29 – (31), we find

$$| u(x, y) | \leqslant a(x, y) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b_1(s, t) \ | u(s, t) | + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b_2(s, t) \times$$
$$\times \sum_{\tau=0}^{s-1}\sum_{\eta=0}^{t-1} b_3(\tau, \eta)\ W(\ | u(\tau, \eta) |).$$

This is the same form discussed in theorem 8 and hence upper estimate for $| u(x, y) |$ can be obtained.

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A SHAPE THEORY FOR CLOSURE ALGEBRAS

BY

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0. Introduction. The closure algebras (i.e. σ-complete Boolean algebras with a closure operation) were studied, from the topological point of view, by many authors, among which we cite: J. Mc Kinsey and A. Tarski, [1], R. Sikorski, [2], [3], G. Nöbeling, [4], [5], M. R. Bunjatov, [6]. Some problems of homology and homotopy theory for closure algebras was investigated in the last years by M. R. Bunjatov, [7], [8], [9], S. A. Bairamov [10], E. A. Raga Mohamed Baha, [11], [12], N. I. Ahmed Hilmi Muhamed, [13], etc.

In this note we construct a shape theory for closure algebras. This is possible by the results in [16] where the notion of ANR-algebra is defined and it is proved that for each σ-complete closure algebra A, there exists a **H**-ANR-**F** Boolean system, which is **H**-ANR-**B** associated with A [16, Theorem 6.5]. Let us denote by **H** the σ-homotopy category of the σ-complete closure algebras and by **A** the σ-homotopy category of the σ-ANR algebras.

We shall use the definitions and results from [16].

1. The Boolean shape category BS.

1.1. Following S. Mardešić, [14], and K. Morita, [15], we shall define the Boolean shape category **BS**: The objects of **BS** are all σ-complete closure algebras. The morphisms of **BS** are called Boolean shapings (or Boolean shape homomorphisms) and are defined as follows: let A and B be σ-complete closure algebras. A Boolean shaping $f \in \mathbf{BS}(A, B)$ is a function which assigns to every $[h] \in \mathbf{H}(Q, A)$, $Q \in \mathrm{Ob}\,\mathbf{A}$, a σ-homotopy class $f([h]) \in \mathbf{H}(Q, B)$, in such a way that for $Q' \in \mathrm{Ob}\,\mathbf{A}$, $[h'] \in \mathbf{H}(Q', A)$ and $[k] \in \mathbf{H}(Q, Q')$, the equality $[h']\,[k] = [h]$ implies $f([h'])\,[k] = f([h])$:

$$\begin{tikzcd}
 & A & & & & B & \\
Q \arrow[ur, "{[h]}"] \arrow[rr, "{[k]}"] & & Q' \arrow[ul, "{[h']}"'] & \Longrightarrow & Q \arrow[ur, "{f([h])}"] \arrow[rr, "{[k]}"] & & Q' \arrow[ul, "{f([h'])}"']
\end{tikzcd}$$

If A, B, C are σ-complete closure algebras and $f \in \mathbf{BS}(A, B)$, $g \in \mathbf{BS}(B, C)$ are Boolean shapings, then the composite Boolean shaping $g \circ f$ is defined as follows: if $[h] \in \mathbf{H}(Q, A)$, $Q \in \mathrm{Ob}\mathbf{A}$, then $(g \circ f)([h]) = g(f([h]))$. If $[h'] \in \mathbf{H}(Q', A)$, $Q' \in \mathrm{Ob}\mathbf{A}$, $[k] \in \mathbf{H}(Q, Q')$, and $[h']\,[k] = [h]$, then $f([h'])\,[k] = f([h])$ and therefore $g(f([h']))\,[k] = g(f([h]))$, that is $(g \circ f)([h'])\,[k] = (g \circ f)([h])$, which proves that $g \circ f$ is indeed a Boolean shaping.

The identical Boolean shaping $1_A \in \mathbf{BS}(A, A)$ is defined by $1_A([h]) = [h]$, for $[h] \in \mathbf{H}(Q, A)$, $Q \in \mathrm{Ob}\mathbf{A}$. It easily results that for every Boolean shaping $f \in \mathbf{BS}(A, B)$, $1_B \circ f = f$ and $f \circ 1_A = f$.

In order to show that **BS** is a category, it remains to verify that **BS** (A, B) is a set, for every two σ-complete closure algebras A, B. For this, we observe that a Boolean shaping $f \in$ **BS** (A, B) is fully determined as soon as $f([h])$ is known for all $[h] \in$ **H** (Q, A), where Q is a σ-ANR-algebra. Obviously **H** (Q, A) and **H**(Q, B) are sets, for any $Q \in$ Ob**A**. We have **BS**(A, B) a subclass of the class $\prod_{Q \in ANRB}$ Ens(**H**(Q, A), **H**(Q, B)). But, for a continuous homomorphism $h: Q \to A$ we can consider the closure subalgebra generated by $h(Q)$ and for $f \in$ **BS** (A, B) we denote by f_Q the component f_Q : **H**$(Q, A) \to$ **H**(Q, B). If $h': Q' \to A$ is another continuous homomorphisms having the same image as well as h, then because Q and Q' are C-algebras, we can define a continuous homomorphism $k: Q \to Q'$ such that $h' o k = h$, which implies $f([h]) = f([h'])ok$ and which proves that f_Q is fully determined by $f_{Q'}$. In this way we can consider the above cartesian product only for the closure subalgebras of the closure algebra A. It follows that **BS** (A, B) is a set and therefore **BS** is a category.

2.1. Now we define the Boolean shape functor BS : **H** $\to$ **BS**. For every σ-complete closure algebras A, B we set BS$(A) = A$ and if $[h] \in$ **H**(A, B), we define BS$([h]) \in$ **BS**(A, B) by BS$([h])([k]) = [h]\,[k]$, where $[k] \in$ **H**(Q, A), $Q \in$ Ob**A**.

Indeed BS $([h])$ is a Boolean shaping, because if $[k']\,[l] = [k]$, then BS $([h])\,([k'])\,[l] = [h]\,[k']\,[l] = [h][k] =$ BS$([h])\,([k])$.
Then BS is a covariant functor because BS $([1_A])\,([k]) = [1_A]\,[k] = [k]$, and for $[h] \in$ **H** (A, B), $[k] \in$ **H** (B, C), we have BS$([k]\,[h])\,([l]) = ([k][h])\,[l] = [k]\,([h]\,[l]) =$ BS$([k])$ BS$([h])\,([l])$, for $[l] \in$ **H** (Q, A), $Q \in$ Ob**A**.

3.1. Theorem. *Let A be a σ-complete closure algebra, having the σ-homotopy type of a σ-ANR-algebra and let $f \in$ **BS** (A, B) a Boolean shaping. Then, there is a unique homotopy class $[h] \in$ **H** (A, B) such that* BS$([h]) = f$.

Proof. If BS$([h]) = f$, then, for $[k] = [1_A] \in$ **H** (A, A), we have BS$([h])\,([k]) = [h]\,[1_A] = [h]$, that is $[h] = f([1_A])$. Conversely, if we set $[h] = f([1_A])$, then, for $k \in$ **H** (Q, A), $Q \in$ Ob**A**, BS$([h])\,([k]) = [h]\,[k] = f([1_A])[k] = f([k])$, that is BS$([h]) = f$.

4.1. Lemma. *Let $f \in$* **BS**(A, B) *a Boolean shaping and let $[h] \in$* **H**(Q, A), *$Q \in$ Ob**A**. Then* $f o$BS$([h]) =$ BS$(f([h]))$.

Proof. For $[k] \in$ **H** (Q', Q), $Q' \in$ Ob**A** we have BS$(f([h]))\,([k]) = f([h])\,[k]$ and $(f o$BS $([h]))\,([k]) = f($BS$(h])\,([k])) = f([h]\,[k]) = f([h])\,[k]$, because f is a Boolean shaping.

5.1. Theorem. *Let* **C** *be a category whose objects are all σ-complete closure algebras and let $\overline{BS}$:* **H** $\to$ **C** *be a covariant functor having the following properties:*

(i) $\overline{\text{BS}}\,(A) = A$, *for every σ-complete closure algebra A,*

(ii) *for every $Q \in$ Ob**A** and every morphism $f' \in$* **C** (Q, A) *there exists a unique homotopy class $[h'] \in$* **H** (Q, A) *such that $\overline{BS}([h']) = f'$. Then there exists a unique covariant functor Φ :* **C** $\to$ **BS** *such that $\Phi o \overline{BS} =$ BS.*

Proof. We define Φ on objects by setting $\Phi(A) = A$. Then, for each morphism $f' \in$ **C**(A, B) we define a Boolean shaping $\Phi(f') \in$ **BS** (A, B) as

follows: for each $[h] \in$ **H** (Q, A), $Q \in$ Ob**A**, we consider the morphism $f'_0\overline{BS}([h]) \in$ **C**(Q, B). By the condition (ii), there is a unique $[k] \in$ **H**(Q, B) such that $f' o \overline{BS}([h]) = \overline{BS}([k])$ and we set $f([h]) = [k]$. We prove that f is a Boolean shaping. If $[h']\,[l] = [h]$ and $[k']$ is chosen in a such a way that $f' o \overline{BS}([h']) = \overline{BS}([k'])$, then $f([h']) = [k']$. Since also $\overline{BS}([h'])\,\overline{BS}([l]) = \overline{BS}([h])$, one obtains $\overline{BS}(f([h'])\,[l]) = \overline{BS}(f([h']))\;\overline{BS}([l]) = \overline{BS}\,([k'])\,\overline{BS}([l]) = f' o \overline{BS}([h'])\overline{BS}([l]) = f' o \overline{BS}([h]) = \overline{BS}([k]) = \overline{BS}(f([h]))$, which by the uniqueness in (ii) implies $f([h'])\,[l] = f([h])$.

We define $\Phi(f') = f$. Then, if $f' \in$ **C**(A, B), $g' \in$ **C** (B, C) and $[h] \in$ **H** (Q, A), $Q \in$ Ob**A**, there exist uniquely $[k] \in$ **H**(Q, B), $[l] \in$ **H**(Q, C) such that $f' o \overline{BS}([h]) = \overline{BS}([k])$, $g' o \overline{BS}([k]) = \overline{BS}([l])$. It follows $g' o f' o \overline{BS}([h]) = \overline{BS}([l])$, and we obtain $(\Phi(g' o f'))\,([h]) = [l]$. On the other hand $(\Phi(g')\,\Phi(f'))\,([h]) = \Phi(g')\,([k]) = [l]$, so that $\Phi(g' o f') = \Phi(g')\,\Phi(f')$. It is easy to see that $\Phi(1_A) = 1_A$. Thus Φ is a covariant functor.

Now we prove that $(\Phi o \overline{BS})\,([h]) = BS([h])$, for every $[h] \in$ **H**(A, B). If $[k] \in$ **H** (Q, A), $Q \in$ Ob**A**, then $BS([h])\,([k]) = [h]\,[k]$ and because $\overline{BS}([h]) o \overline{BS}([k]) = \overline{BS}([h]\,[k])$, it follows $(\Phi o \overline{BS})\,([h])\,([k]) = [h]\,[k]$. In order to prove the uniqueness of the functor Φ, we consider a covariant functor Ψ : **C** $\to$ **BS** satisfying $\Psi o \overline{BS} = BS$. We have $A = BS\,(A) = (\Psi o \overline{BS})\,(A) = \Psi(A)$, for every σ-complete closure algebra A. Now, let $f' \in$ **C** (A, B) and we consider $[h] \in$ **H**(Q, A), $Q \in$ Ob**A**. If $f' o \overline{BS}([h]) = \overline{BS}([k])$, then $\Psi(f' o \overline{BS}([h])) = \Psi(\overline{BS}([k])$that is $\Psi(f') o \Psi o \overline{BS}([h]) = (\Psi o \overline{BS})\,([k])$, and thus $\Psi(f') o BS\,([h]) = BS\,([k])$. Then, by Lemma 4.1 it follows that $BS([k]) = BS(\Psi(f')\,[h])$ and by Theorem 3.1 we obtain $\Psi(f')\,[h] = [k]$. Comparing with the definition of Φ, we obtain $\Psi(f') = \Phi(f')$. This completes the proof of the theorem.

By Theorem 3.1 and Theorem 5.1 we obtain:

6.1. Theorem. *There exist a unique (up to isomorphism) category* **BS** *having all σ-complete closure algebras for objects and a unique covariant functor BS :* **H** $\to$ **BS** *having the properties (i), (ii) of 5.1, such that the universal property of Theorem 5.1 holds.*

2. Boolean shapings and HB-systems.

2.1. Theorem. *Let B be a σ-complete closure algebra and $\mathcal{A} = \{A_j, [q_{jj'}], J\}$ a* **A**-*system which is* **A**-*associated with antother σ-complete closure algebra A and let $q_j : A_j \to A$ be continous homomorphisms satisfying the conditions (i), (ii), (iii) of the definition of associated systems with closure algebras [16, Definition 6.4]. Then there is an one-to-one correspondence between the Boolean shapings $f \in$* **BS** (A, B) *and the systems of homotopy classes $[h_j] \in$* **H**(A_j, B), *satisfying $[h_{j'}]\,[q_{jj'}] = [h_j]$, if $j < j'$.* The correspondence is given by $f\,o BS([q_j]) = BS([h_j])$, $j \in J$, using Theorem 3.1.

Proof. Because A_j are of the homotopy type of ANR-algebras, it follows by Theorem 3.1 that there exists a unique homotopy class $[h_j] \in$ **H**(A_j, B) such that $f o BS([q_j]) = BS([h_j])$. We consider the correspondence $f \to \{[h_j]\}_{j \in J}$. If $j < j'$ let $f o BS([q_{j'}]) = BS([h_{j'}])$. We have, [16, Definition 6.4], $[q_{j'}]\,[q_{jj'}] = [q_j]$. Thus $f o BS([q_j]) = f o BS([q_{j'}]\,[q_{jj'}]) = f o BS([q_{j'}]) o$ $o BS([q_j]) = BS([h_{j'}])\,BS([q_{jj'}])$, that is $BS([h_j]) = BS([h_{j'}]\,[q_{jj'}])$. By Theorem 3.1, it follows $[h_j]=[h_{j'}]\,[q_{jj'}]$. We prove that the correspondence so defined is a bijection. Let $f, g \in \mathbf{BS}(A, B)$, on which corresponding systems coincide. Then we have $f\circ BS([q_j]) = g\circ BS([q_j])$, $\forall j\in J$. For $[h]\in\mathbf{H}(Q, A_j)$, $Q\in\mathrm{Ob}\mathbf{A}$, we have $(f\circ \mathrm{BS}([q_j]))([h]) = (g\circ BS([q_j])[h] \Rightarrow f(BS([q_j]\,[h]) = g(\mathrm{BS}([q_j]\,[h]) \Rightarrow f([q_j]\,[h]) = g([q_j]\,[h])$ and by the definition of Boolean shapings, it follows $f([q_j]) = g([q_j])$, $\forall j\in J$. Now let $Q\in\mathrm{Ob}\mathbf{A}$ and $[k]\in\mathbf{H}(Q, A)$. Then, there exists $k_j : Q\to A_j$ a σ-complete continuous homomorphism, such that $[q_j]\,[k_j]=[k]$. By this we obtain $f([k]) = f([q_j]\,[k_j]) = f([q_j])\,[k_j] = g([q_j])\,[k_j] = g([q_j]\,[k_j]) = g([k])$, that is $f=g$. Thus the above defined correspondence $f\to\{[h_j]\}_{j\in J}$ is injective.

Now let $\{[h_j]\}_{j\in J}$ be a system of homotopy classes, $[h_j]\in\mathbf{H}(A_j, B)$, satisfying $[h_{j'}]\,[q_{jj'}]=[h_j]$, if $j<j'$. Then we define $f\in\mathbf{BS}(A, B)$ as follows: if $[k]\in\mathbf{H}(Q, A)$, $Q\in\mathrm{Ob}\,\mathbf{A}$, there exist $j_0\in J$ and a continuous homomorphism $k_{j_0} : Q\to A_{j_0}$ such that $[q_{j_0}]\,[k_{j_0}]=[k]$. Then $BS([h_{j_0}])\in\mathbf{BS}(Q, A_{j_0})$ and we define $f([k]) = BS([h_{j_0}])\,[k_{j_0}]$. First we prove that f is well-defined. In fact, if $k_{j_1} : Q\to A_{j_1}$ is a σ-complete continuous homomorphism, such that $[q_{j_1}]\,[k_{j_1}]=[k]$, then there exists $j\in J$, with $j_0<j$, $j_1<j$ and $[q_j] = [q_j]\,[q_{j_0j}]$, $[q_{j_1}] = [q_j]\,[q_{j_1j}]$ and therefore $[q_j]\,[q_{j_0j}]\,[k_{j_0}] = [q_j]\,[q_{j_1j}]\,[k_{j_1}]$. Thus it follows [16, Definition 6.4, (ii)] that there exists $j'\in J$, $j<j'$ such that $[q_{jj'}]\,[q_{j_0j}]\circ[k_{j_0}] = [q_{jj'}]\,[q_{j_1j}]\,[k_{j_1}]$. This implies $[q_{j_0j'}]\,[k_{j_0}] = [q_{j_1j'}]\,[k_{j_1}]$. Then $BS([h_{j_0}])([k_{j_0}]) = BS([h_{j'}]\,[q_{j_0j'}])([k_{j_0}]) = BS([h_{j'}])\;[q_{j_0j'}]\,[k_{j_0}] = BS([h_{j'}])[q_{j_1j'}]\,[k_{j_1}] = BS([h_{j'}]\circ[q_{j_1j'}])\,[k_{j_1}] = BS([h_{j_1}])([k_{j_1}])$, that is f is well-defined.

Now we prove that f is a Boolean shaping. Let $[k']\in\mathbf{H}(Q', A)$, $Q'\in\mathrm{Ob}\mathbf{A}$ and $[l]\in\mathbf{H}(Q, Q')$ such that $[k']\,[l]=[k]$. If $[q_{j^0}]\,[k'_{j_0}]=[k']$, it follows $[q_{j_0}]\,[k'_{j_0}]\,[l]=[k]$ and we have $f([k'])=\mathrm{BS}([h_{j_0}])\,[k'_{j_0}]$ and $f([k]) = \mathrm{BS}([h_{j_0}])\,[k'_{j_0}]\,[l] = f([k'])\,[l]$.

Finally, we prove that $f\circ BS([q_j]) = BS([h_j])$. In fact, if $[k]\in\mathbf{H}(Q, A_j)$, $Q\in\mathrm{Ob}\mathbf{A}$, then $(f\circ BS([q_j]))\;([k]) = f([q_j]\,[k]) = f([q_j])\,[k]$; $BS([h_j])([k]) = [h_j]\,[k]$. But $[q_j] = [q_j]\,[1_{A_j}]$ and therefore $f([q_j]) = BS([h_j])([1_{A_j}]) = [h_j]$. This finishes the proof of the theorem.

2.2. Theorem. *Let* $\mathcal{A}=\{A_j, [q_{jj'}], J\}$ *be a* **A**-*system which is* **A**-*associated with a σ-complete closure algebra A. Then, in the Boolean shape category* **BS** *the object A is the inductive limit of the directed inductive system* $BS(\mathcal{A})=\{A_j, BS([q_{jj'}]), J\}$.

Proof. Because BS is a covariant functor, it follows that $BS(\mathcal{A})$ is indeed a directed inductive system. Then, if $q_j : A_j\to A$ are continuous homomorphisms satisfying Definition 6.4 of [16], then the Boolean shapings $BS([q_j])\in\mathbf{BS}(A_j, A)$ satisfy the condition $BS([q_{j'}])\,BS([q_{jj'}]) = BS([q_j])$, if $j<j'$. The universal property of inductive limit results by the surjectivity of the correspondence given by Theorem 1.2.

2.3. Let $\mathcal{A}=\{A_i, [p_{ii'}], I\}$ and $\mathcal{B}=\{B_j, [q_{jj'}], J\}$ be two **A**-systems. A morphism (or a Boolean system homomorphism) from $\mathcal{A}$ to $\mathcal{B}$ consists on a map $\varphi : I\to J$ and homotopy classes of σ-complete continuous homomorphisms $[h_i] : A_i\to B_{\varphi(i)}$, for every $i\in I$, such that $i<i'$ in I and $\varphi(i)<j$, $\varphi(i')<j$ in J, then $[q_{\varphi(i)j}]\,[h_i] = [q_{\varphi(i')j}]\,[h_{i'}]\,[p_{ii'}]$.

2.4. Two morphisms $(\varphi, [h_i], I)$ and $(\psi, [k_i], I)$ from the **A**-system $\mathcal{A}=\{A_i, [p_{ii'}], I\}$ to the **A**-system $\mathcal{B}=\{B_j, [q_{jj'}], J\}$ are said to be equivalent, if for each $i\in I$, there exists $j\in J$ such that $\varphi(i)<j$, $\psi(i)<j$ and $[q_{\varphi(i)j}]\,[h_i] = [q_{\psi(i)j}]\,[k_i]$.

2.5. Theorem. *Suppose that the* **A**-*systems* $\mathcal{A}=\{A_i, [p_{ii'}], I\}$, $\mathcal{B}=\{B_j, [q_{jj'}], J\}$ *are* **A**-*associated with the σ-complete closure algebras A and B respectively. Then there is an one-to-one correspondence between the Boolean shapings $f\in\mathbf{BS}(A, B)$ and the equivalence classes of morphisms from $\mathcal{A}$ to $\mathcal{B}$.*

Proof. By Theorem 1.2 there exists a one-to-one correspondence between the Boolean shapings $f\in\mathbf{BS}(A, B)$ and the systems of homotopy classes $[h_i]\in\mathbf{H}(A_i, B)$ satisfying $[h_{i'}]\,[p_{ii'}]=[h_i]$, if $i<i'$. Because $A_i\in\mathrm{Ob}\mathbf{A}$ and $\mathcal{B}$ is **A**-associated with B, it follows [16, Definition 6.4] that for each $i\in I$ there exists $j\in J$ and a σ-complete continuous homomorphism $k_i : A_i\to B_j$ such that $[q_j][k_i]=[h_i]$. We prove that $\varphi : I\to J$, $\varphi(i)=j$ and $k_i : A_i\to B_{\varphi(i)}$ define a class of morphisms from $\mathcal{A}$ to $\mathcal{B}$. In fact, if $j'\in J$ and $k'_i : A_i\to B_{j'}$, is a σ-complete continuous homomorphism such that $[q_{j'}]\,[k'_i]=[h_i]$, then let be $j''\in J$ such that $j<j''$, $j'<j''$. We have $[q_j] = [q_{j''}]\,[q_{jj''}]$ and $[q_{j'}]=[q_{j''}]\,[q_{j'j''}]$. Thus we obtain $[q_{j''}]\,[q_{j'j''}]\,[k'_i] = [h_i]$ and $[q_{j''}]\,[q_{jj''}]\,[k_i]=[h_i]$. By the condition (ii) of Definition 6.4, [16], there is $j'''\in J$ with $j''<j'''$ such that $[q_{j''j'''}]\,[q_{jj''}]\,[k_i] = [[q_{j''j'''}]\,[q_{j'j''}]\,[k'_i]$. This implies $[q_{jj'''}]\,[k_i] = [q_{j'j'''}]\,[k'_i]$, which shows that the class of morphisms corresponding to the Boolean shaping f is well-defined. The correspondence $f\to\{\varphi, [k_i], I\}$ is injective. In fact, if $g\to\{\psi, [k'_i], I\}$ and if $(\varphi, [k_i], I)$, (ψ, k'_i, I) are equivalent, then we have $[q_{\varphi(i)j}]\,[k_i]=[q_{\psi(i)j}]\,[k'_i]$, for $\varphi(i)<j$, $\psi(i)<j$ and if $\{[h_i]\}_{i\in I}$, $\{[h'_i]\}_{i\in I}$ are the homotopy classes corresponding by Theorem 1.2 to f and respectively g, then $[q_{\varphi(i)}]\,[k_i]=[h_i]$, $[q_{\psi(i)}]\,[k'_i]=[h'_i]$. Then we have $[q_{\varphi(i)}]=[q_j]\,[q_{\varphi(i)j}]$ and $[q_{\psi(i)}] = [q_j]\,[q_{\psi(i)j}]$. Therefore $[h_i] = [q_j]\,[q_{\varphi(i)j}]\,[k_i] = [q_j]\,[q_{\psi(i)j}]\,[k'_i] = [q_{\psi(i)j}]\,[k'_i] = [q_{\psi(i)}]\,[k'_i] = [h'_i]$, and by Theorem 1.2 it follows $f=g$.

Finally if $\{\varphi, [k_i], I\}$ is a class of morphisms from $\mathcal{A}$ to $\mathcal{B}$, we consider the homotopy classes $[h_i] = [q_{\varphi(i)}]\,[k_i]$. If $i<i'$, then $[h_{i'}]\,[p_{ii'}] = [q_{\varphi(i)}]\,[k_{i'}]\,[p_{ii'}]$, and for $\varphi(i)<j$, $\varphi(i')<j$, $[q_{\varphi(i')}] = [q_j]\,[q_{\varphi(i')j}]$ and $[q_{\varphi(i)}] = [q_j]\,[q_{\varphi(i)j}]$. We obtain $[h_{i'}]\,[p_{ii'}] = [q_j]\,[q_{\varphi(i')j}]\,[k_{i'}]\,[p_{ii'}] = [q_j]\,[q_{\varphi(i)j}]\,[k_j] = [q_{\varphi(i)}]\,[k_i] = [h_i]$, that is, the homotopy classes $\{[h_i]\}_{i\in I}$ satisfy the condition of Theorem 1.2. Moreover, if $(\psi, [k'_i], I)$ is equivalent to $(\varphi, [k_i], I)$, then $[h'_i] = [q_{\psi(i)}]\,[k'_i]$ and there is $j\in J$ such that $\varphi(i)<j$, $\psi(i)<j$ and $[q_{\varphi(i)j}]\,[k_i] = [q_{\psi(i)j}]\,[k'_i]$. By $[q_{\psi(i)}] = [q_j]\,[q_{\varphi(i)j}]$, we obtain $[h'_i] = [q_j]\,[q_{\psi(i)j}]\,[k'_i] = [q_j]\,[q_{\varphi(i)j}]\,[k_i] = [q_{\varphi(i)}]\,[k_i] = [h_i]$. Thus the class $\{(\varphi, [k_i], I)\}$ defines uniquely the system of homotopy classes $\{[h_i]\}_{i\in I}$, satisfying the condition of Theorem 1.2, and therefore it defines a unique Boolean shaping $f\in\mathbf{BS}(A, B)$. This finishes the proof of the theorem.

2.6. Two σ-complete closure algebras A, B are said to have the same Boolean shape if A and B are isomorphic as objects of the category **BS**. We denote this by $\mathrm{BSh}(A)=\mathrm{BSh}(B)$, and this holds iff there exist $f\in\mathbf{BS}(A, B)$, $g\in\mathbf{BS}(B, A)$ such that $g\circ f=1_A$, $f\circ g=1_B$.

2.7. **Corollary.** *Two σ-complete closure algebras A, B which have the same σ-homotopy type, have also the same Boolean shape.*

Proof. The Boolean shape functor $\mathrm{BS} : \mathbf{H}\to\mathbf{BS}$ preserves isomorphisms.

2.8. Corollary. *Two σ-ANR-algebras A, B have the same homotopy type iff A, B have the same Boolean shape.*

Proof. The necessity follows from Corollary 7.2.

Converselly, suppose that A and B have the same Boolean shape, i.e. there exist $f\in \mathbf{BS}(A, B)$ and $g\in\mathbf{BS}(B, A)$, such that $g\circ f=1_A$, $f\circ g=1_B$. By Theorem 3.1 there exist a unique $[u]\in\mathbf{H}(A, B)$ and a unique $[v]\in\mathbf{H}(B, A)$ such that $\mathrm{BS}([u])=f$ and $\mathrm{BS}([v])=g$. Then $\mathrm{BS}([v]\,[u])=\mathrm{BS}([v])\,\mathrm{BS}([u])=g\circ f=1_A$ and by the uniqueness, it follows $[v]\,[u]=[1_A]$. Similarly we obtain $[u]\,[v]=[1_B]$.

3. **Homology and cohomology for closure algebras related with the shape theory.** 3.1. Two variants of Alexandrov-Čech homology and cohomology for complete closure algebras has been introduced by M. R. B u n j a t o v [8], and N. I. A h m e d H i l m i M u h a m e d, [13]. The second author has proved the homotopical invariance of these groups. Now we prove that two σ-complete closure algebras which have the same Boolean shape have the same homology and cohomology (up to isomorphisms).

First we recall from [13] the definitions of the spectral homology and cohomology for closure algebras.

Let A be a complete closure algebra and let $\Sigma(A)$ be the set of all partitions of unity of A. For a $\lambda\in\Sigma(A)$, let $A(\lambda)$ be the closure subalgebra generated by λ. Let $\mathrm{Cov}(\lambda)$ be the set of all open coverings of unity of the closure algebra $A(\lambda)$, that is $\mathrm{Cov}(\lambda)=\{\alpha=(\alpha_i)_{i\in I}/\bigvee_{i\in I}\alpha_i=1,\ \alpha_i$ open elements in $A(\lambda)$, i.e. $\alpha_i\in A(\lambda)$ and α_i are open elements in $A\}$. Let $\mathrm{Cov}(A)$ be the directed set (with respect to the refinement relation) of all coverings of unity of the closure algebra A. Now we set: $\mathrm{Cov}(\Sigma(A))=\cup\{\mathrm{Cov}(\lambda)\subset\mathrm{Cov}(A)/\lambda\in\Sigma(A)\}$. Then $\mathrm{Cov}(\Sigma(A))$ is a directed subset of the directed set $\mathrm{Cov}(A)$. For an open covering of unity of $A(\lambda)$, $\alpha\in\mathrm{Cov}(\lambda)$, $\alpha=(\alpha_i)_{i\in I}$, we consider the simplicial complex $\mathrm{nerv}\,\alpha=\{s\subset I/\bigwedge_{i\in s}\alpha_i\neq 0_A\}$ and one defines the Σ-spectral Alexandrov-Čech homology and cohomology by:

$$(1)\qquad H_n(A)=\varprojlim_{\alpha\in\mathrm{Cov}(\Sigma(A))}H_n(\mathrm{nerv}\,\alpha),\qquad H^n(A)=\varinjlim_{\alpha\in\mathrm{Cov}(\Sigma(A))}H^n(\mathrm{nerv}\,\alpha)$$

and one obtains, [13]:

$$(2)\qquad H_n(A)=\varprojlim_{\lambda\in\Sigma(A)}H_n(\lambda),\qquad H^n(A)=\varinjlim_{\lambda\in\Sigma(A)}H^n(\lambda),$$

where $H_n(\lambda)$ and $H^n(\lambda)$ are respectively the spectral Čech homology and cohomology groups of the topological space $(\lambda,\ 0^A_\lambda)$.

3.2. If A is a σ-complete closure algebra (without being necessary complete) then we can define the Σ_c-spectral homology and cohomology groups by:

$$(3)\qquad H^\sigma_n(A)=\varprojlim_{\lambda\in\Sigma_c(A)}H_n(\lambda),\qquad H^n_\sigma(A)=\varinjlim_{\lambda\in\Sigma_c(A)}H^n(\lambda),$$

where $\Sigma_c(A)$ is the set of all at most countable partitions of the unity of A, and $H_n(\lambda)$, $H^n(\lambda)$ are the n^{th} Čech homology (resp. cohomology) group based on locally finite normal open coverings.

3.3. Let A be a σ-complete closure algebra. For $\lambda\in\Sigma(A)$ we have:

$$(4)\qquad H_n(\lambda)=\varprojlim_{\alpha(\lambda)\in\mathcal{A}(\lambda)}H_n(K_{\alpha(\lambda)}),\qquad H^n(\lambda)=\varinjlim_{\alpha(\lambda)\in\mathcal{A}(\lambda)}H^n(K_{\alpha(\lambda)}),$$

where $\mathcal{A}(\lambda)$ denotes the set of all locally finite normal open coverings of the topological space $(\lambda, 0^A_\lambda)$, [15].

By the formulas (3), we obtain:

$$(5)\qquad H^\sigma_n(A)=\varprojlim_{(\lambda,\alpha(\lambda))\in\mathcal{A}}H_n(K_{\alpha(\lambda)}),\qquad H^n_\sigma(A)=\varinjlim_{(\lambda,\alpha(\lambda))\in\mathcal{A}}H^n(K_{\alpha(\lambda)}),$$

where $\mathcal{A}=\{(\lambda,\alpha(\lambda))/\lambda\in\Sigma_c(A),\ \alpha(\lambda)\in\mathcal{A}(\lambda)\}$ is a directed set, [16].

4.3. Theorem. *Let A, B be two σ-complete closure algebras and let $f\in\mathbf{BS}(A, B)$ be a Boolean shaping. Then f induces homomorphismes $f_*: H^\sigma_n(A)\to H^\sigma_n(B)$, $f^*: H^n_\sigma(B)\to H^n_\sigma(A)$ such that* $(1_A)_*=1_{H^\sigma_n(A)}$, $(gf)_*=g_*f_*$ *and* $(1_A)^*=1_{H^n_\sigma(A)}$, $(g\circ f)^*=f^*g^*$.

Proof. Let $\mathcal{A}=\{\sigma(K_{\alpha(\lambda)}),\ [q_{\alpha(\lambda)\alpha'(\lambda')}],\ \mathcal{A}^*\}$ be the $\mathbf{H}$-ANR-$\mathbf{F}$ system which is $\mathbf{A}$-associated with A, [16, Theorem 6.5] and let $\mathcal{B}=\{\sigma(L_{\beta(\mu)}),\ [q_{\beta(\mu)\beta'(\mu')}],\ \mathcal{B}^*\}$ be the associated system with B.

If $f\in\mathbf{BS}(A, B)$, then by Theorem 5.2 there exists a unique equivalence class $\{(\varphi, [h_{\alpha(\lambda)}], \mathcal{A}^*)\}$ of morphisms from $\mathcal{A}$ to $\mathcal{B}$, $[h_{\alpha(\lambda)}]:\sigma(K_{\alpha(\lambda)})\to\sigma(L_{\varphi(\alpha(\lambda))})$, corresponding to f, and by the theorem of homotopical invariance, [13, Theorem 3], the homomorphisms:

$$[h_{\alpha(\lambda)}]_*: H_n(\sigma(K_{\alpha(\lambda)}))\to H_n(\sigma(L_{\varphi(\alpha(\lambda))})),$$
$$[h_{\alpha(\lambda)}]^*: H^n(\sigma(L_{\varphi(\alpha(\lambda))}))\to H^n(\sigma(K_{\alpha(\lambda)})),$$

are induced, that is:

$$(6)\qquad [h_{\alpha(\lambda)}]_*: H_n(K_{\alpha(\lambda)})\to H_n(L_{\varphi(\alpha(\lambda))}),\qquad [h_{\alpha(\lambda)}]^*: H^n(L_{\varphi(\alpha(\lambda))})\to H^n(K_{\alpha(\lambda)}).$$

Passing to limit in the formulas (6) we obtain the homomorphisms f_* and respectively f^* of the theorem.

The second part of the theorem follows by Theorem 5.2 and by the definitions of the composite and the identical Boolean shapings.

3.5. Corollary. *If A, B are two σ-complete closure algebras and* $\mathrm{BSh}(A)=\mathrm{BSh}(B)$, *then*:

$$H^\sigma_n(A)\approx H^\sigma_n(B)\ \text{and}\ H^n_\sigma(A)\approx H^n_\sigma(B).$$

3.6. *Remark.* For complete closure algebras Theorem 4.3 and Corollary 5.3 holds also for Σ-spectral Alexandrov-Čech homology and cohomology, using an adequate statement of Theorem 6.5 of [16] and a corresponding shape theory for complete closure algebras.

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HIPERSURFACES OF A RIEMANNIAN MANIFOLD AND A PINCHING PROBLEM ON THE SECOND FUNDAMENTAL FORM

BY

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1. Introduction. Let M be a hypersurface in a Riemannian manifold $\overline{M}$ with metric $<,>$, and let ∇ and $\overline{\nabla}$ be the covariant differentiations on M and $\overline{M}$, respectively. Now, let e be the C^∞ unit normal on M. Then the Weingarten map and Gauss equation become.

$$\overline{\nabla}_X e = -A_e X, \tag{1.1}$$

$$\overline{\nabla}_X Y = \nabla_X Y + a(X, Y), \tag{1.2}$$

where, X,Y are vector fields tangent to M, and a the second fundamental form, which is symmetric. Obviously $A_e = A$ is a self-adjoint linear map and $<A_e(X), Y> = <a(X, Y), e>$. The Gauss curvature equation and Codazzi-Mainardi equation now become

$$\tan \overline{R}(X, Y) Z = R(X, Y) Z + <AX, Z> AY - <AY, Z> AX \tag{1.3}$$

$$\text{nor } \overline{R}(X, Y) Z = <\overline{\nabla}_X AY - \overline{\nabla}_Y AX - A[X,Y], Z> e, \tag{1.4}$$

where, $\overline{R}$ and R are the curvature tensors of $\overline{\nabla}$ and ∇ respectively.

Definition. *A hypersurface M in a Riemannian manifold $\overline{M}$ such that everywhere on M,* nor $\overline{R}(X, Y) Z = 0$ *for X, Y, Z vector fields tangent to M is called TF-hypersurface.*

Remark. Every hypersurface of a space form is TF-hypersurface. Also, every umbilical hypersurface with constant mean curvature in a Riemannian manifold is TF-hypersurface ([2], p. 266, equation 3.2). A TF-hypersurface satisfies the classical form of Codazzi-Mainardi equation $(\nabla_X A) Y = (\nabla_Y A) X$.

Now, let M be a TF-hypersurface with constant mean curvature $H = 1/n$ trace A, in a Riemannian manifold $\overline{M}$. We set $f = \text{trace } A^2$ and we compute Δf, where Δ is the Laplacian of the induced metric on M. To begin with

$$(1/2)\, \Delta f = <\Delta' A, A> + <\nabla A, \nabla A>, \tag{1.5}$$