Now, using formula (2.3), we prove the following theorem

Theorem 2·3. *Let M be an n-dimensional ($n \geq 3$), complete TF-hypersurface, with constant mean curvature in a Riemannian manifold $\overline{M}$ (not necessarily complete). If* trace $A^2 \leqq (\text{trace } A)^2/(n-1) + 2 \inf \overline{K}$, *with* $\inf \overline{K} > 0$, *then M is totally umbilical in $\overline{M}$ (i.e an extrinsic hypersphere)*

Proof. By assumption, we conclude that the sectional curvatures of M are bounded below (lemma 2.1) and also g^2 is bounded above. Then using (2.3) and the same method as in [3] (proof of theorem A) we complete the proof of this theorem.

Finally we obtain the following result for complete minimal TF-hypersurfaces of a Riemannian manifold.

Theorem 2.4. *Let M be a complete minimal TF-hypersurface of a Riemannian manifold $\overline{M}$ (not necessarily complete) with* $\inf \overline{K} > 0$. *Then* suptrace $A^2 \geqq n \inf \overline{K}$ *or M is totally geodesic.*

Proof. By puting in (1.8) trace $A = 0$ we get

$$(1/2)\,\Delta f \geqq f(n \inf \overline{K} - f) + \langle \nabla A, \nabla A \rangle, \text{ where } f = \text{trace } A^2$$

If suptrace $A^2 < n \inf \overline{K}$, then there exists a positive number ε such that trace $A^2 \leqq n \inf \overline{K} - \varepsilon$, everywhere on M and thus

$$(1/2)\Delta f \geqq f\varepsilon. \tag{2.4}$$

On the other hand, the sectional, curvature of M is bounded below. In fact, if $X \wedge Y$ is a 2-plane of T_pM then from $K(X \wedge Y) = \overline{K}(X \wedge Y) + \langle a(X, X), a(Y, Y) \rangle - \langle a(X, Y) \rangle^2$ we get

$$K(X \wedge Y) \geqq \inf \overline{K} - 2 \text{ trace } A^2 \geqq (1-2n) \inf \overline{K} + 2\varepsilon = const. > -\infty,$$

since trace $A^2 \leqq n \inf \overline{K} - \varepsilon$. Then using the same method as in [3], we conclude from (2.4) that $f = 0$ everywhere on M (i. e M is totally geodesic) and this completes the proof.

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HYPERSURFACES OF QUATERNION SPACE FORMS

BY

AUREL BEJANCU

§ 1. **Introduction.** Let N be a $4(n+1)$-dimensional manifold and g be a Riemannian metric on N. Then N is said to be a quaternion manifold (see [2]) if there exists a 3-dimensional vector bundle V of tensors of type (1, 1) with local basis of almost Hermitian structures J_1, J_2, J_3 satisfying

$$J_1 \circ J_2 = -J_2 \circ J_1 = J_3 \quad \text{and} \tag{1.1}$$

$$\tilde{\nabla}_X J_r = \sum_{s=1}^{3} Q_{rs}(X) J_s \quad (r = 1, 2, 3), \tag{1.2}$$

for all vector fields X tangent to N, where $\tilde{\nabla}$ is the Riemannian connection determined by g on N and Q_{rs} are certain local 1-forms on N and $Q_{rs} + Q_{sr} = 0$.

A quaternion manifold N is a quaternion space form of quaternion sectional curvature c if and only if the curvature tensor $\tilde{R}$ of N is given by [2]:

$$\tilde{R}(X, Y)Z = \frac{c}{4}\Big\{g(Y, Z)X - g(X, Z)Y + \sum_{r=1}^{3} \{g(J_rY, Z)J_rX - g(J_rX, Z)J_rY + 2g(X, J_rY)J_rZ\}\Big\}, \tag{1.3}$$

for all X, Y, Z tangent to N. From now on, we denote by $N(c)$ a quaternion space form of quaternion sectional curvature c.

In [1] we have initiated a study of the geometry of hypersurfaces in quaternion manifolds. The purpose of the present paper is to obtain results on the geometry of hypersurfaces in quaternion space forms. First, in § 2 we describe the geometrical configuration inherited by any hypersurface in a quaternion manifold and give fundamental formulas. Next, in § 3 we give characterizations of some classes of hypersurfaces in $N(c)$ by means of their sectional curvatures. Finally, in § 4 we obtain theorems of decomposition for hypersurfaces in a quaternion manifold.

§ 2. **The geometrical configuration on a hypersurface in a quaternion manifold.** Let N be a $4(n+1)$-dimensional quaternion manifold and $\{J_1, J_2, J_3\}$ be a local basis for the vector bundle V on a coordinate neighborhood $\mathcal{U}$ on N. We take another coordinate neighborhood $\tilde{\mathcal{U}}$ with the corresponding local basis $\{\tilde{J}_1, \tilde{J}_2, \tilde{J}_3\}$ and $\mathcal{U} \cap \tilde{\mathcal{U}} \neq \varnothing$. Then we have

$$\tilde{J}_r = \sum_{s=1}^{3} a_{rs} J_s \quad (r=1,\ 2,\ 3). \tag{2.1}$$

where (a_{rs}) is an element of the special orthogonal group $SO(3)$ of dimension 3.

Now, let M be an orientable hypersurface of N and ξ be a unit normal vector field defined on M. Then, on the coordinate neighborhood $\mathcal{V} = \mathcal{U} \cap M$ we define three vector fields $X_r = J_r\xi$ tangent to M. In a similar way we define $\tilde{X}_r = \tilde{J}_r\xi$ on $\tilde{\mathcal{V}} = \tilde{\mathcal{U}} \cap M$. Suppose $\mathcal{V} \cap \tilde{\mathcal{V}} \neq \varnothing$. Then by (2.1) we have

$$\tilde{X}_r = \sum_{s=1}^{3} a_{rs} X_s \ \text{ on } \mathcal{V} \cap \tilde{\mathcal{V}}, \qquad (r=1,\ 2,\ 3). \tag{2.2}$$

From (2.2) it follows that we have obtained a globally defined distribution $D^{\perp}$ on M which is locally represented by $\{X_1, X_2, X_3\}$ on the coordinate neighborhood $\mathcal{V}$. We denote by D the orthogonal complementary distribution to $D^{\perp}$ in the tangent bundle TM. It is easily seen that D is invariant by J_r, that is, we have $J_r(D_x) = D_x$ for each $x \in M$ and $r = 1, 2, 3$. That why D is called the quaternion distribution on M.

We denote by P the projection morphism of TM to the quaternion distribution D. Then, for any vector field Y tangent to M we put

$$Y = PY + u_1(Y)X_1 + u_2(Y)X_2 + u_3(Y)X_3, \tag{2.3}$$

where u_1, u_2, u_3 are 1-forms locally defined on M by

$$u_r(Y) = g(Y,\ X_r),\ \ r=1,\ 2,\ 3. \tag{2.4}$$

Applying J_1 to (2.3) and taking account of (1.1) we obtain

$$J_1Y = J_1PY - u_1(Y)\xi + u_2(Y)X_3 - u_3(Y)X_2. \tag{2.5}$$

The Riemannian connection induced by g on M is denoted by ∇. Then the equations of Gauss and Weingarten are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,\ Y)\xi \ \text{ and respectively} \tag{2.6}$$

$$\tilde{\nabla}_X \xi = -AX, \tag{2.7}$$

for all X, Y tangent to M, where h is the second fundamental form of M and A is the fundamental tensor of Weingarten with respect to ξ. It is well known that we have

$$h(X,\ Y) = g(AX,\ Y), \quad \text{for all } X,\ Y \text{ tangent to } M. \tag{2.8}$$

Differentiating (2.5) with respect to an arbitrary vector field X tangent to M and comparing the tangential and the normal parts, we obtain

$$\begin{aligned} &\nabla_X J_1PY = J_1P\nabla_X Y + (u_1Y)\{AX + Q_{12}(X)X_3 + Q_{31}(X)X_2\} + \\ &+ (u_2Y)\{\nabla_X X_3 + Q_{13}(X)X_1\} + (u_3Y)\{Q_{21}(X)X_1 - \nabla_X X_2\} + \\ &+ \{X(u_2Y) - u_2(\nabla_X Y)\}X_3 - \{X(u_3Y) - u_3(\nabla_X Y)\}X_2 + Q_{21}(X)J_2PY + \\ &+ Q_{31}(X)J_3PY - h(X,\ Y)X_1 = 0, \ \text{and respectively} \end{aligned} \tag{2.9}$$

$$\begin{aligned} &h(X,\ J_1PY) - X(u_1Y) + (u_2Y)h(X,\ X_3) - (u_3Y)h(X,\ X_2) + \\ &+ u_1(\Delta_X Y) + Q_{12}(X)(u_2Y) + Q_{13}(X)(u_3Y) = 0, \end{aligned} \tag{2.10}$$

for all Y tangent to M.

We say that a vector field X tangent to M belongs to the distribution D (resp. $D^{\perp}$) if we have $X_x \in D_x$ (resp. $X_x \in D^{\perp}_x$) for each $x \in M$. We denote by $\Gamma(D)$ (resp. $\Gamma(D^{\perp})$) the module of all vector fields on M which belong to D (resp. $D^{\perp}$).

The hypersurface M is called D-geodesic (resp. $D^{\perp}$-geodesic) if we have

$$\begin{aligned} &h(X,\ Y) = 0 \ \text{ for all } X,\ Y \in \Gamma\ (D) \ \ (\text{resp. } h(U,\ V) = 0, \\ &\qquad\qquad \text{for all } U,\ V \in \Gamma\ (D^{\perp}). \end{aligned} \tag{2.11}$$

Also, we say that M is $(D,\ D^{\perp})$-geodesic if we have

$$h(X,\ Y) = 0 \quad \text{for all } X \in \Gamma(D) \text{ and } Y \in \Gamma(D^{\perp}). \tag{2.12}$$

Lemma 1. *Let M be an orientable hypersurface of a quaternion manifold N. Then we have*

1. *The quaternion distribution is involutive if and only if M is D-geodesic.*
2. *The distribution $D^{\perp}$ is involutive if and only if M is $(D,\ D^{\perp})$-geodesic.*
3. *If D is involutive, then any leaf of D is totally geodesic immersed in N.*
4. *If $D^{\perp}$ is involutive, then any leaf of $D^{\perp}$ is totally geodesic immersed in M.*

Thus we have a complete characterization of the integrability of both distributions D and $D^{\perp}$ by means of the second fundamental form of M. The proof of lemma is given in [1].

§ 3. **Hypersurfaces of quaternion space forms.** Let M be an orientable hypersurface of the quaternion space form $N(c)$. Denote by R the curvature tensor field of M. Then, taking account of (1.3) the structure equation of Gauss is given by

$$\begin{aligned} g(R(X,\ Y)Z,\ W) = \frac{c}{4}\Big\{&g(Y,\ Z)g(X,\ W) - g(X,\ Z)g(Y,\ W) + \\ &+ \sum_{r=1}^{3}\{g(J_rY,\ Z)g(J_rX,\ W) - g(J_rX,\ Z)g(J_rY,\ W) + \\ &+ 2g(X,\ J_rY)g(J_rZ,\ W)\}\Big\} + h(X,\ W)h(Y,\ Z) - h(X,\ Z)h(Y,\ W), \end{aligned} \tag{3.1}$$

for all X, Y, Z, W, tangent to M.

From (3.1) we obtain that the sectional curvature K_M of M determined by orthonormal vectors $\{X, Y\}$ is given by

$$(3.2)\quad K_M(X\wedge Y)=c/4\left\{1+3\sum_{r=1}^{3} g(X, J_rY)^2\right\}+h(X, X)h(Y, Y)-h(X, Y)^2.$$

Now, we denote $K_r(X)=K_M(X\wedge J_rX)$ and $K_{rs}(X)=K_M(J_rX\wedge J_sX)$ for any unit vector field $X\in\Gamma(D)$, $r\neq s$, $r, s=1, 2, 3$ and call them the quaternion sectional curvatures of the hypersurface M. Then we have

Proposition 1. *Let M be an orientable hypersurface of $N(c)$. If the quaternion distribution is involutive, then we have*

$$(3.3)\quad K_r(X)=K_{rs}(X)=c.$$

for all unit vector fields $X\in\Gamma(D)$ and $s, r=1, 2, 3, s\neq r$.

The *proof* follows by a direct computation using Lemma 1 and (3.2).

The hypersurface M is called $D\perp$-minimal if the second fundamental form of M satisfies

$$(3.4)\quad \sum_{r=1}^{3}\{h(X_r, X_r)\}=0.$$

Theorem 1. *Let M be a $D\perp$-minimal orientable hypersurface of $N(c)$. Then M is $D\perp$-geodesic if and only if we have*

$$(3.5)\quad K_M(X_1\wedge X_2)=K_M(X_2\wedge X_3)=K_M(X_3\wedge X_1)=c.$$

Proof. Suppose M is $D\perp$-geodesic. Then from (3.2) follows (3.5) by using (1.1).

Conversely, from (3.5) taking account of (3.2) we obtain (3.6)

$$(3.6)\quad h(X_r, X_r)h(X_s, X_s)-h(X_r, X_s)^2=0 \text{ for all } r\neq s,\ r, s=1, 2, 3.$$

Then from (3.6) and (3.4) follows $h(X_r, X_s)=0$ for all $r, s=1, 2, 3$. Hence M is $D\perp$-geodesic.

Theorem 2. *Let M be an orientable hypersurface of $N(c)$. Suppose*

$$(3.7)\quad K_M(X\wedge Y)=c/4 \text{ for all } X\in\Gamma(D),\ Y\in\Gamma(D\perp),$$

and M be $D\perp$-minimal. Then M is $(D, D\perp)$-geodesic.

Proof. We take $X\in\Gamma(D)$ and $Y\in\Gamma(D\perp)$ in (3.2) and by using (3.7) obtain

$$(3.8)\quad h(X, X)h(Y, Y)-h(X, Y)^2=0.$$

Next, substituting Y by X_r in (3.8) for $r=1, 2, 3$ and taking account of (3.4) we obtain $h(X, X_r)=0$. Thus M is $(D, D\perp)$-geodesic.

The hypersurface M is called totally umbilical if we have

$$(3.9)\quad AX=\alpha X,$$

for all X tangent to M, where α is a differentiable function on M. The equation of Codazzi for the immersion of M in $N(c)$ is given by

$$(3.10)\quad g((\nabla_XA)Y-(\nabla_YA)X, Z)=g(\tilde{R}(X, Y)Z,\xi), \text{ for all } X, Y, Z \text{ tangent to } M.$$

Now, we state

Theorem 3. *There exist no totally umbilical orientable hypersurfaces in a $4(n+1)$-dimensional quaternion space form $N(c)$ with $n\geq 1$ and $c\neq 0$.*

Proof. By a direct computation using (3.9) we obtain

$$(3.11)\quad X(\alpha)g(Y, Z)-Y(\alpha)g(X, Z)=g((\nabla_XA)Y-(\nabla_YA)X, Z).$$

Next we take $Z=X_1$, $Y=J_1X\in\Gamma(D)$ in (3.11) and obtain

$$(3.12)\quad g((\nabla_XA)J_1X-(\nabla_{J_1X}A)X, X_1)=0.$$

On the other hand, from (1.3) follows

$$(3.13)\quad g(\tilde{R}(X, J_1X)X_1, \xi)=(c/2)\, g(X, X).$$

Thus, from (3.10), (3.12) and (3.13) follows $c=0$ and the proof is complete.

§ 4. Theorems of decomposition for hypersurfaces of quaternion manifolds. Let M be a hypersurface of the quaternion manifold N. We say that M is a foliate hypersurface if both distributions D and $D\perp$ are involutive on M.

Theorem 4. *Let M be a foliate orientable hypersurface of the quaternion manifold N. Then M is locally the Riemannian product $M^\top\times M\perp$, where $M^\top$ is a quaternion submanifold totally geodesic immersed in N and $M\perp$ is a leaf of $D\perp$ and is totally geodesic immersed in M.*

Proof. From (2.10) follows

$$(4.1)\quad h(X, J_1Y)=-u_1(\nabla_XY),$$

for all X tangent to M and $Y\in\Gamma(D)$. By the assertions 1 and 2 of Lemma 1, from (4.1) we have $g(\nabla_XY, X_1)=0$. In a similar way we obtain $g(\Delta_XY, X_r)=0$, for all X tangent to M, $Y\in\Gamma(D)$ and $r=2, 3$. Thus the distribution D is parallel on M. Since ∇ is a Riemannian connection we obtain that $D\perp$ is also parallel. Hence M is locally the Riemannian product $M^\top\times M\perp$, where $M^\top$ is a leaf of D and $M\perp$ is a leaf of $D\perp$. The remaining part of the theorem follows from assertions 2 and 3 of Lemma 1.

On each coordinate neighborhood $(\mathcal{V})$ on M, we define three tensor fields F_r of type (1,1) by

$$(4.2)\quad F_1X=J_1PX+(u_2X)X_3-(u_3X)X_2,$$

$$(4.3)\quad F_2X=J_2PX+(u_3X)X_1-(u_1X)X_3,$$

$$(4.4)\quad F_3X=J_3PX+(u_1X)X_2-(u_2X)X_1,$$

for all X tangent to M. It is a simple verification to see that F_r are f-structures on $\mathcal{V}$ (see [3] for the definition of an f-structure).

We say that F_r are D-parallel on $\mathcal{V}$ if we have

$$(4.5)\quad (\nabla_XF_r)Y=\sum_{s=1}^{3}Q_{rs}(X)F_sY,$$

for all X tangent to M and $Y\in\Gamma(D)$. Now, we state

Theorem 5. *Let M be an orientable hypersurface of the quaternion manifold N. If one of the f-structures F_r $(r=1, 2, 3)$ is D-parallel, then M is locally represented as in Theorem 4.*

Proof. By using (2.9) we obtain that each one from the f-structures F_r is D-parallel if and only if we have $(h(X, Y)=0$, for all X tangent to M and $Y \in \Gamma(D)$. But this is equivalent to the integrability of both distributions D and $D\perp$. Hence the proof follows from Theorem 4.

Now we prove the non-existence of foliate orientable hypersurfaces in a quaternion space form of non-null quaternion sectional curvature and give a complete characterization of them in $R^{4(n+1)}$.

Theorem 6. *Let M be a foliate complete orientable hypersuface of a simply connected complete $4(n+1)$-dimensional quaternion space form $N(c)$ with $n \geq 1$. Then we have $c=0$ and M is $R^{4n} \times M\perp$ in $R^{4(n+1)}$ where M is a leaf of $D\perp$ totally geodesic immersed in M.*

Proof. Taking into account that both distributions are involutive, for each $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D\perp)$ we obtain

$$g((\nabla_X A)Y-(\nabla_Y A)X, Z)=g(AX, \nabla_Y Z)-g(AY, \nabla_X Z)\,. \tag{4.6}$$

Since M is $(D, D\perp)$-geodesic we obtain that D is invariant by A. Then by using (1.2) we have

$$\nabla_X X_1=-J_1AX+Q_{12}(X)X_2+Q_{13}(X)X_3\,, \tag{4.7}$$

for all $X \in \Gamma(D)$. Now, substituting Z by X_1 in (4.6) and taking account of (4.7) we obtain

$$g((\nabla_X A)Y-(\nabla_Y A)X, X_1)=g(AY, J_1AX)-g(AX, J_1AX), \tag{4.8}$$

$$\text{for all } X, Y \in \Gamma(D).$$

By the assertion 1 of Lemma 1 we see that (4.8) becomes

$$g((\nabla_X A)Y-(\nabla_Y A)X, X_1)=0. \tag{4.9}$$

By using (1.3) and (4.9) in (3.10) we obtain $cg(X, X)=0$, which implies $c=0$ since we have $n \geq 1$. Thus M is a foliate orientable hypersurface of $R^{4(n+1)}$ and the assertion of the theorem follows from Theorem 4.

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QUASI-CONNECTIONS IN THE SEMIHOLONOMIC FRAME BUNDLE OF SECOND ORDER AND THEIR DIFFERENTIAL INVARIANTS

BY

W. MOZGAWA

Introduction. The purpose of the present paper is to study quasi-connections on the principal bundle of the semiholonomic frames of second order.

The notion of quasi-connection arises as a generalization of a Oproiu connection [17] and a Wong quasi-connection [24]. We shall recall in §1 the definition of semiholonomic frame bundle and some facts closely related to the Oproiu connection. In §2 we shall give the definition of quasi-connection in the semiholonomic frame bundle and their fundamental properties for the latter use. In §3 we shall consider the covariant derivative of extensor with respect to the quasi-connection. §4 is devoted to the induction of quasi-connection on a framed immersed submanifold. §5 is divided into four parts: in the first we give some natural tensors connected with quasi-connection, in the second one we determine all skew-symmetric torsion tensors, curvature extensors and their fundamental properties. In the third part we give the structure equations and Bianchi's identities of quasi-connection. The fourth part is devoted to study the curvature of the distribution determined by quasi-connection. §6 contains some examples.

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§ 1. Preliminaries and notations. In the present paper the indices $a, b, \ldots, i, j, \ldots$ will run over the range $1, 2, \ldots, n$, if we don't mark another convention.

Our considerations are in the category C^∞. Let us consider a bundle (P^1M, M, π) of linear frames on the manifold M dim $M=n$. We can define P^1M as $J_0^1(\mathbf{R}^n, M)$ — a set of the first order jets of local diffeomorphisms of $\mathbf{R}^n$ in M, where source of these all jets is $O \in \mathbf{R}^n$ and the target mapping is just the projection π. We can assume coordinates of jet for coordinates of frame i.e. if (U, x^i) is a coordinate neighbourhood in M then every frame $j_0^1\psi \in \pi^{-1}(U)$ can be uniquely expressed as well as (u^i, u^i_j) where $u^i = x^i \circ \psi(0)$, $u^i_j=(x^i \circ \psi)_{,j}(0)$ and the matrix u^i_j is non-singular. If V is neighbourhood of zero in $\mathbf{R}^n$ and $\varphi: V \to P^1M$ is the mapping such that $\pi\circ\varphi: V \to M$ is the local diffeomorphism, then the jet $j^1\circ\varphi$ is said to be a non-holonomic frame of second order in the point $\pi\circ\varphi(0)$. The space $\tilde{J}_0^2(\mathbf{R}^n, M)$