Theorem 5. *Let M be an orientable hypersurface of the quaternion manifold N. If one of the f-structures F_r $(r=1, 2, 3)$ is D-parallel, then M is locally represented as in Theorem 4.*

Proof. By using (2.9) we obtain that each one from the f-structures F_r is D-parallel if and only if we have $(h(X, Y)=0$, for all X tangent to M and $Y\in\Gamma(D)$. But this is equivalent to the integrability of both distributions D and $D^{\perp}$. Hence the proof follows from Theorem 4.

Now we prove the non-existence of foliate orientable hypersurfaces in a quaternion space form of non-null quaternion sectional curvature and give a complete characterization of them in $R^{4(n+1)}$.

Theorem 6. *Let M be a foliate complete orientable hypersuface of a simply connected complete $4(n+1)$-dimensional quaternion space form $N(c)$ with $n\geq 1$. Then we have $c=0$ and M is $R^{4n}\times M^{\perp}$ in $R^{4(n+1)}$ where M is a leaf of $D^{\perp}$ totally geodesic immersed in M.*

Proof. Taking into account that both distributions are involutive, for each X, $Y\in\Gamma(D)$ and $Z\in\Gamma(D^{\perp})$ we obtain

$$g((\nabla_X A)Y-(\nabla_Y A)X,\ Z)=g(AX,\ \nabla_Y Z)-g(AY,\ \nabla_X Z)\,. \tag{4.6}$$

Since M is $(D, D^{\perp})$-geodesic we obtain that D is invariant by A. Then by using (1.2) we have

$$\nabla_X X_1=-J_1AX+Q_{12}(X)X_2+Q_{13}(X)X_3\,, \tag{4.7}$$

for all $X\in\Gamma(D)$. Now, substituting Z by X_1 in (4.6) and taking account of (4.7) we obtain

$$g((\nabla_X A)Y-(\nabla_Y A)X,\ X_1)=g(AY,\ J_1AX)-g(AX,\ J_1AX), \tag{4.8}$$

$$\text{for all } X,\ Y\in\Gamma(D).$$

By the assertion 1 of Lemma 1 we see that (4.8) becomes

$$g((\nabla_X A)Y-(\nabla_Y A)X,\ X_1)=0. \tag{4.9}$$

By using (1.3) and (4.9) in (3.10) we obtain $cg(X, X)=0$, which implies $c=0$ since we have $n\geq 1$. Thus M is a foliate orientable hypersurface of $R^{4(n+1)}$ and the assertion of the theorem follows from Theorem 4.

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Received 7. X. 1980

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Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXVII, s. I a, 1981 f. 2

QUASI-CONNECTIONS IN THE SEMIHOLONOMIC FRAME BUNDLE OF SECOND ORDER AND THEIR DIFFERENTIAL INVARIANTS

BY

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Introduction. The purpose of the present paper is to study quasi-connections on the principal bundle of the semiholonomic frames of second order.

The notion of quasi-connection arises as a generalization of a Oproiu connection [17] and a Wong quasi-connection [24]. We shall recall in §1 the definition of semiholonomic frame bundle and some facts closely related to the Oproiu connection. In §2 we shall give the definition of quasi-connection in the semiholonomic frame bundle and their fundamental properties for the latter use. In §3 we shall consider the covariant derivative of extensor with respect to the quasi-connection. §4 is devoted to the induction of quasi-connection on a framed immersed submanifold. §5 is divided into four parts: in the first we give some natural tensors connected with quasi-connection, in the second one we determine all skew-symmetric torsion tensors, curvature extensors and their fundamental properties. In the third part we give the structure equations and Bianchi's identities of quasi-connection. The fourth part is devoted to study the curvature of the distribution determined by quasi-connection. §6 contains some examples.

The author expresses his thanks to Mr. A. Szybiak for his kind suggestions and criticisms.

§ 1. Preliminaries and notations. In the present paper the indices $a, b, \ldots, i, j, \ldots$ will run over the range $1, 2, \ldots, n$, if we don't mark another convention.

Our considerations are in the category C^{∞}. Let us consider a bundle (P^1M, M, π) of linear frames on the manifold M $\dim M=n$. We can define P^1M as $J_0^1(\mathbf{R}^n, M)$ — a set of the first order jets of local diffeomorphisms of $\mathbf{R}^n$ in M, where source of these all jets is $O\in\mathbf{R}^n$ and the target mapping is just the projection π. We can assume coordinates of jet for coordinates of frame i.e. if (U, x^i) is a coordinate neighbourhood in M then every frame $j_0^1\psi\in\pi^{-1}(U)$ can be uniquely expressed as well as (u^i, u^i_j) where $u^i=x^i\circ\psi(0)$, $u^i_j=(x^i\circ\psi)_{,j}(0)$ and the matrix u^i_j is non-singular. If V is neighbourhood of zero in $\mathbf{R}^n$ and $\varphi: V\to P^1M$ is the mapping such that $\pi\circ\varphi: V\to M$ is the local diffeomorphism, then the jet $j^1\circ\varphi$ is said to be a non-holonomic frame of second order in the point $\pi\circ\varphi(0)$. The space $\tilde{J}_0^2(\mathbf{R}^n, M)$

of nonholonomic frames of second order is principal frame bundle on M with the structural group $\tilde{L}^2_n = J^1_0(\mathbf{R}^n, J^1_0(\mathbf{R}^n, \mathbf{R}^n))_0$. This principal bundle is usual denoted by $(\tilde{P}^2M, \tilde{M}, \tilde{\pi})$. Let β denote the target projection $\beta : \tilde{P}^2M \to P^1M$. A frame $X \in \tilde{P}^2M$ is said to be semiholonomic if

$$\beta(X) = j^1_0(\pi \circ \varphi). \tag{1}$$

We can introduce a differential structure in a set $\overline{P}^2M$ of semiholonomic frames with a help of coordinates (u^i, u^i_j) in $\pi^{-1}(U) \subset P^1M$. Every nonholonomic frame $r = j^1_0\varphi$ can be uniquely expressed as well as $(v^i, v^i_j, w^i_j, w^i_{jk})$, where $v^i = (u^i \circ \varphi)(0)$ $v^i_j = (u^i_j \circ \varphi)(0)$, $w^i_j = (u^i \circ \varphi)_{,j}(0)$, $w^i_{jk} = (u^i_j \circ \varphi)_{,k}(0)$.
The condition (1) implies that $v^i_j = w^i_j$, thus if a frame of second order $r = j^1_0\varphi$ is semiholonomic then it has coordinates $r = (v^i, v^i_j, v^i_{jk})$. In similar way, we can introduce coordinates for the elements of the group $\overline{L}^2_n$ (the group of semiholonomic jets of local diffeomorphisms in $\mathbf{R}^n$): element $g \in \tilde{L}^2_n$ has coordinates $g = (g^i_j, \dot{g}^i_j, g^i_{jk})$ however (1) implies that $g^i_j = \dot{g}^i_j$ thus for every $g \in \overline{L}^2_n$ we can write $g = (g^i_j, g^i_{jk})$. For any $g, h \in \overline{L}^2_n$, the multiplications law is given by

$$(gh)^i_j = g^i_k h^k_j,$$

$$(gh)^i_{jk} = g^i_l h^l_{jk} + g^i_{lp} h^l_j h^p_k.$$

The right action of group $\overline{L}^2_n$ on $\overline{P}^2M$ is given by:

$$Xg = (x^i, x^i_j g^j_k, x^i_j g^j_{kt} + x^i_{jp} g^j_k g^p_t),$$

where $X = (x^i, x^i_j, x^i_{jk}) \in \overline{P}^2M$, $g = (g^i_j, g^i_{jk}) \in \overline{L}^2_n$. The set of semi-holonomic frames of second order is principal fibre bundle $(\overline{P}^2M, M, \overline{\pi})$ with the structure group $\overline{L}^2_n$.
A local trivialisation of the bundle $\overline{P}^2M$ is given by

$$h_u : \overline{\pi}^{-1}(U) \to x(U) \times L^2_n,$$

$$j^1 \circ \varphi \to (v^i, v^i_j, v^i_{jk}).$$

If (V, x^i) is another coordinate neighbourhood and $U \cap V \neq \emptyset$, then we would like to derive the transformation law of coordinates of semiholonomic frames of second order. We have the following diagram:

$$\begin{tikzcd} & \overline{\pi}^{-1}(U \cap V) \arrow[dl] \arrow[dr] & \\ X(U \cap V) \times \overline{L}^2_n \arrow[rr] & & X(U \cap V) \times L^2_n \end{tikzcd}$$

$$h_v \circ h^{-1}_U(x^i, x^i_j, x^i_{jk}) = (x^{i'}, x^{i'}_j, x^{i'}_{jk}).$$

It is easy to verify that

$$x^{i'} = (x^{i'} \circ x^{-1}) \circ x, \quad x^{i'}_j = x^k_j p^{i'}_k, \quad x^{i'}_{jk} = x^t_{jk} p^{i'}_t + x^t_j x^r_k p^{i'}_{tr},$$

$$\text{where } p^{i'}_k = \frac{\partial(x^{i'} \circ x^{-1})}{\partial x^k}, \quad p^{i'}_{tr} = \frac{\partial^2(x^{i'} \circ x^{-1})}{\partial x^t\, \partial x^r}.$$

The family of mappings (2) $p^{i'}_k, p^{i'}_{tr}$ are called transition functions of the bundle $(\overline{P}^2M, M, \overline{\pi})$. As usual the set $(\partial/\partial x^1)_p, \ldots, (\partial/\partial x^n)_p$ will denote the basis of tangent space T_pM with respect to the coordinate neighbourhood (u, x^i), $p \in pU$. Similarly $(\partial/\partial x^i, \partial/\partial x^i_j, \partial/\partial x^i_{jk})_z$ is the basis of $T_z\overline{P}^2M$ and $(\partial/\partial g^i_j, \partial/\partial g^i_{jk})_e$ is the basis of $T_e\overline{L}^2_n$.
By simple calculations we obtain the following propositions:
Proposition 1. *The basis of left invariant vector fields on L^2_n is given by*

$$\mathbf{E}^i_j = g^t_j \frac{\partial}{\partial g^t_i} + g^t_{ip} \frac{\partial}{\partial g^t_{jp}} + g^t_{pj} \frac{\partial}{\partial g^t_{pi}} \qquad \mathbf{E}^{ik}_j = g^t_j \frac{\partial}{\partial g^t_{ik}}. \tag{3}$$

Let $(\partial/\partial x^i, \partial/\partial x^i_j, \partial/\partial x^i_{jk})_z$ be the basis of $T_z\overline{P}^2M$ with respect to the coordinate neighbourhood (U, x^i) and $(\partial/\partial x^{i'}, \partial/\partial x^{i'}_j, \partial/\partial x^{i'}_{jk})$ is the another basis of $T_z\overline{P}^2M$ with respect to the another coordinate neighbourhood $(V, x^{i'})$, $z \in U \cap V$.
Proposition 2. *If (U, x^i) and $(V, x^{i'})$ are the charts such that $U \cap V \neq \emptyset$ then the following relations hold*

$$\frac{\partial}{\partial x^k} = p^{i'}_k \frac{\partial}{\partial x^{i'}} + x^i_t p^{j'}_{ik} \frac{\partial}{\partial x^{j'}_t} + (x^u_t x^l_r p^{j'}_{ulk} + x^u_{tr} p^{j'}_{uk}) \frac{\partial}{\partial x^{j'}_{tr}},$$

$$\frac{\partial}{\partial x^i_j} = p^{t'}_i \frac{\partial}{\partial x^{t'}_j} + x^i_t p^{j'}_{il} \frac{\partial}{\partial x^{j'}_{jt}} + x^l_t p^{j'}_{li} \frac{\partial}{\partial x^{j'}_{tj}}, \tag{4}$$

$$\frac{\partial}{\partial x^h_{ij}} = p^{i'}_h \frac{\partial}{\partial x^{i'}_{ij}}.$$

where $p^{i'}_{j'}, p^{i'}_{jk}$ are transition functions of the bundle $\overline{P}^2M$.
V. Oproiu [17] obtained a connection form in the principal fibre bundle $\overline{P}^2M$

$$\omega^i_j = x^{-1i}_k (dx^k_j + \Gamma^k_{pt} x^t_j dx^p),$$

$$\omega^i_{jk} = x^{-1i}_t (dx^t_{jk} - x^t_{pk}\omega^p_j - x^t_{jp}\omega^p_k + \Gamma^t_{spr} x^p_j x^r_k dx^s + \Gamma^t_{rp} x^p_{jk} dx^r), \tag{5}$$

where Γ^i_{jk} and Γ^i_{jkl} are the connection coefficients in the chart (U, x^i). Let $(\Gamma^i_{jk}, \Gamma^i_{jkl})$ and $(\Gamma^{i'}_{j'k'}, \Gamma^{i'}_{j'k'l'})$ be the connection coefficients in the chart (U, x^i) and $(V, x^{i'})$, respectively. In the intersection $U \cap V$ we have

$$\Gamma^a_{jk} p^{i'}_a = p^{i'}_{jk} + \Gamma^{i'}_{a'b'} p^{a'}_j p^{b'}_k,$$

$$\Gamma^i_{ju} p^{j'}_{il} + \Gamma^i_{jl} p^{j'}_{ui} + \Gamma^i_{jul} p^{i'}_i = p^{i'}_{jul} + \Gamma^{i'}_{j'k'l'} p^{j'}_j p^{k'}_u p^{l'}_l + \Gamma^{i'}_{j'k'} p^{j'}_j p^{k'}_{ul}. \tag{6}$$

Remark. Above connection coefficients with minus sign are exactly coefficients of Oproiu connection [17].

We denote by θ^i the canonical form on semiholonomic frame bundle. Given a connection $\omega^i_j, \omega^i_{jk}$ on $\overline{P}^2M$ we introduce n-tuple of vector fields E_z, such that

$$\theta^i(E_z)=\delta^i_z,\quad \omega^i_j(E^z)=0,\quad \omega^i_{jk}(E_z)=0.$$

We call E_z the standard horizontal vector fields.

V. Oproiu [17] found the structure equations of the connection $\omega^i_j \omega^i_{jk}$,

$$d\theta^i=\theta^k\wedge\omega^i_k+\frac{1}{2}T^i_{kt}\theta^k\wedge\theta^t,\quad d\omega^i_j=\omega^k_j\wedge\omega^i_k+\frac{1}{2}R^i_{jkt}\theta^k\wedge\theta^t,$$

$$\begin{aligned}(7)\qquad d\omega^i_{jk}&=\omega^t_{jk}\wedge\omega^i_t+\omega^t_j\wedge\omega^i_{tk}+\omega^t_k\wedge\omega^i_{jt}+\\&+\frac{1}{2}(R^i_{trjk}+R^i_{sjk}x^{-1s}_{\ \ p}x^p_{tr}-x^{-1i}_{\ \ p}x^p_{sp}R^s_{tjk}-x^{-1i}_{\ \ p}x^p_{ts}R^s_{rjk})\theta^t\wedge\theta^r.\end{aligned}$$

where

$$T^i_{kt}=\Gamma^i_{kt}-\Gamma^i_{tk},\qquad R^i_{jk|t}=\Gamma^i_{jk|t}-\Gamma^i_{jt|k}+\Gamma^i_{rk}\Gamma^r_{jt}-\Gamma^i_{rt}\Gamma^r_{jt},$$

$$R^i_{jkls}=\Gamma^i_{jkl|s}-\Gamma^i_{jks|l}+\Gamma^i_{rkl}\Gamma^r_{js}-\Gamma^i_{rks}\Gamma^r_{jt}+\Gamma^i_{jrl}\Gamma^r_{ks}-\Gamma^i_{jrs}\Gamma^r_{kl}-\Gamma^i_{rs}\Gamma^r_{jkl}+\Gamma^i_{rl}\Gamma^r_{jks}.$$

By Proposition 1 the fundamental vector fields on $\overline{P}^2M$ corresponding to $\partial/\partial g^i_j, \partial/\partial g^i_{jk}\in\pi^2_n$ are

$$(8)\qquad E^j_i=x^t_i\frac{\partial}{\partial x^t_j}+x^t_{ip}\frac{\partial}{\partial x^t_{jp}}+x^t_{pi}\frac{\partial}{\partial x^t_{pj}},\qquad E^{jk}_i=x^t_i\frac{\partial}{\partial x^t_{jk}}.$$

§2. **Quasi-connection in the semiholonomic frame bundle of second order.** The structural equations of the connection $\omega^i_j, \omega^i_{jk}$ has a dual formulation in terms of vector fields. If E_j, E^i_j, E^{ij}_k are the standard horizontal vector fields and the fundamental vector fields, then the structural equations become

$$(9.1)\qquad [E^i_j, E^k_i]=\delta^i_i E^k_j-\delta^k_j E^i_i,$$

$$(9.2)\qquad [E_j, E^i_k]=-\delta^i_j E_k,$$

$$\begin{aligned}(9.3)\qquad [E_j, E_k]&=-T^i_{jk}E_i-R^t_{rjk}E^r_t-(R^t_{rsjk}+R^t_{pjk}x^{-1p}_{\ \ z}x^z_{rs}-\\&-x^{-1t}_{\ \ z}x^z_{ps}R^p_{rjk}-x^{-1t}_{\ \ z}x^z_{rp}R^p_{sjk})E^{rs}_t,\end{aligned}$$

$$(9.4)\qquad [E_j, E^{kt}_i]=0,$$

$$(9.5)\qquad [E^i_j, E^{tp}_r]=\delta^t_j E^{ip}_j-\delta^t_j E^{ip}_r-\delta^p_j E^{ti}_r,$$

$$(9.6)\qquad [E^{jk}_i, E^{tp}_r]=0.$$

The equations (9.1), (9.5), (9.6) contain only the information about the group $\overline{L}^2_n$. Taking into account (9.2), (9.3), (9.4) we see that the conditions (9.2) and (9.4) admit to put on any assumptions which are independent on the structure of the semiholonomic frame bundle $\overline{P}^2M$. Y. Ch-Wong [24] has asked under what assumptions vector fields $\overline{E}_j$ which satisfy the condition (9.2) determine a linear connection. We will ask under what assumptions vector fields $\overline{E}_j$ which satisfy the conditions (9.2) and (9.4) determine Oproiu connection. We prove the following

Theorem 3. *Let E^i_j, E^{jk}_i be the fundamental vector fields on $\overline{P}^2M$; then every n-tuple of vector fields $\overline{E}_j$ on semiholonomic frame bundle $\overline{P}^2M$ such that*

$$(10.1)\qquad [\overline{E}_j, E^i_k]=-\delta^i_j\overline{E}_k,$$

$$(10.2)\qquad [\overline{E}_j, E^{kt}_i]=0,$$

is locally given by

$$\overline{E}_j=x^t_j\left(C^i_t\frac{\partial}{\partial x^i}-x^k_z\Phi^i_{tk}\frac{\partial}{\partial x^i_z}-x^k_p x^l_r\Phi^i_{tkl}\frac{\partial}{\partial x^i_{pr}}-x^k_{pr}\Phi^i_{tk}\frac{\partial}{\partial x^i_{p,r}}\right).$$

where $C^i_j, \Phi^i_{jk}, \Phi^i_{jkl}$ are family of functions which are subject to the following transformation rule

$$\begin{aligned}(11)\qquad &p^{a'}_j C^{i'}_{a'}=C^a_j p^{r'}_a,\quad \Phi^a_{jk}p^{i'}_a=C^a_j p^{i'}_{ak}+p^{a'}_j p^{b'}_k\Phi^{i'}_{a'b'},\\&\Phi^i_{ju}p^{j'}_{il}+\Phi^i_{jl}p^{j'}_{ui}+\Phi^i_{jul}p^{i'}_i-C^i_j p^{j'}_{ul|i}=p^{j'}_j p^{k'}_u p^{l'}_l\Phi^{i'}_{j'k'l'}+p^{j'}_j p^{k'}_{ul}\Phi^{i'}_{j'k'},\end{aligned}$$

and $p^{i'}_j, p^{i'}_{kt}$ are transition functions of the bundle $\overline{P}^2M$.

Proof. Let us consider vector fields

$$\overline{E}_j=f^i_j\frac{\partial}{\partial x^i}+g^i_{jt}\frac{\partial}{\partial x^i_t}+h^i_{jtr}\frac{\partial}{\partial x^i_{tr}},$$

where $f^i_j, g^i_{jt}, h^i_{jtr}$ are unknown functions of variables (x^i, x^i_j, x^i_{jk}). From the identities (10.1) and (10.2) we obtain the following partial differential equations

$$(12.1)\qquad x^j_t\frac{\partial f^i_k}{\partial x^j_s}=\partial^s_k f^i_t,\qquad (12.2)\qquad \partial^t_r g^i_{sp}-x^j_p\frac{\partial g^i_{sr}}{\partial x^j_t}=-\partial^t_s g^i_{pr}.$$

$$(12.3)\qquad h^i_{spz}\delta^t_r+h^i_{srp}\delta^t_z-x^j_p\frac{\partial h^i_{srz}}{\partial x^j_t}-x^j_{pu}\frac{\partial h^i_{srz}}{\partial x^j_{tu}}-x^j_{up}\frac{\partial h^i_{srz}}{\partial x^j_{ut}}=-\delta^t_s h^i_{prz}.$$

$$(12.4)\qquad \delta^t_z\delta^r_u g^j_{sp}=x^i_p\frac{\partial h^j_{szu}}{\partial x^i_{tr}},$$

$$(12.5)\qquad \frac{\partial f^j_t}{\partial x^i_{kr}}=O,\qquad (12.6)\qquad \frac{\partial g^j_{kt}}{\partial x^i_{ps}}=O.$$

By virtue of (12.5) and (12.6) we have that f^j_t, g^j_{kt} are independent on x^i_{pr}. Y. Ch-Wong proved in [24] that the equations (12.1) and (12.2) imply

$$(13.1)\qquad f^i_j=x^k_j C^i_k,\qquad (13.2)\qquad g^i_{jk}=-x^p_j x^t_k\Phi^i_{pr}$$

where C^i_j, Φ^i_{jk} are functions depending on points of M only.

By equation (12.4) it follows that, if $t\neq z$ and $r\neq u$ then

$$(14)\qquad \frac{\partial h^j_{szu}}{\partial x^i_{tr}}=O$$

Hence we have that h^j_{szu} depends only on x^i_{zu}. Thus if $t=z$ and $r=u$ then

$$x^i_p \frac{\partial h^j_{szu}}{\partial x^i_{zu}} = g^j_{sp},$$

and by virtue of (13.2) we have

$$\frac{\partial h^j_{szu}}{\partial x^i_{zu}} = -x^t_s \Phi^j_{ti}\,. \tag{15}$$

Since Φ^j_{ti} depends only on x^k, h^j_{szu} has the following form

$$h^j_{szu} = -x^t_s x^b_{zu} \Phi^j_{tb} + K^j_{szu}(x^k, x^k_p). \tag{16}$$

Let $t=r$, $t\neq z$ and $t\neq s$. If we substitute the right hand side of (16) into equation (12.3) then

$$K^j_{spz} - x^i_p \frac{\partial k^j_{srz}}{\partial x^i_r} = 0 \qquad (\textit{no summation with respect to } r) \tag{17}$$

If we put $t=z$, $t\neq r$, $t\neq s$ and $t=s$, $t\neq r$, $t\neq z$, then we obtain

$$k^j_{srp} - x^i_p \frac{\partial k^j_{srz}}{\partial x^i_z} = 0, \qquad (\textit{no summation with respect to } z) \tag{18}$$

$$k^j_{prz} - x^i_p \frac{\partial k^j_{srz}}{\partial x^i_s} = 0, \qquad (\textit{no summation with respect to } s) \tag{19}$$

respectively. Taking into account (12.3), we have

$$\frac{\partial h^i_{srz}}{\partial x^j_t} = 0. \tag{20}$$

Let us differentiate the equality (17) with respect to x^i_p $(p\neq s, r, z)$; we obtain

$$\frac{\partial k^j_{spz}}{\partial x^i_p} = \frac{\partial k^j_{srz}}{\partial x^i_r}\,.$$

Thus we have that $\partial k^j_{spz}/\partial x^i_p$ depends only on x^i, x^i_s, x^i_z. Similarly taking into account (18) and (19) we obtain

$$K^i_{spz} = -x^a_s x^b_p x^c_z \Phi^i_{abc} + D^i_{spz}\,, \tag{21}$$

where Φ^i_{abc}, D^i_{spz} depend only on x^j. Taking into account (17), (18), (19) and (21), we obtain

$$h^i_{spz} = -x^a_s x^b_p x^c_z \Phi^i_{abc} - x^a_s x^b_{pz} \Phi^i_{ab}\,. \tag{22}$$

Thus the vector fields $\overline{E}_j$ are given by

$$\overline{E}_j = x^i_j\left(C^k_i \frac{\partial}{\partial x^k} - x^t_p \Phi^z_{it} \frac{\partial}{\partial x^z_p} - x^k_p x^l_r \Phi^t_{ikl} \frac{\partial}{\partial x^t_{pr}} - x^k_{pr} \Phi^t_{ik} \frac{\partial}{\partial x^t_{pr}}\right). \tag{23}$$

We shall deduce the transformation rule of local vector fields $\overline{E}_j$ in the intersection of the two coordinate neighbourhoods. Given two coordinate neighbourhoods (U, x^i) and $(V, x^{i'})$ such that $U\cap V\neq\emptyset$ we have

$$\overline{E}_j = x^{j'}_j\left(C^{i'}_{j'} \frac{\partial}{\partial x^{i'}} - x^{k'}_t \Phi^{i'}_{j'k'} \frac{\partial}{\partial x^{i'}_t} - x^{k'}_t x^{l'}_p \Phi^{i'}_{j'k'l'} \frac{\partial}{\partial x^{i'}_{tp}} - x^{k'}_{tp} \Phi^{i'}_{j'k'} \frac{\partial}{\partial x^{i'}_{tp}}\right) \tag{24.1}$$

in $\overline{\pi}^{-1}(U\cap V)$. On the other hand, by Proposition 2 we have

$$\begin{aligned}\overline{E}_j = x^z_j \Big(& C^i_z p^{i'}_i \frac{\partial}{\partial x^{i'}} + C^i_z x^t_r p^{j'}_{ti} \frac{\partial}{\partial x^{j'}_r} + C^i_z (x^u_r x^t_s p^{j'}_{uti} + x^u_{rs} p^{j'}_{ui}) \frac{\partial}{\partial x^{j'}_{rs}} - \\ & - x^k_r \Phi^i_{zk} p^{i'}_i \frac{\partial}{\partial x^{i'}_r} - x^t_s \Phi^i_{zr} x^u_w p^{j'}_{iu} \frac{\partial}{\partial x^{j'}_{sw}} - x^k_r \Phi^i_{zk} p^{j'}_{ti} x^t_s \frac{\partial}{\partial x^{j'}_{sr}} + \\ & + x^k_r x^l_s \Phi^i_{zkl} p^{i'}_i \frac{\partial}{\partial x^{i'}_{rs}} - x^k_{rs} \Phi^i_{zk} p^{i'}_i \frac{\partial}{\partial x^{i'}_{rs}} \Big).\end{aligned} \tag{24.2}$$

By comparing the coefficients in (24.1) and (24.2), we obtain easily

$$p^{j'}_j C^{i'}_{j'} = C^i_j p^{i'}_i\,, \tag{25.1}$$

$$\Phi^a_{jk} p^{i'}_a = C^a_j p^{i'}_{ka} + p^{a'}_j p^{b'}_k \Phi^{i'}_{a'b'}\,, \tag{25.2}$$

$$\Phi^i_{ju} p^{i'}_{it} + \Phi^i_{jt} p^{i'}_{ui} + \Phi^i_{jut} p^{i'}_i = C^i_j p^{i'}_{uti} + p^{j'}_j p^{k'}_u p^{l'}_t \Phi^{i'}_{j'k'l'} + p^{j'}_j p^{k'}_{ut} \Phi^{i'}_{j'k'} \tag{25.3}$$

Corollaries.

(26.1) It follows from (25.2) that C^i_j is a tensor.

(26.2) If $z\in\overline{P}^2M$, then there are $n^3+n^2+\operatorname{rank} C^i_j(\overline{\pi}(z))$ linearly independent vectors in the family $\overline{E}_j(z)$, $E^i_j(z)$, $E^{ik}_j(z)$.

(26.3) If rank of tensor C is constant everywhere in the manifold M and is equal to $n=\dim M$, then we can put

$$\Gamma^i_{jk} := C^{-1a}_{\ j}\Phi^i_{ak}, \quad \Gamma^i_{jkl} := C^{-1a}_{\ j}\Phi^i_{akl}\,. \tag{27}$$

If this is the case, we can rewrite (23) in the form

$$\overline{E}_j = x^r_j C^i_r\left(\frac{\partial}{\partial x^t} - x^k_s \Gamma^t_{ik} \frac{\partial}{\partial x^t_s} - x^k_s x^l_z \Gamma^t_{ikl} \frac{\partial}{\partial x^t_{sz}} - x^k_{sz} \Gamma^t_{ik} \frac{\partial}{\partial x^t_{sz}}\right). \tag{28}$$

It is easily seen from the definition of the standard horizontal vector fields and (28) that $\overline{E}_j = \tilde{C}^i_j E_i$, where $\tilde{C}^i_j := x^r_j C^s_r x^{-1i}_{\ \ s}$. Thus we have:

Theorem 4. *If $\overline{E}_j$ are arbitrary vector fields on $\overline{P}^2M$ satisfying*

$$[\overline{E}_j, E^i_k] = -\delta^i_j \overline{E}_k, \quad [\overline{E}_j, E^{ik}_t] = 0,$$

and if $\overline{E}_j(z)$, $E^i_j(z)$, $E^{ik}_j(z)$ are linearly independent, for each $z\in\overline{P}^2M$, then there exists a unique connection in the sense of Oproiu (5) such that its standard horizontal vector fields are given by

$$E_j = \tilde{C}^{-1i}_{\ \ j}\overline{E}_i \text{ where } \tilde{C}^i_j = x^r_j C^s_r x^{-1i}_{\ \ s}\,.$$

Taking into account above consideration we can state the following

Definition. *The triple* $(c^i_j, \Phi^i_{jk}, \Phi^i_{jkl})$ *is said to be quasi-connection if it obeys the transformation rule* (25) *and tensor* C^i_j *is singular on* M.

We shall now consider the identity (9.3)

Proposition 5. *The following conditions are equivalent:*

a)
$$[\overline{E}_j, \overline{E}_k] = -\overline{T}^i_{jk}\overline{E}_i - \overline{R}^t_{rjk}E^r_t - (\overline{R}^t_{rsjk} + \overline{R}^t_{pjk}x^{-1p}_{\ z}x^z_{rs} - \tag{29.1}$$
$$- x^{-1t}_{\ z}x^z_{ps}\overline{R}^p_{rjk} - x^{-1t}_{\ z}x^z_{rp}\overline{R}^p_{sjk})E^{rs}_t$$

b) *there exists a family of functions* λ^a_{kl} *in every coordinate neighbourhood* (U, x^i) *such that*

$$C^a_{[k}C^i_{l]|a} = \lambda^a_{kl}C^i_a. \tag{29.2}$$

Proof. It is easily verified that

$$[\overline{E}_j, \overline{E}_k] = x^t_j x^s_k(C^i_{[t}C^z_{s]|i} - \Phi^i_{[ts]}C^z_i)\frac{\partial}{\partial x^z} - x^t_j x^p_k x^r_u(C^a_{[t}\Phi^i_{p]r|a} - \Phi^a_{[tp]}\Phi^i_{ar} -$$
$$-\Phi^a_{[t|r|}\Phi^i_{p]a})\frac{\partial}{\partial x^i_u} + x^t_j x^l_k x^p_{uz}(\Phi^a_{[tl]}\Phi^i_{ap} + \Phi^a_{[t|p|}\Phi^i_{l]a} - C^a_{[t}\Phi^i_{l]p|a})\frac{\partial}{\partial x^i_{uz}} + \tag{29.3}$$
$$+ x^t_j x^l_k x^p_u x^r_z(-C^a_{[t}\Phi^i_{l]pr|a} + \Phi^a_{[tl]}\Phi^i_{apr} + \Phi^a_{[t|p|}\Phi^i_{l]ar} + \Phi^a_{[t|r|}\Phi^i_{l]pa} + \Phi^a_{[j|pr|}\Phi^i_{l]a})\frac{\partial}{\partial x^i_{uz}}.$$

On the other hand, we have

$$-\overline{T}^i_{jk}\overline{E}_i - \overline{R}^t_{rjk}E^r_t - (\overline{R}^t_{rsjk} + \overline{R}^t_{pjk}\overline{x}^{1p}_{\ z}x^z_{rs} - \overline{x}^{1t}_{\ z}x^z_{ps}\overline{R}^p_{rjk} -$$
$$-\overline{x}^{1t}_{\ z}x^z_{rp}\overline{R}^p_{sjk})E^{rs}_t = -\overline{T}^i_{jk}x^t_iC^z_t\frac{\partial}{\partial x^z} + (\overline{T}^i_{jk}x^p_jx^r_u\Phi^z_{pr} -$$
$$-\overline{R}^s_{ujk}x^z_s)\frac{\partial}{\partial x^z_u} + (\overline{T}^i_{jk}x^a_i x^b_u x^l_z\Phi^c_{abl} - \overline{R}^i_{uzjk}x^c_i)\frac{\partial}{\partial x^c_{uz}} + \tag{29.4}$$
$$+(\overline{T}^i_{jk}x^a_i x^b_{uz}\Phi^c_{ab} - \overline{R}^f_{ejk}x^e_h x^h_{uz}x^c_f)\frac{\partial}{\partial x^c_{uz}} = [\overline{E}_j, \overline{E}_k].$$

Taking into account (29.3) and (29.4), we find

$$C^a_{[k}C^i_{l]|a} - \Phi^a_{[kl]}C^i_a = -T^a_{kl}C^i_a. \tag{30.1}$$

$$C^a_{[k}\Phi^i_{l]j|a} - \Phi^a_{[k|j|}\Phi^i_{l]a} - \Phi^a_{[kl]}\Phi^i_{aj} = -R^i_{jkl} - T^a_{kl}\Phi^i_{aj} \tag{30.2}$$

$$-C^a_{[j}\Phi^i_{l]pr|a} + \Phi^a_{[jl]}\Phi^i_{apr} + \Phi^a_{[j|p|}\Phi^i_{l]ar} + \Phi^a_{[j|r|}\Phi^i_{l]pa} + \Phi^a_{[j|pr|}\Phi^i_{l]a} = T^t_{jl}\Phi^i_{tpr} - R^i_{prjl}, \tag{30.3}$$

where

$$\overline{T}^t_{jl} = T^p_{rs}x^t_p\overline{x}^{1r}_{\ j}\overline{x}^{1s}_{\ l}, \quad R^i_{jkl} = \overline{R}^d_{abc}x^i_d\overline{x}^{1a}_{\ j}\overline{x}^{1b}_{\ k}\overline{x}^{1c}_{\ l},$$
$$\overline{R}^i_{jkln} = R^e_{abcd}x^i_e\overline{x}^{1a}_{\ j}\overline{x}^{1b}_{\ k}\overline{x}^{1c}_{\ l}\overline{x}^{1d}_{\ n}.$$

Taking account of (30.1) we see that (29.1) is equivalent to (29.2). If we

take account of (30.2) and (30.3) we see that R^i_{jkl}, R^i_{jkln} are uniquely determined by C^i_j, T^i_{jk}, Φ^i_{jk}, Φ^i_{jkl}.

§ 3. **Covariant differentiation with respect to the quasi-connection.** Let us consider a bundle B^p_q of (p, q) tensors on M and the bundle $J^1(B^p_q)$ which is the space of 1-jets of the local sections in B^p_q ([15], [17]). The cross-section in $J^1(B^p_q)$ is said to be a (p, q) extensor on M. An element T of $J^1(B^p_q)$ has the coordinates

$$(T^{i_1 \dots i_p}_{j_1 \dots j_q}, \quad T^{i_1 \dots i_p}_{j_1 \dots j_{q,k}})$$

in a local chart (U, x^i) of M.

In this section we shall consider only cases when (p, q) is equal to $(1,0)$ and $(1,1)$. The theory of covariant differentiation of extensors can be given in the general case but it is outside of our considerations. Let φ be the quasi-connection given by $(C^i_j, \Phi^i_{jk}, \Phi^i_{jkl})$. Then we can define the covariant differentials for $X \in J^1(B^1_0) = J^1(TM)$ and $Z \in J^1(B^1_1)$ with the coordinates $X = (x^i, x^i_{.k})$ and $Z = (z^i_j, z^i_{j.k})$ by

$$\tilde{\nabla}_j(x^i, x^i_{.k}) := (C^a_j x^i_{|a} + x^b\Phi^i_{jb},\ C^a_j x^i_{.k|a} + x^b_{.k}\Phi^i_{jb} - \Phi^b_{jk}x^i_{.b} + \Phi^i_{jbk}x^b), \tag{31.1}$$

$$\tilde{\nabla}_j(Z^c_d, Z^c_{d,l}) := (C^a_jZ^c_{d|a} + \Phi^c_{jt}Z^t_d - \Phi^t_{jd}Z^c_t,\quad C^a_jZ^c_{d,l|a} + \Phi^c_{jq}Z^q_{d,l} - \tag{31.2}$$
$$-Z^c_{q,l}\Phi^q_{jp} - Z^c_{d,q}\Phi^q_{jl} + Z^a_d\Phi^c_{jal} - Z^c_e\Phi^e_{jdl}).$$

Theorem 6. *Let* φ *be the quasi-connection on manifold* M. *Then for any extensors* $X \in J^1(TM)$ *and* $Y \in J^1(B^1_1)$ *expressions* $\nabla_j(X^i, X^i_{,k})$ *and* $\nabla_j(Y^c_d, Y^c_{d,k})$ *are the coordinates of extensors belonging to* $J^1(TM) \otimes TM^*$ *and* $J^1(B^1_1) \otimes TM^*$, *respectively.*

The proof can be obtained by straightforward but tedious checking of the transformation rule of the covariant differential, and is ommited here.

If the tensor C has the maximal rank on M then the quasi-connection reduces to the Oproiu connection (27) and the formulas (31.1) and (31.2) give covariant differential $\nabla_j = C^{-1i}_{\ j}\nabla$, with respect to the O p r o i u connection.

V. O p r o i u [17] derived only the covariant derivative of elements of $J^1(TM)$ i.e. we have

$$\nabla_j(x^i, x^i_{.k}) = (x^i_{|j} + x^a\Gamma^i_{ja},\ x^i_{.k|j} + x^b_{.k}\Gamma^i_{jb} - \Gamma^b_{jk}x^i_{.b} + \Gamma^i_{jbk}x^b). \tag{32}$$

Using the method given in [21], one can obtain the covariant differentials of arbitrary extensor with respect to Oproiu connection $(\Gamma^i_{jk}, \Gamma^i_{jkl})$. For example the covariant differential of (1.1) extensor with local coordinates $(X^c_d, X^c_{d,l})$ is given by

$$\nabla_j(X^c_d, X^c_{d,l}) = (X^c_{d|j} + \Gamma^c_{jt}X^t_d - \Gamma^t_{jd}X^c_t,\quad X^c_{d,l|j} + \Gamma^c_{jq}X^q_{d,l} - \tag{33}$$
$$-X^c_{q,l}\Gamma^q_{jd} - X^c_{d,q}\Gamma^q_{jl} + \Gamma^c_{jal}X^a_d - X^c_e\Gamma^e_{jdl}).$$

§ 4. **Induced quasi-connection.** Let M and N be the manifolds, $\dim M = n$, $\dim N = m$, $m \leqslant n$ and the manifold N is immersed in manifold M. That is, we have the pair (N, F), $F: N \to M$ where rank $F_* = m$. In this section the indices $a, b, \ldots$ run over the range $1, \ldots, n$ the indices $i, j, \ldots$ the range $1, \ldots, m$ and the indices $s, t, \ldots$, the range $m+1, \ldots, n$.

Let (U, φ) and (V, ψ) be the coordinate neighbourhoods of points $y \in N$ and $F(y) \in M$, respectively. Then we define $f^a(x) := \psi^a \circ F \circ \varphi^{-1}(x)$. Let be given a vector $v = j^1_0 \tilde{u}(t) \in T_y N$ $\tilde{u}(0) = y$ and a vector field $\tilde{v}$ such that $\tilde{v}(y) = v$. Then $V = j^1_y \tilde{v} \in J^1(TN)$ with the local coordinates $V = (v^i, v^i_{,j})$. We shall compute the exvector on M which is determined by the exvector V on N: if $\tilde{u}: R \to N$, $v = j^1_0 \tilde{u}$ then $F \circ \tilde{u}: R \to M$ determines the vector $j^1_0 F \circ \tilde{u}$ with the coordinates u^a. In such a case we have

$$(34) \qquad u^a = \left.\frac{d}{dt}\right|_{t=0} \psi^a \circ F \circ \tilde{u}(t) = f^a_{|i} v^i .$$

Differentiating the above relation (34), we obtain:

$$(35) \qquad u^a_{.b} f^b_j = f^a_{i|j} v^i + f^a_i v^i_{.j} .$$

In order to induce the quasi-connection, we need the concept of the projection of extensor and covectors from M to N. We assume that manifold M is framed, i.e. there exist exvectors $n^a_s, n^a_{s,t} \notin J^1(TN)$, which together with $f^a_i, f^a_{i,j}$ form the basis of $J^1(TM)$. Exvectors of the dual basis are uniquely determined from the following conditions:

$$(36) \qquad \begin{aligned} &(\bar{f}^i_a, \bar{f}^i_{a.b}) \cdot (f^a_j, f^a_{j.k}) = (\delta^i_j, 0), \qquad (\bar{n}^s_a, \bar{n}^s_{a.b}) \cdot (f^a_i, f^a_{i.j}) = (0, 0) \\ &(\bar{f}^i_a, \bar{f}^i_{a.b}) \cdot (n^a_s, n^a_{s,t}) = (0, 0), \qquad (\bar{n}^t_a, \bar{n}^t_{a.b}) \cdot (n^a_s, n^a_{s,r}) = (\delta^t_s, 0). \end{aligned}$$

Now we can project the exvector along the space spanned on exvectors $(n^a_s, n^a_{s,t}) \in J^1(TM)$. Given a covector x_a, then its projection on N, is expressed by

$$(37) \qquad x_j = x_a f^a_j .$$

Given a vector y^a, then its projection depends on framing $(n^a_s, n^a_{s,t})$ and is expressed by

$$(38) \qquad y^j = y^a \bar{f}^j_a .$$

In order to obtain the projection of a exvector we differentiate both sides of identity (38)

$$y^j_{.k} = y^a_{.b} f^b_k \bar{f}^j_a + y^a \bar{f}^j_{a.b} f^b_k .$$

Taking into account (36), we have

$$(39) \qquad y^j_{.k} = y^a_{,b} f^b_k \bar{f}^j_a - y^a \bar{f}^i_a f^e_{i.k} \bar{f}^j_e .$$

Summing up, the projection of extensor $(X^a_d, X^a_{d.b}) \in J^1(TM) \otimes TM^*$ is given by

$$(40) \qquad (x^i_j, x^i_{j.k}) = (f^b_j x^a_b \bar{f}^i_a,\ x^a_{d.b} f^b_k \bar{f}^i_a f^d_j - x^a_b \bar{f}^i_a f^e_{i.k} \bar{f}^i_e f^b_j).$$

Theorem 7. *If $\varphi = (C^a_b, \Phi^a_{bc}, \Phi^a_{bcd})$ is a quasi-connection on the manifold M, then there exists the unique quasi-connection $(C^i_j, \psi^i_{jk}, \psi^i_{jkl})$ induced φ on the framed manifold N immersed in M. This quasi-connection is given by*

$$(41.1) \qquad C^i_j = f^b_j C^a_b \bar{f}^i_a ,$$

$$(41.2) \qquad \Psi^p_{ki} = C^r_k f^a_{i,r} \bar{f}^p_a + \Phi^a_{c,b} f^b_i f^c_k \bar{f}^p_a ,$$

$$(41.3) \qquad \begin{aligned} \Psi^r_{kij} = {} & C^p_k f^a_{i,j,p} \bar{f}^r_a + \Phi^a_{cb} f^b_{i,j} f^c_k \bar{f}^r_a - C^p_k f^b_{i,p} \bar{f}^e_b f^a_{e,j} \bar{f}^r_a - \Phi^e_{cb} f^b_i \bar{f}^p_e f^a_{p,j} \bar{f}^r_a f^c_k - \\ & - C^p_k f^b_{j,p} \bar{f}^l_b f^a_{l,j} \bar{f}^r_a - \Phi^e_{cb} f^b_j f^c_k \bar{f}^p_e f^a_{i,p} \bar{f}^r_a + \Phi^a_{cbd} f^b_i f^c_k f^d_j \bar{f}^r_a . \end{aligned}$$

Proof. Given a exvector field $(v^a, v^a_{.b}) \in J^1(TN)$ let us extend it regularly to the extensor field $(v^i, v^i_{.j})$ on M. In such a case we have

$$(42.1) \qquad v^i = v^a \bar{f}^i_a ,$$

$$(42.2) \qquad v^i_{.j} = v^a_{.b} f^b_j \bar{f}^i_a - f^a_{p.j} \bar{f}^p_c \bar{f}^i_a v^c .$$

Let be given the covariant differential

$$(43) \qquad \nabla_g(v^a, v^a_{.b}) = (C^e_g v^a_{.e} + \Phi^a_{ge} v^e,\ C^e_g v^a_{.b|e} + \Phi^a_{ge} v^e_{.b} - \Phi^e_{gb} v^a_{.e} + \Phi^a_{geb} v^e).$$

Taking into account (40), the projection of the covariant differential (43) is given by

$$(44) \qquad \bar{\nabla}_j(v^i, v^i_{.k}) = (f^g_j \nabla_g v^a \bar{f}^i_a,\ \nabla_c v^a_{.d} f^c_j f^d_k \bar{f}^i_a - \nabla_g v^c f^g_j \bar{f}^p_c f^a_{p.k} \bar{f}^i_a).$$

Let us consider the first term in the formula (44)

$$(45) \qquad f^g_j \nabla_g v^a \bar{f}^i_a = f^g_j C^e_g v^a_{.e} \bar{f}^i_a + f^g_j \Phi^a_{ge} v^e \bar{f}^i_a .$$

Then from (42) we find

$$\begin{aligned} \tilde{\nabla}_j v^i &= f^g_j \nabla_g v^a \bar{f}^i_a = f^g_j C^e_g f^p_e v^i_{.p} + f^g_j C^e_g f^p_e f^a_{l.s} \bar{f}^i_a v^l + f^g_j \Phi^a_{ge} \bar{f}^i_a f^e_p v^p = \\ &= C^p_j v^i_{|p} + \{C^p_j f^a_{l.p} \bar{f}^i_a + \Phi^a_{ge} f^g_j \bar{f}^i_a f^e_l\} v^l = C^p_j v^i_{|p} + \psi^i_{lp} v^p . \end{aligned}$$

Hence we have

$$\psi^i_{jp} = C^l_j f^a_{p.l} \bar{f}^i_a + \Phi^a_{ge} f^g_j f^e_p \bar{f}^i_a .$$

The proof of (41.3) is similar. If we take account of the second term of (44) and (42.2), then we obtain

$$\tilde{\nabla}_c v^a_{.d} f^c_j f^d_k \bar{f}^i_a - \tilde{\nabla}_g v^c f^g_j \bar{f}^p_c f^a_{p.k} \bar{f}^i_a = \tilde{\nabla}_j v^i_{.k} = C^p_j v^i_{.k|p} + \psi^i_{jp} v^p_{.k} - \psi^p_{jk} v^i_{.p} + \psi^i_{jpk} v^p .$$

It is easily seen that, the term at v^p is equal to the right side of (41.3).

Corollaries. 1. If $(f^a_j, f^a_{i,j})$ are the transition functions of the bundle $\bar{P}^2M$, then the formulas (41) are just the transformation rule of the quasi-connection.

2. If rank $C = \dim M$ then there exists the unique Oproiu connection $(\Gamma^l_{ij}, \Gamma^p_{mij})$ induced from the connection (27) on the framed manifold N immersed in M. This is given by

$$(46.1) \qquad \Gamma^p_{ji} = f^a_{i,j} \bar{f}^p_a + \Gamma^a_{db} f^d_j f^b_i \bar{f}^p_a ,$$

$$(46.2)\quad \begin{aligned}\Gamma^p_{mij} &= f^a_{i,j,m}\bar f^p_a - \Gamma^a_{db} f^d_n f^b_{i,j}\bar f^p_a - f^b_{i,n}\bar f^r_b f^a_{r,j}\bar f^p_a - \Gamma^c_{cb} f^c_n f^b_i \bar f^r_c f^a_{r,j}\bar f^p_a - \\ &- f^l_{j,n}\bar f^r_b f^a_{i,r}\bar f^p_a - \Gamma^r_{cb}\bar f^k_r f^a_{i,k} f^b_j f^c_n \bar f^p_a + \Gamma^a_{cbd} f^b_i f^c_n f^d_j \bar f^p_a .\end{aligned}$$

Proposition 8. If $(C^a_b, \Phi^a_{bc}, \Phi^a_{bcd})$ is the quasi-connection on the manifold M and $(C^i_j, \psi^i_{jk}, \psi^i_{jkl})$ is the induced quasi-connection on the framed submanifold $N \subset M$, then the covariant differential of (1.1) mixed extensor is given by

$$(47)\quad \begin{aligned}\nabla_{;}(t^a_i, t^a_{i,j}) = (c^p_k t^a_{i|p} + \Phi^a_{cb} t^b_i f^c_k - \psi^p_{ki} t^a_p ,\quad & C^p_k t^a_{i,j|p} + \Phi^a_{cb} t^b_{i,j} f^c_k - \\ - \psi^p_{ki} t^a_{p,j} - \psi^p_{kj} t^a_{i,p} + \Phi^a_{cbd} t^b_i f^c_k f^d_j - \psi^p_{kij} t^a_p).\end{aligned}$$

The proof can be obtained by straightforward but tedius checking of the transformation rule of the covariant differential.

Analogously the general case is outside of our considerations.

§ 5. **The differential invariants of the quasi-connection.** All considerations of this section are in the fixed coordinate neighbourhood (U, x^i) of M.

a) **Naive invariants.** We shall give a few tensors naturally determined by the quasi-connection $\varphi = (C^i_j, \Phi^i_{jk}, \Phi^i_{jkl})$.

(48.1) $C^t_{[k}\Phi^a_{l]t}$ are the coordinates of (1,2)-tensor, [24]

(48.2) $C^u_{tp} C^t_r \Phi^j_{jul}$ are the coordinates of (1,3)-tensor,

(48.3) $C^b_{[k} C^h_{l]|b} - \Phi^b_{[kl]} C^h_b$ are the coordinates of (1,2)-tensor, [24]

W being a (r, s)-tensor, $r \geqslant 1$ such that

$$(48.4)\quad W^{i_1 \ldots i_t a i_{t+1} \ldots i_{r-1}}_{j_1 \ldots j_s} C^h_a = 0,$$

then

(48.5) $W^{i_1 \ldots i_t a i_{t+1} \ldots i_{r-1}}_{j_1 \ldots j_s} \Phi^h_{al}$ are the coordinates of $(r, s+1)$-tensor, [24]

W being a (r, s)-tensor, $r \geqslant 1$ such that the condition (48.4) and

$W^{i_1 \ldots i_t a i_{t+1} \ldots i_{r-1}}_{j_1 \ldots j_s} \Phi^h_{ak} = 0$, are satisfied, then

(48.6) $W^{i_1 \ldots i_t a i_{t+1} \ldots i_{r-1}}_{j_1 \ldots j_s} \Phi^h_{akl}$ are the coordinates of $(r, s+2)$-tensor,

(48.7) $\Phi^i_{j[kl]}$ are the coordinates of (1,3)-tensor,

(48.8) $\Phi^a_{[jk]} + C^a_{[j|k]}$ are the coordinates of (1,2)-tensor.

b) **Torsion tensors and curvature extensors.** We shall give the classical theorem of Frobenius in the following form.

Proposition 9. *C being an arbitrary tensor of type* (1,1) *such that* rank $C =$ $= m$ *on M, the distribution* im C *is integrable iff there exist a family of functions λ^a_{kl} such that*

$$C^a_{[k} C^i_{l]|a} = \lambda^a_{kl} C^i_a ,$$

in (U, x^i).

Y. Ch-Wong proved in [24]

Proposition 10. *Let C and S be* (1,1) *and* (1,2) *tensors, respectively and* rank $C = const$ *on M. If there exist a family of function θ^h_{kl} such that $C^i_h \theta^h_{kl} =$ $= S^i_{kl}$, then there exists globaly a* (1,2) *tensor T such that $C^i_h T^h_{kl} = S^i_{kl}$ is satisfied.*

Corollary [24]. *If φ is the quasi-connection,* rank $C = const = m$ *and distribution* im C *is integrable, then there exists* (1 2) *tensor T such that the following condition is satisfied*

$$(49)\quad T^h_{kl} C^i_h = (\Phi^h_{[kl]} - \lambda^h_{kl})\, C^i_h .$$

Definition. *A tensor T is said to be a torsion tensor of quasi-connection if the condition* (49) *is satisfied.* Let be given a quasi-connection $\varphi = (C^i_j, \Phi^i_{jk}, \Phi^i_{jkl})$ and denote by $(R^i_{\cdot kl}, R^i_{prjl})$ the following expressions

$$(50.1)\quad R^i_{jkl} := C^a_{[k}\Phi^i_{l]j|a} - \Phi^a_{[k|j|}\Phi^i_{l]a} - \Phi^a_{[kl]}\Phi^i_{aj} + T^a_{kl}\Phi^i_{aj} ,$$

$$(50.2)\quad R^i_{prjl} := C^a_{[j}\Phi^i_{l]pr|a} - \Phi^a_{[jl]}\Phi^i_{apr} - \Phi^a_{[j|p|}\Phi^i_{l]ar} - \Phi^a_{[j|r|}\Phi^i_{l]pa} - \Phi^a_{[j|pr|}\Phi^i_{l]a} + T^t_{jl}\Phi^i_{tpr} .$$

Let $\bar{\mathbf{L}}^2_n$ denote the vector bundle with fibre the Lie algebra $\bar{L}^2_n$ of the group $\bar{\mathbf{L}}^2_n$, associated with the principal bundle $\bar{P}^2 M$.

Theorem 11. *If φ is the quasi-connection,* rank $C = m$ *and distribution* im C *is integrable, then for an arbitrary tensor T of the expressions* (50.1) *and* (50.2) *are the coordinates of extensor* $R \in \bar{\mathbf{L}}^2_n \otimes \wedge^2 T^* M$.

The proof is ommitted here, because it can be obtained by straightforward but tedious checking of the transformation rule of (50.1) and (50.2).

Definition. *An extensor R with coordinates (R^i_{jkl}, R^i_{ikln}) is said to be a curvature extensor of quasi-connection φ.*

Taking into account (23), we obtain the following geometric interpretation of curvature extensor R.

Theorem 12. *A necessary and sufficient condition for the distribution* Lin$(\bar E_j)$ *to be integrable is that there exists an extensor R with the vanishing coordinates among all curvature extensors of φ.*

Proof. Let the distribution Lin$(\bar E_j)$ be integrable i.e.

$$(51)\quad [\bar E_j, \bar E_k] = \bar\lambda^i_{jk}\bar E_i .$$

Taking into account (30) we have

$$(52.1)\quad C^i_{[j} C^t_{k]|i} - \Phi^i_{[jk]} C^t_i = \lambda^z_{jk} C^t_z ,$$

$$(52.2)\quad C^a_{[k}\Phi^t_{l]j|a} - \Phi^a_{[kl]}\Phi^t_{aj} - \Phi^a_{[k|j|}\Phi^i_{l]a} = \lambda^t_{kl}\Phi^i_{tj} ,$$

$$(52.3)\quad C^a_{[j}\Phi^i_{l]pr|a} - \Phi^a_{[jl]}\Phi^i_{apr} - \Phi^a_{[j|p|}\Phi^i_{l]ar} - \Phi^a_{[j|r|}\Phi^i_{l]pa} - \Phi^a_{[j|pr|}\Phi^i_{l]a} = \lambda^t_{jl}\Phi^i_{tpr}$$

where

$$\lambda^z_{jk} := \bar\lambda^c_{ab} x^{-1a}_{\ \ j} x^{-1b}_{\ \ k} \lambda^z_c .$$

Thus taking account of (52.1), it follows from Proposition 10, that there exists a tensor T such that

$$(53)\quad C^i_{[j} C^t_{k]|i} - \Phi^i_{[jk]} C^t_i = -T^z_{jk} C^t_z .$$

Hence

$$C^a_{[k}\Phi^i_{l]j|a} - \Phi^a_{[kl]}\Phi^i_{aj} - \Phi^a_{[k|j|}\Phi^i_{l]a} + T^z_{kl}\Phi^i_{zj} = 0, \tag{54.1}$$

$$C^a_{[j}\Phi^i_{l]pr|a} - \Phi^a_{[jl]}\Phi^i_{apr} - \Phi^a_{[j|p|}\Phi^i_{l]ar} - \Phi^a_{[j|r|}\Phi^i_{l]pa} - \Phi^a_{[j|pr|}\Phi^i_{l]a} + T^t_{jl}\Phi^i_{tpr} = 0. \tag{54.2}$$

If the coordinates (R^i_{jkl}, R^i_{jkln}) of a curvature extensor are vanishing for some torsion tensor T, then from (29.1) we have

$$[\bar{E}_j, \bar{E}_k] = -\bar{T}^i_{jk}\bar{E}_i .$$

Now, we shall involve the Nijenhuis tensor of the tensor C

$$N(X, Y) = [CX, CY] - C(X, CY] - C(CX, Y] + C^2[X, Y], \tag{55.1}$$

with the coordinates

$$N^i_{kl} = C^a_{[k}C^i_{l]|a} - C^a_{[l|k]}C^i_a . \tag{55.2}$$

Let K_c be the following set

$$K_c := \{P \in TM \otimes \wedge^2 TM^* ; \ \operatorname{im} P = \ker C\}.$$

Theorem 13. *Every skew-symmetric tensor T with the coordinates*

$$T^a_{kl} = \Phi^a_{[kl]} + C^a_{[k|l]} - S^a_{kl} + P^a_{kl} \tag{56}$$

is torsion tensor of the quasi-connection φ iff the tensor S satisfies the condition

$$N^i_{kl} = S^a_{kl}C^i_a , \tag{57}$$

and tensor $P \in K_c$.

Proof. Given a torsion tensor T of the quasi-connection φ, it satisfies the condition

$$T^a_{kl}C^i_a = \Phi^a_{[kl]}C^i_a - C^a_{[k}C^i_{l]|a} . \tag{58}$$

Subtituting the equation (56) in (58), we obtain

$$\Phi^a_{[kl]}C^i_a + C^a_{[k|l]}C^i_a - S^a_{kl}C^i_a + P^a_{kl}C^i_a = \Phi^a_{[kl]}C^i_a - C^a_{[k}C^i_{l]|a} ,$$

and hence $N^i_{kl} = S^a_{kl}C^i_a$ and $P \in K_c$.

Conversely, if (58) is satisfied, we have

$$(T^a_{kl} - \Phi^a_{[kl]} - C^a_{[k|l]})\, C^i_a = -N^i_{kl} ; \tag{59.1}$$

since the right side of (59.1) does not depend on $\Phi^a_{[kl]}$, therefore T^a_{kl} should have the form

$$T^a_{kl} = \Phi^a_{[kl]} + W^a_{kl} \tag{59.2}$$

where W^a_{kl} obeys such a transformation rule, that $\Phi^a_{[kl]} + W^a_{kl}$ are just the coordinate of some tensor. It is easily seen that W^a_{kl} obeys the same transformation rule as $C^a_{[k|l]}$; therefore we have:

$$(W^a_{kl} - C^a_{[k|l]})\, C^i_a = -N^i_{kl} = -S^a_{kl}C^i_a$$

and in the sequel

$$(W^a_{kl} - C^a_{[k|l]} + S^a_{kl})\, C^i_a = 0.$$

Hence $W^a_{kl} - C^a_{[k|l]} + S^a_{kl}$ are the coordinates of the tensor $P \in K_c$. Finally, we have

$$P^a_{kl} = W^a_{kl} - C^a_{[k|l]} + S^a_{kl} .$$

Suming up, the coordinates of the tensor T are given by:

$$T^a_{kl} = \Phi^a_{[kl]} + C^a_{[k|l]} - S^a_{kl} + P^a_{kl} .$$

We shall now recall the Rao-Mitra's theorem which will be used here.

Theorem 14. ([19]), [13]). *Let A be an arbitrary matrix and A^- a matrix such that $AA^-A = A$. A necessary and sufficient condition for the equation*

$$Ax = y \tag{60}$$

to be consistent is $AA^-y = y$. Then all solutions of the equation (60) are given by

$$x = A^-y + (I - A^-A)z,$$

where z is an arbitrary vector belonging to the range of A.

Corollary (Obata's theorem). *A being a projection operator, a necessary and sufficient condition for the equation (60) to be consistent is $Ay = y$, then all solutions of the equation (60) are given by*

$$x = y + (I - A)z.$$

By Rao-Mitra's theorem and Theorem 13, we can propose a following simple classification of manifolds with a quasi-connection:
type α – when $N^i_{kl} = 0$; (cf [12]);
type β – when the Nijenhuis tensor N^i_{kl} is a priori given in the form $N^i_{kl} = S^a_{kl}C^i_a$,
type γ_1 – when the tensor C is the projection operator (i.e. $C^2 = C$) an the distribution $\operatorname{im} C$ is integrable;
type γ_2 – when there is given a tensor $\bar{C}$, suh that $C\bar{C}C = C$ and the distribution $\operatorname{im} C$ is integrable.

We can give all torsion tensors and curvature extensors in all above cases, but we shall present all these ones only for a space of type γ_2. Thus by virtue of Theorems 14 and 13 we have

$$T^a_{kl} = \Phi^a_{[kl]} + C^a_{[k|l]} - N^a_{kl}\bar{C}^i_a + B^i_{kl} - B^a_{kl}\bar{C}^b_a C^i_b .$$

$$R^i_{jkl} = C^a_{[k}\Phi^i_{l]j|a} - \Phi^a_{[k|j|}\Phi^i_{l]a} + C^a_{[k|l]}\Phi^i_{aj} - N^a_{kl}\bar{C}^b_a\Phi^i_{bj} - B^a_{kl}\Phi^i_{aj} - B^a_{kl}\bar{C}^b_a\Phi^i_{cj}C^c_b ,$$

$$R^i_{prjl} = C^a_{[j}\Phi^i_{l]pr|a} - \Phi^a_{[j|p|}\Phi^i_{l]ar} - \Phi^a_{[j|r|}\Phi^i_{l]pa} - \Phi^a_{[j|pr|}\Phi^i_{l]a} + C^a_{[j|l]}\Phi^i_{apr} - \tag{61}$$
$$- N^a_{jl}\bar{C}^b_a\Phi^i_{bpr} - B^a_{jl}\Phi^i_{apr} - B^a_{jl}\bar{C}^b_a C^e_b\Phi^i_{epr} .$$

where B^i_{jk} is an arbitrary skew-symmetric tensor.

c) **The structure equations of a quasi-connection.** In order to present the structure equations of a quasi-connection, we should give an exterior differentiation on a manifold of an arbitrary type α, β, γ_1 or γ_2. We shall introduce it only for smooth functions, 1-forms and 2-forms.

If f is an arbitrary smooth function on M, ω – a 1-form and Φ a 2-form, then we put

$$\delta_z f := Z^j C_j^k f_{|k}, \quad Z = Z^j \frac{\partial}{\partial x^j} . \tag{62.1}$$

$$\begin{aligned} \delta\omega(X, Y) := \delta_X \omega(Y) - \delta_Y \omega(X) - \omega([X, CY] + [CX, Y] - \\ - C[X, Y] + S(X, Y) - P(X, Y)) \end{aligned} \tag{62.2}$$

$$\begin{aligned} \delta\Omega(X, Y, Z) := \delta_X\Omega(Y, Z) - \delta_Y\Omega(X, Z) + \delta_Z\Omega(X, Y) - \Omega(X, CY] + [CX, Y] - \\ - C[X, Y] + S(X, Y) - P(X, Y), Z) + \Omega([X, CZ] + [CX, Z] - C[X, Z] + \\ + S(X, Z) - P(X, Z), Y) - \Omega([Y, CZ] + [CY, Z] - C[Y, Z] + \\ + S(Y, Z) - P(Y, Z), X) \end{aligned} \tag{62.3}$$

where X, Y, Z are the local vector fields on M.

By the very definition of K_c and (57) we obtain easily, that $\delta\delta = 0$.

Let be given a quasi-connection $\varphi = (C_j^i, \Phi_{jk}^i, \Phi_{jkl}^i)$. We put $\omega_j^i := \Phi_{tj}^i dx^t$ $\omega_{jk}^i := \Phi_{tjk}^i dx^t$. By the definition of torsion tensor, curvature extensor and (62) we obtain

Theorem 15. *If φ is the quasi-connection on M, then the following structure equations*

$$\delta(dx^i) + \omega_t^i \wedge dx^t = \frac{1}{2} T_{jk}^i dx^j \wedge dx^k,$$

$$\delta\omega_j^i + \omega_t^i \wedge \omega_j^t = \frac{1}{2} R_{jkm}^i dx^k \wedge dx^m, \tag{63}$$

$$\delta\omega_{jk}^i + \omega_t^i \wedge \omega_{jk}^t + \omega_{tk}^i \wedge \omega_j^t + \omega_{jt}^i \wedge \omega_k^t = \frac{1}{2} R_{jktp}^i dx^t \wedge dx^p$$

are valid.

If we denote by Ω^i, Ω_j^i, Ω_{jk}^i the right hand members of (63), then differentiating the structure equations (63), we obtain the Bianchi identities

$$\begin{aligned} \delta\Omega^i - = \Omega_k^i \wedge \omega^k - \omega_k^i \wedge \Omega^k ; \quad \delta\Omega_j^i = \Omega_k^i \wedge \omega_j^k - \omega_k^i \wedge \Omega_j^k ; \\ \delta\Omega_{jk}^i = \Omega_t^i \wedge \omega_{jk}^t - \omega_{jt}^i \wedge \Omega_k - \omega_{tk}^i \wedge \Omega_j^t - \omega_t^i \wedge \Omega_{jk}^t + \Omega_{tk}^i \wedge \omega_j^t + \Omega_{jt}^i \wedge \omega_k^t . \end{aligned} \tag{64}$$

d) **The curvature of the distribution of a quasi-connection.** In this section the indices $a, b, c, \ldots$ run over the range $1, \ldots, m$ the indices $i, j, k, \ldots$ under $1, \ldots, n$ and the indices $A, B, C, \ldots$ under $m+1, \ldots, n$. We consider the vector fields $\overline{E}_j$, (23). As was seen by the corollary (26.2), the number of the linearly independent vector fields is equal to $m = \operatorname{rank} C$. We may assume, without loss of generality, that the matrix C_b^a is non-singular. Let us consider the matrix $A_b^i = \begin{bmatrix} C^{-1}{}_b^a \\ O \end{bmatrix}$ and the vector fields

$$\overline{F}_j := \overline{E}_r x^{-1}{}_j^r = C_j^i \frac{\partial}{\partial x^i} - x_p^k x_t^l \Phi_{jkl}^i \frac{\partial}{\partial x_{pt}^i} - x_p^k \Phi_{jk}^i \frac{\partial}{\partial x_p^i} - x_{pt}^k \Phi_{jk}^i \frac{\partial}{\partial x_{pt}^i} . \tag{65}$$

Let $G_b := \overline{F}_j A_b^j$. Then G_b has the form

$$G_b = A_b^j C_j^i \frac{\partial}{\partial x^i} - x_p^k A_b^j \Phi_{jk}^i \frac{\partial}{\partial x_p^i} - x_p^k x_r^l A_b^j \Phi_{jkl}^i \frac{\partial}{\partial x_{pr}^i} - x_{pr}^k A_b^j \Phi_{jk}^i \frac{\partial}{\partial x_{pr}^i} \tag{66.1}$$

or

$$G_b = \frac{\partial}{\partial x^b} + B_b^A \frac{\partial}{\partial x^A} - x_p^k \psi_{bk}^i \frac{\partial}{\partial x_p^i} - x_p^k x_r^l \psi_{bkl}^i \frac{\partial}{\partial x_{pr}^i} - x_{pl}^k \psi_{bk}^i \frac{\partial}{\partial x_{pl}^i} \tag{66.2}$$

where

$$C_j^i A_b^j = \begin{bmatrix} \delta_b^a \\ B_b^A \end{bmatrix}$$

and

$$\psi_{bk}^i := A_b^j \Phi_{jk}^i, \quad \psi_{bkl}^i := A_b^j \Phi_{jkl}^i$$

moreover the G_b form a set of linearly independent vector fields of $\operatorname{Lin}(\overline{E}_j)$ which is transversal to the fibres of $\overline{P}^2 M$. Consider the dual Pfaff system

$$\begin{aligned} \omega^A = dx^A - B_b^A dx^b, \quad \omega_j^i = dx_j^i + x_j^k \psi_{bk}^i dx^b, \\ \omega_{jk}^i = dx_{jk}^i + x_j^t x_k^l \psi_{btl}^i dx^b + x_j^t \psi_{bt}^i dx^b, \end{aligned} \tag{67}$$

that is $\ker \omega = \operatorname{Lin}(G_b)$. The forms ω^A measure the deviation of the quasi-connection from the Oproiu connection.

In order to calculate the curvature of (67), we shall briefly recall the general method (cf. [7], [18], [14]).

Given a submersion $\pi : Y \to X$, the x^i, y^α, $i, j, \ldots, = 1, \ldots, n$ $\alpha, \beta, \ldots = 1, \ldots, m$ denote some local fibre coordinates on Y, where $n = \dim X$, $m = \dim Y - n$.

Consider the subspace of $T_u Y$ given by

$$\omega = dy^\alpha - F_i^\alpha(x, y)\, dx^i = 0. \tag{68}$$

Let VY denotes the bundle of all vertical tangent vectors of Y. Then we define the curvature Ω of the system (68) by

$$\begin{aligned} \Omega : \wedge^2 \ker \omega \to VY \\ \Omega = (F_{i|j}^\alpha + F_j^\beta F_{i|\beta}^\alpha) dx^i \wedge dx^j, \end{aligned} \tag{69}$$

Coming back to our consideration, we calculate the forms for the system (67)

$$\begin{aligned} \Omega^A = (B_{b|e}^A + B_e^D B_{b|D}^A)\, dx^b \wedge dx^e, \\ \Omega_j^i = x_j^k (\psi_{bk|e}^i - \psi_{ek}^t \psi_{bt}^i)\, dx^b \wedge dx^e, \\ \Omega_{jk}^i = x_j^t x_k^l (\psi_{btl|e}^i - \psi_{etl}^r \psi_{br}^i)\, dx^b \wedge dx^e + x_{jk}^t (\psi_{bte}^i - \psi_{et}^r \psi_{br}^i)\, dx^b \wedge dx^e. \end{aligned} \tag{70}$$

We have from the above construction and Theorem of Frobenius (i.e. third form in [14] p. 120).

Theorem 16. *The system (67) is integrable iff the forms $\Omega^A, \Omega_j^i, \Omega_{jk}^i$ vanish for vectors belonging to $\operatorname{Lin}(G_b)$.*

It is easily seen that the distribution $\operatorname{Lin}(G_b)$ satisfies the definition of a partial connection (cf. [5]). Then we would use the theory of foliated bundles and their characteristic classes in the investigations of the quasi-connections, but it is outside of our present considerations.

§6. **Examples and final remarks** a) Given a space M of type α with $P=0$, and a quasi-connection (C^i_j, Φ^i_{jk}) in the bundle of linear frames P^1M, the torsion tensor is given by $T^a_{kl}=\Phi^a_{[kl]}+C^a_{[k|l]}$. Let (M, g) be a Riemannian manifold.

Definition. *A quasi-connection $\tilde\nabla$ is said to be Riemannian if the following conditions hold*

$$\delta_{z g}(X, Y)=g(\tilde\nabla_z X, Y)+g(X, \tilde\nabla_z Y) \quad T(X, Y)=0. \tag{71}$$

In [12] we have proved that such the connection is given by

$$\Phi^i_{kl}=\frac{1}{2}g^{it}(C^a_l g_{kt|a}+C^a_k g_{tl|a}-C^a_t g_{kl|a}+g_{at}C^a_{[k|l]}+g_{ak}C^a_{[t|l]}+g_{al}C^a_{[k|t]}). \tag{72}$$

b) In this section we shall give an example of quassi-connection induced on hypersurfaces in Euclidean space E^{n+1}.

We follow notations of §4, i.e. $a, b \ldots =1, \ldots, n+1$, $i, j, k \ldots =1, \ldots, n$. Let C^a_b be an arbitrary singular $n+1\times n+1$ matrix. Consider the quasi-connection C^a_b, $\Phi^a_{bc}=0$, $\Phi^a_{bcd}=0$ on the trivial enveloping manifold $M=E^{n+1}$. Consider an orientable hypersurface S and let $f: R^n\to B\to S\subset E^{n+1}$ be a coordinate patch.

If $f^a{}_{|j}$ denote the basis of tangent plane to S then unique vector field m^a perpendicular to $\mathrm{Lin}\,(f^a{}_{|j})$ gives the framing of S.

Thoerem 17. *If $\varphi=(C^a_b, \Phi^a_{bc}, \Phi^a_{bcd})$ is the above mentioned quasi-connection on E^{n+1}, then there exist the unique quasi-connection $(C^i_j, \Phi^p_{ki}, \Phi^p_{kij})$ induced by φ on the naturally framed hypersurface S in E^{n+1}. This quasi-connection is given by*

$$C^i_j=f^b_j C^a_b \tilde f^i_a, \quad \Phi^p_{ki}=C^r_k\{^p_{ir}\}, \quad \Phi^r_{kij}=C^p_k(\{^r_{ip}\}_{|j}-\{^l_{i,p}\}\{^r_{l,j}\}). \tag{73}$$

where $\{^p_{ir}\}$ denote the Christoffel symbols of S.

Proof. The scalar product on S is given by

$$g_{ij}=\sum_a f^a_{|i} f^a_{|j}. \tag{74}$$

Taking into account (36.1), (36.2) and (74), we obtain

$$\tilde f^i_a=g^{ik} f^a_k. \tag{75}$$

Thus by virtue of (41.2) and (75) we have

$$\Phi^p_{ki}=C^j_k \sum_a f^a_{i,j} g^{ps} f^a_s$$

Using the fact, that $f^a_{i,j}=f^a_{j,i}$ we obtain

$$\begin{aligned}\Phi^p_{si} &=\frac{1}{2}C^j_s g^{pk}\sum_a (f^a_{i|j} f^a_k+f^a_{j,i} f^a_k)=\frac{1}{2}C^j_s g^{pk}\sum_a ((f^a_i f^a_k)_{|j}-f^a_i f^a_{k,j}+\\ &+(f^a_j f^a_k)_{|i}-f^a_j f^a_{k,i})=\frac{1}{2}C^j_s g^{pk}(g_{ik|j}+g_{jk|i}-\sum_a (f^a_i f^a_{j,k}+f^a_j f^a_{i,k}))=\\ &=\frac{1}{2}C^j_s g^{pk}(g_{ik|j}+g_{jk|i}-g_{ij|k})=C^j_s\{^p_{ij}\}.\end{aligned} \tag{76}$$

We have from (41.3)

$$\Phi^r_{kij}=C^p_k f^a_{i,j,p}\tilde f^r_a-C^p_k f^b_{i,p}\tilde f^l_b f^a_{l,j}\tilde f^r_a-C^p_k f^b_{j,p}\tilde f^l_b f^a_{l,i}\tilde f^r_a.$$

Taking account of (76) and the fact that $f^a_{i,j,p}=f^a_{i,p,j}$ we obtain

$$\Phi^r_{kij}=C^p_k(\{^r_{ip}\}_{|j}-\{^l_{i,p}\}\{^r_{l,j}\}). \tag{77}$$

It is remarkable that the expression $\{^r_{ip}\}_{|j}-\{^l_{i,p}\}\{^r_{l,j}\}$ is the prolongation of the connection $\{^r_{ip}\}$ in the sense of Oproiu [17].

c) In [13] we considered the problem of existence of a quasi-connection in the parabolic Podkowyrin space.

d) A. J. A. Morgan has applied the Oproiu's results (cf. [11]) in the theoretical mechanics. We see the possibility of similar applications of the present paper.

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Received 15. VI. 1980

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Analele ştiinţifice ale Universităţii „Al.I.Cuza" din Iaşi
Tomul XXVII, s.I a, 1981, f.2

APPLICATIONS QUASI-CONFORMES SUR DES PRODUITS FINIS D'ESPACES NORMÉS

PAR

SILVIU CRĂCIUNAŞ

Dans cet ouvrage, on établit, à partir d'un n-uple d'homéomorphismes sur des espaces normés réels quelques résultats concernant la liaison entre la quasi-conformité des applications composantes dans le sens de la définition métrique et de la définition avec des dérivées scalaires) et la quasi-conformité de l'application produit.

Soient E_i, $i=\overline{1, n}$ des espaces normés réels et $E=\prod_{i=1}^{n} E_i$ l'espace produit sur lequel on considère la norme définie par

$$x\in E\to \|x\|=\max_{1\leq i\leq n} \|x_i\| \qquad \text{où } x_i\in E_i \quad \text{et} \quad x=(x_1, \dots, x_n),$$

et où pour ne pas compliquer la notation on a utilisé $\|\cdot\|$ pour toutes les normes. Sur E dans la topologie produit compatible avec la norme introduite, on peut considérer comme une base, les ensembles ouverts, de la forme $D=\prod_{i=1}^{n} D_i$ où D_i sont des ensembles ouverts de E_i.

On note aussi

$$f: D=\prod_{i=1}^{n} D_i\to D^*=\prod_{i=1}^{n} D_i^*$$

l'application définie par

$$x\in D\to f(x)=(f_1(x_1), \dots, f_n(x_n)),$$

où $f_i: D_i\to D_i^*$ est un homéomorphisme de l'ouvert $D_i\in E_i$ sur l'ouvert $D_i^*\in E_i$ $(i=\overline{1, n})$. Il est aisé de voir:

Proposition 1. *L'application*

$$f: D=\prod_{i=1}^{n} D_i\to D^*=\prod_{i=1}^{n} D_i^*$$

est un homéomorphisme sur D.

Soit maintenant l'application f de la proposition précédente; on considère la dilatation locale linéaire et les dérivées scalaires définies respectivement par les relations