3. Sikorski, Roman — *On the inducing of homomorphisms*, Fund. Math. Tom 36 (1949), 7—22.
4. Nöbeling, G. — *Topologie der Vereine und Verbände*, Arch. der Math. 1 (1949), 154—159.
5. Nöbeling, G. — *Grundlagen der analytischen Topologie*, Berlin—Göttingen—Heidelberg, 1954.
6. Bunjatov, M.R. — *Uniform Boolean algebras* (Russian), Dokl. Akad. Nauk S.S.S.R., Tom 224 (1975), No. 2, 265—268.
7. Bunjatov, M.R. — *Kolmogorov cohomology for abstract Boolean closure algebras* (Russian), Dokl. Akad. Nauk S.S.S.R., Tom 224 (1975), No. 1, 15—18.
8. Bunjatov, M.R. — *Alexandrov-Čech cohomology for abstract Boolean closure algebras* (Russian), Dokl. Akad. Nauk S.S.S.R., Tom 237 (1977), No. 1, 17—20.
9. Bunjatov, M.R. — *Basic concepts of the homotopy theory of continuous homomorphisms of closure algebras*, Soviet Math. Dokl. Vol. 18 (1977), No. 5, 1 346—1 350.
10. Bairamov, S.A. — *Thom-Gysin's homological sequence for inverse spectra of topological spaces* (Russian), Dokl. Akad. Nauk-Azer. S.S.R., Tom 35 (1979), No. 3, 3—8.
11. Raga Mohamed Baha, E.A. — *Cohomologies for uniform Boolean algebras* (Russian), Dokl. Akad. Nauk Azer. S.S.R., Tom 34 (1978), No. 3, 7—10.
12. Raga Mohamed Baha, E.A. — *The cohomology for inverse spectra of chain complexes and Boolean algebras* (Russian), Dokl. Akad. Nauk Azer. S.S.R., Tom 34 (1978), No. 4, 10—13.
13. Ahmed Hilmi Muhamed, N.I. — *Homotopy in the spectral homology of Boolean closure algebras* (Russian), Dokl. Akad. Nauk Azer. S.S.R., Tom 34 (1978), No. 8, 3—6.
14. Mardešić, Sibe — *Shapes for topological spaces*, General Topology and its Applications 3 (1973), 265—282.
15. Morita, Kiiti — *On shapes of topological spaces*, Fund. Math. Tom 86 (1975), 251—259.
16. Pop, I. — *A K. Morita type theorem for closure algebras*, An. şt. Univ. Iaşi, Tom XXVI, s Ia, 1980, f. 2, 369—380.

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HIPERSURFACES OF A RIEMANNIAN MANIFOLD AND A PINCHING PROBLEM ON THE SECOND FUNDAMENTAL FORM

BY

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1. Introduction. Let M be a hypersurface in a Riemannian manifold $\overline{M}$ with metric $<\, ,\, >$,and let ∇ and $\overline{\nabla}$ be the covariant differentiations on M and $\overline{M}$, respectively. Now, let e be the C^∞ unit normal on M. Then the Weingarten map and Gauss equation become.

$$(1.1) \qquad \overline{\nabla}_X e = -A_e X,$$

$$(1.2) \qquad \overline{\nabla}_X Y = \nabla_X Y + a(X, Y),$$

where, X,Y are vector fields tangent to M, and a the second fundamental form, which is symmetric. Obviously $A_e = A$ is a self-adjoint linear map and $<A_e(X), Y> = <a(X, Y), e>$. The Gauss curvature equation and Codazzi-Mainardi equation now become

$$(1.3) \qquad \tan \overline{R}(X, Y) Z = R(X, Y) Z + <AX, Z>AY - <AY, Z>AX$$

$$(1.4) \qquad \text{nor } \overline{R}(X, Y) Z = <\overline{\nabla}_X AY - \overline{\nabla}_Y AX - A\,[X,Y], Z>e,$$

where, $\overline{R}$ and R are the curvature tensors of $\overline{\nabla}$ and ∇ respectively.

Definition. *A hypersurface M in a Riemannian manifold $\overline{M}$ such that everywhere on M,* nor $\overline{R}\,(X, Y)\, Z = 0$ *for X, Y, Z vector fields tangent to M is called TF-hypersurface.*

Remark. Every hypersurface of a space form is TF-hypersurface. Also, every umbilical hypersurface with constant mean curvature in a Riemannian manifold is TF-hypersurface ([2], p. 266, equation 3.2). A TF-hypersurface satisfies the classical form of Codazzi-Mainardi equation $(\overline{\nabla}_X A)Y = (\overline{\nabla}_Y A)X$.

Now, let M be a TF-hypersurface with constant mean curvature $H = 1/n$ trace A, in a Riemannian manifold $\overline{M}$. We set $f =$ trace A^2 and we compute Δf, where Δ is the Laplacian of the induced metric on M. To begin with

$$(1.5) \qquad (1/2)\, \Delta f = <\Delta' A, A> + <\nabla A, \nabla A>,$$

where Δ' is the restricted Laplacian and we also recall that

$$\Delta' A(X) = \sum_{i=1}^{n} [R(E_i, X), A]\, E_i, \tag{1.6}$$

where $\{E_1,\dots,E_n\}$ is an orthonormal frame at a point of M. Here the reader should consult [4] and observe that (1.6) is valid because A is symmetric with constant trace and satisfies the classical Codazzi-Mainardi equation. Now, since M is a TF-hypersurface of $\overline{M}$ we have from (1.3)

$$R(X, Y)\, Z = \overline{R}(X\, Y)\, Z + <AY, Z> AX - <AX, Z> AY$$

and thus

$$<\Delta' A, A> = \sum_{j=1}^{n} <\Delta' A(E_j), AE_j> = \sum_{i,j} <[R(E_i, E_j), A]\, E_i, AE_j>$$

or

$$<\Delta' A, A> = \sum_{i,j} \{<R(E_i, E_j) A E_i, AE_j> - <R(E_i, E_j) E_i, A^2 E_j>\}.$$

Now, by assuming that E_i $(i=1,\dots,n)$ is eigenvector of A with corresponding eigenvalue λ_i, that is $AE_i = \lambda_i E_i$, we get

$$<\Delta' A, A> = \sum_{i,j} \{\lambda_i \lambda_j <R(E_i, E_j)\, E_i, E_j> - \lambda_j^2 <R\,(E_i, E_j) E_i, E_j>\}$$

$$= \sum_{i \neq j} (\lambda_j^2 - \lambda_i \lambda_j)\, K_{ij} = \sum_{i<j} (\lambda_i - \lambda_j)^2 K_{ij},$$

where $K_{ij} = <R\,(E_i, E_j)\, E_j, E_i>$ is the sectional curvature of M for the 2-plane spanned by E_i and E_j, $i \neq j$. Thus we have

$$(1/2) \Delta f = \sum_{i<j} (\lambda_i - \lambda_j)^2 K_{ij} + <\nabla A, \nabla A>. \tag{1.7}$$

Now, since $K_{ij} = \overline{K}_{ij} + \lambda_i \lambda_j$, where $\overline{K}_{ij}$ is the sectional curvature of $\overline{M}$ for the 2-plane spanned by E_i and E_j, $i \neq j$, we get

$$(1/2) \Delta f = \sum_{i<j} (\lambda_i - \lambda_j)^2 \overline{K}_{ij} + \sum_{i<j} (\lambda_i - \lambda_j)^2 \lambda_i \lambda_j + <\nabla A, \nabla A>$$

$$\geqq \inf \overline{K} \{n \operatorname{trace} A^2 - (\operatorname{trace} A)^2\} + \operatorname{trace} A \,.\, \operatorname{trace} A^3 -$$

$$- (\operatorname{trace} A^2)^2 + <\nabla A, \nabla A>,$$

where

trace $A = \Sigma \lambda_i$, trace $A^2 = \Sigma \lambda_i^2$, trace $A^3 = \Sigma \lambda_i^3$ and $\inf \overline{K}$ is the infimum of the sectional curvature of $\overline{M}$. Thus

$$(1/2) \Delta f \geqq n \inf \overline{K} \{\operatorname{trace} A^2 - (1/n) (\operatorname{trace} A)^2\} + \operatorname{trace} A \,.\, \operatorname{trace} A^3 - (\operatorname{trace} A^2)^2 + <\nabla A, \nabla A>. \tag{1.8}$$

2. Main result. The following two theorems are very simple consequences of the maximum principal and formula (1.7).

Theorem 2.1. *A compact TF-hypersurface with constant mean curvature and positive sectional curvature, in a Riemannian manifold (not necessarily complete), is totally umbilical. In particular, if M is minimal, then it is totally geodesic.*

Theorem 2.2. *A complete, non-compact TF-hypersurface with constant mean curvature, positive sectional curvature and trace A^2=const., in a Riemannian manifold (not necessarily complete), is totally umbilical. In particular; If M is minimal, then it is totally geodesic.*

Remark. Theorems 1 and 2 give necessary conditions, for a hypersurface of a Riemannian manifold to be an extrinsic hypersphere (for the definition c. f. [2]).

We prove the following lemma

Lemma 2.1. *Let M be a hypersurface of a Riemannian manifold $\overline{M}$, so that at a point $p \in M$*

$$\operatorname{trace} A^2 \leqq \frac{(\operatorname{trace} A)^2}{n-1} + 2 \inf \overline{K},$$

then the sectional curvatures of M at p are nonnegative.

Proof. Let X and Y are two orthonormal vectors at $p \in M$. We choose an orthonormal frame $e_1,\dots,e_n$ in T_pM such that $e_1 = X$, $e_2 = Y$. By assumption we get

$$(n-1) \sum_{i,j} h_{ij}^2 \leqq \Big(\sum_i h_{ii}\Big)^2 + 2\,(n-1) \inf \overline{K}, \quad \text{where } h_{ij} = <A(e_i), e_j>,$$

or

$$\Big(\sum_i h_{ii}\Big)^2 \geqq (n-1) \sum_i h_{ii}^2 + (n-1) \sum_{i \neq j} h_{ij}^2 - 2(n-1) \inf \overline{K}. \tag{2.1}$$

Applying the well-known algebraic lemma [1], we obtain

$$2\, h_{11}\, h_{22} \geqq \sum_{i \neq j} h_{ij}^2 - 2 \inf \overline{K} \geqq 2 h_{12}^2 - 2 \inf \overline{K},$$

or

$$h_{11}\, h_{22} - h_{12}^2 \geqq - \inf \overline{K},$$

then

$$K(X \wedge Y) = \overline{K}(X \wedge Y) + h_{11}\, h_{22} - h_{12}^2 \geqq \overline{K}(X \wedge Y) - \inf \overline{K} \geqslant 0.$$

Now, we put $LX = AX - (1/n)$ (trace A) X. Then

$$\begin{gathered} \operatorname{trace} L = 0, \quad \operatorname{trace} L^2 = \operatorname{trace} A^2 - (1/n) (\operatorname{trace} A)^2, \\ \operatorname{trace} L^3 = \operatorname{trace} A^3 - (3/n) (\operatorname{trace} A) \operatorname{trace} L^2 - (1/n^2) (\operatorname{trace} A)^3. \end{gathered} \tag{2.2}$$

Moreover we set $g^2 = \operatorname{trace} L^2$. Obviously $g^2 = 0$, everywhere on M, if and only if M is totally umbilical. By assuming $H = (1/n)$ trace A = const. > 0 and substituting (2.2) to (1.8) we get, by repeating the same calculations as M. Okumura in [5]

$$(1/2)\, \Delta g^2 \geqslant g^2 \left[n \inf \overline{K} + \frac{(\operatorname{trace} A)^2}{n} - \frac{(n-2)}{\sqrt{n(n-1)}} \operatorname{trace} A \,.\, g - g^2 \right] + <\nabla A, \nabla A>. \tag{2.3}$$

Now, using formula (2.3), we prove the following theorem

Theorem 2·3. *Let M be an n-dimensional ($n\geq 3$), complete TF-hypersurface, with constant mean curvature in a Riemannian manifold $\overline{M}$ (not necessarily complete). If* trace $A^2\leqq(\text{trace } A)^2/(n-1)+2\inf \overline{K}$, *with* $\inf\overline{K}>0$, *then M is totally umbilical in $\overline{M}$ (i.e an extrinsic hypersphere)*

Proof. By assumption, we conclude that the sectional curvatures of M are bounded below (lemma 2.1) and also g^2 is bounded above. Then using (2.3) and the same method as in [3] (proof of theorem A) we complete the proof of this theorem.

Finally we obtain the following result for complete minimal *TF*-hypersurfaces of a Riemannian manifold.

Theorem 2.4. *Let M be a complete minimal TF-hypersurface of a Riemannian manifold $\overline{M}$ (not necessarily complete) with* $\inf\overline{K}>0$. *Then* suptrace $A^2\geqq n\inf\overline{K}$ *or M is totally geodesic.*

Proof. By puting in (1.8) trace $A=0$ we get

$$(1/2)\,\Delta f\geqq f(n\inf\overline{K}-f)+<\nabla A,\nabla A>,\ \text{where } f=\text{trace } A^2$$

If suptrace $A^2<n\inf\overline{K}$, then there exists a positive number ε such that trace $A^2\leqq n\inf\overline{K}-\varepsilon$, everywhere on M and thus

$$(1/2)\Delta f\geqq f\varepsilon. \tag{2.4}$$

On the other hand, the sectional, curvature of M is bounded below. In fact, if $X\wedge Y$ is a 2-plane of T_pM then from $K(X\wedge Y)=\overline{K}(X\wedge Y)+<a(X,X),a(Y,Y)>-<a(X,Y)>^2$ we get

$$K(X\wedge Y)\geqq \inf\overline{K}-2\ \text{trace } A^2\geqq(1-2n)\inf\overline{K}+2\varepsilon=const.>-\infty,$$

since trace $A^2\leqq n\inf\overline{K}-\varepsilon$. Then using the same method as in [3], we conclude from (2.4) that $f=0$ everywhere on M (i. e M is totally geodesic) and this completes the proof.

REFERENCES

1. Chen B-Y and M. Okumura – *Scalar curvature, inequality and submanifold*, Proc. Amer. Math. Soc. 38 (1973), 605–608.
2. Chen, B-Y – *Extrinsic spheres in compact symmetric spaces are intrinsic spheres*, Michigan Math. J. 24 (1977), 26 –271.
3. Hasanis Th. – *Characterization of totally umbilical hypersurfaces*, Proc. Amer. Math. Soc. (to appear-Jaw 346).
4. Nomizu K. and B. Smyth, – *A formula of Simon's type and hypersurfaces with constant mean curvature*, Journal of Dif. Geometry 3(1969), 367–377.
5. Okumura M. – *Hypersurfaces and a pinching problem on the second fundamental tensor*, Amer. J. of Math. XCVI (1974), 207–213.

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HYPERSURFACES OF QUATERNION SPACE FORMS

BY

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§ 1. **Introduction.** Let N be a $4(n+1)$-dimensional manifold and g be a Riemannian metric on N. Then N is said to be a quaternion manifold (see [2]) if there exists a 3-dimensional vector bundle V of tensors of type (1, 1) with local basis of almost Hermitian structures J_1, J_2, J_3 satisfying

$$J_1\circ J_2=-J_2\circ J_1=J_3 \quad \text{and} \tag{1.1}$$

$$\tilde{\nabla}_X J_r=\sum_{s=1}^{3} Q_{rs}(X)J_s \quad (r=1,\ 2,\ 3), \tag{1.2}$$

for all vector fields X tangent to N, where $\tilde{\nabla}$ is the Riemannian connection determined by g on N and Q_{rs} are certain local 1-forms on N and $Q_{rs}+Q_{sr}=0$.

A quaternion manifold N is a quaternion space form of quaternion sectional curvature c if and only if the curvature tensor $\tilde{R}$ of N is given by [2]",

$$\tilde{R}(X,\ Y)Z=\frac{c}{4}\Big\{g(Y,\ Z)X-g(X,\ Z)Y+\sum_{r=1}^{3}\{g(J_rY,\ Z)J_rX-$$
$$-g(J_rX,\ Z)J_rY+2g(X,\ J_rY)J_rZ\}\Big\}, \tag{1.3}$$

for all X, Y, Z tangent to N. From now on, we denote by $N(c)$ a quaternion space form of quaternion sectional curvature c.

In [1] we have initiated a study of the geometry of hypersurfaces in quaternion manifolds. The purpose of the present paper is to obtain results on the geometry of hypersurfaces in quaternion space forms. First, in § 2 we describe the geometrical configuration inherited by any hypersurface in a quaternion manifold and give fundamental formulas. Next, in § 3 we give characterizations of some classes of hypersurfaces in $N(c)$ by means of their sectional curvatures. Finally, in § 4 we obtain theorems of decomposition for hypersurfaces in a quaternion manifold.

§ 2. **The geometrical configuration on a hypersurface in a quaternion manifold.** Let N be a $4(n+1)$-dimensional quaternion manifold and $\{J_1,\ J_2,\ J_3\}$ be a local basis for the vector bundle V on a coordinate