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ALMOST SEMI-INVARIANT SUBMANIFOLDS IN SASAKIAN SPACE FORMS

BY

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1. Introduction. Let $\tilde{M}$ be a $(2n+1)$-dimensional almost contact metric manifold. We denote by (φ, ξ, η, g) the almost contact metric structure on $\tilde{M}$, where φ is a tensor field of type (1, 1) on $\tilde{M}$, ξ is a vector field η is a 1-form and g is the Riemannian metric on $\tilde{M}$. As it is known, these tensor fields are related by

(1.1) $$\varphi^2X=-X+\eta(X)\xi\ ;\ \varphi\xi=0\ ;\ \eta(\varphi X)=0\ ;\ \eta(\xi)=1 \text{ and}$$

(1.2) $$g(\varphi X, \varphi Y)=g(X, Y)-\eta(Y)\eta(X),$$

for all vector fields X, Y tangent to $\tilde{M}$.

When φ satisfies

(1.3) $$(\tilde{\nabla}_X\varphi)Y=g(X, Y)\xi-\eta(Y)X$$

for all X, Y tangent to $\tilde{M}$, where $\tilde{\nabla}$ is the Riemannian connection on $\tilde{M}$, we say that $\tilde{M}$ is a Sasakian manifold. Also, from (1.3) we have

(1.4) $$\tilde{\nabla}_X\xi=-\varphi X \text{ for any } X \text{ tangent to } \tilde{M}.$$

Now, let M be an m-dimensional Riemannian manifold isometrically immersed in $\tilde{M}$ and suppose the structure vector field ξ is tangent to M. For each vector field X tangent to M we put

(1.5) $$\varphi X=bX+cX,$$

where bX is the tangential part of φX and cX is the normal part of φX.

Definition. *The submanifold M is called an almost semi-invariant submanifold of $\tilde{M}$ if its tangent bundle TM has the decomposition $TM=D\oplus D^{\perp}\oplus\tilde{D}\oplus\{\xi\}$ where:*

(i) *$\{\xi\}$ is the distribution spanned on M by the vector field ξ,*

(ii) *D is an invariant distribution on M, that is, $\varphi(D_x)=D_x$ for each $x\in M$*

(iii) *$D^{\perp}$ is an anti-invariant distribution on M, that is, $\varphi(D^{\perp}_x)\subset T_xM^{\perp}$ for any $x\in M$, where $T_xM^{\perp}$ is the space normal to M in x,*

(iv) *$\tilde{D}$ is neither an invariant nor an anti-invariant distribution on M, that is, $bX_x\neq 0$ and $cX_x\neq 0$ for each $x\in M$ and $X_x\in\tilde{D}_x$.*

Then, there exist the following particular cases of almost semi-invariant submanifolds:

1. *semi-invariant submanifolds when* $\tilde{D}=\{0\}$, [2].
2. *invariant submanifolds* when $\tilde{D}=D^{\perp}=\{0\}$, [4].
3. *anti-invariant submanifolds* when $D=\tilde{D}=\{0\}$, [5].
4. *almost invariant submanifolds* when $D^{\perp}=\{0\}$, [1].
5. *almost anti-invariant submanifolds* when $D=\{0\}$, [1].
6. *pseudo-invariant submanifolds* when $D=D^{\perp}=\{0\}$, [1].

Fundamental properties of almost semi-invariant submanifolds in a Sasakian manifold have been obtained in [1] and [6].

The purpose of the present paper is to obtain some results on the almost semi-invariant submanifolds in a Sasakian space form.

2. Preliminaries. Let M be an almost semi-invariant submanifold in a Sasakian manifold $\tilde{M}$. We denote by g both Riemannian metrics on $\tilde{M}$ and M. The projection morphisms of TM to the distributions D, $D^{\perp}$ and $\tilde{D}$ are denoted respectively by P, Q and L.

For any vector bundle H on M we denote by $\Gamma(H)$ the module of all differentiable sections of H. Then, for each $X\in\Gamma(TM)$ we have

$$X=PX+QX+LX+\eta(X)\xi, \quad \text{where} \quad \eta(X)=g(X,\xi). \tag{2.1}$$

We take $X\in\Gamma(\tilde{D})$ and note that $bX\neq 0$ and $cX\neq 0$. Thus c defines a vector subbundle $c\tilde{D}: x\to c\tilde{D}_x$ of $T_xM^{\perp}$.

For any $N\in\Gamma(TM^{\perp})$ we put

$$\varphi N=tN+fN, \tag{2.2}$$

where tN and fN are the tangential and normal components of φN, respectively. Then we have.

$$g(\varphi D^{\perp}, c\tilde{D})=0. \tag{2.3}$$

Next, we denote by ν the orthogonal complementary vector bundle to $\varphi D^{\perp}\oplus c\tilde{D}$ on $TM^{\perp}$. Thus that the normal bundle to M has the decomposition

$$TM^{\perp}=\varphi D^{\perp}\oplus c\tilde{D}\oplus\nu. \tag{2.4}$$

Lemma 2.1, [1]. *The morphisms t and f satisfy*

$$t(TM^{\perp})=D^{\perp}\oplus\tilde{D}, \quad t(\varphi D^{\perp})=D^{\perp}, \quad t(c\tilde{D})=\tilde{D} \text{ and } f(c\tilde{D})=c\tilde{D}). \tag{2.5}$$

Now, we denote by ∇ the Riemannian connection on M with respect to the metric g and by $\nabla^{\perp}$ the linear connection induced by $\tilde{\nabla}$ on the normal bundle $TM^{\perp}$. Then, the Gauss and Weingarten formulas for M are given respectively by

$$\tilde{\nabla}_X Y=\nabla_X Y+h(X,Y), \quad \text{and} \tag{2.6}$$

$$\nabla_X N=-A_N X+\nabla^{\perp}_X N, \tag{2.7}$$

for any $X, Y\in\Gamma(TM)$ and $N\in\Gamma(TM)^{\perp}$), where h is the second fundamental form of M and A_N is the fundamental tensor of Weingarten with respect to the normal section N. These tensor fields are related by

$$g(h(X,Y),N)=g(A_N X,Y), \tag{2.8}$$

for all $X, Y\in\Gamma(TM)$ and $N\in\Gamma(TM^{\perp})$.

For any $X, Y\in\Gamma(TM)$ we denote:

$$u(X,Y)=\nabla_X\varphi PY+\nabla_X bLY-A_{\varphi QY}X-A_{cLY}X. \tag{2.9}$$

Then we have

Lemma 2.2. [1]. *Let M be an almost semi-invariant submanifold of the Sasakian manifold $\tilde{M}$. Then we have*

$$P(u(X,Y))=\varphi P(\nabla_X Y)-\eta(Y)PX, \tag{2.10}$$

$$Q(u(X,Y))=Qth(X,Y)-\eta(Y)QX, \tag{2.11}$$

$$L(u(X,Y))=bL(\nabla_X Y)+Lth(X,Y)-\eta(Y)LX, \tag{2.12}$$

$$\eta(u(X,Y))=g(\varphi X,\varphi Y), \tag{2.13}$$

$$\begin{gathered} h(X,\varphi PY)+h(X,bLY)+\nabla^{\perp}_X\varphi QY+\nabla^{\perp}_X cLY=\varphi Q\nabla_X Y+cL\nabla_X Y+ \\ fh(X,Y), \textit{ for all } X, Y\in\Gamma(TM) \textit{ and} \end{gathered} \tag{2.14}$$

$$\nabla_X\xi=-\varphi X;\ h(X,\xi)=0;\ \nabla_\xi X\in\Gamma(D), \textit{ for any } X\in\Gamma(D), \tag{2.15}$$

$$\nabla_Y\xi=0;\ h(Y,\xi)=-\varphi Y;\ \nabla_\xi Y\in\Gamma(D^{\perp}), \textit{ for all } Y\in\Gamma(D^{\perp}), \tag{2.16}$$

$$\nabla_Z\xi=-bZ;\ h(Z,\xi)=-cZ;\ \nabla_\xi Z\in\Gamma(\tilde{D}), \textit{ for all } Z\in\Gamma(\tilde{D}), \tag{2.17}$$

$$\nabla_\xi\xi=0;\ h(\xi,\xi)=0. \tag{2.18}$$

Now, suppose $\tilde{M}(k)$ is a Sasakian space form of constant φ-sectional curvature k and M be an almost semi-invariant submanifold of $\tilde{M}(k)$.

The curvature tensor $\tilde{R}$ of $M(k)$ is given by

$$\begin{gathered} \tilde{R}(X,Y)Z=\frac{k+3}{4}\{g(Y,Z)X-g(X,Z)Y\}+\frac{k-1}{4}\{\eta(X)\eta(Z)Y- \\ -\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi+g(Z,\varphi Y)\varphi X- \\ -g(Z,\varphi X)\varphi Y+2g(X,\varphi Y)\varphi Z\}, \end{gathered} \tag{2.19}$$

for any X, Y, Z tangent to $\tilde{M}(k)$.

Then, the equation of Codazzi is given by

$$\begin{gathered} (D_X h)(Y,Z)-(D_Y h)(X,Z)=\frac{k-1}{4}\{g(Z,bY)cX-g(Z,bX)cY+ \\ +2g(X,bY)cZ\}, \end{gathered} \tag{2.20}$$

for all $X, Y, Z\in\Gamma(TM)$, where $(D_X h)(Y,Z)$ is defined by

(2.21) $(D_X h)(Y, Z) = \nabla^{\perp}_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$.

3. Integrability. In this paragraph, we recall some fundamental results on the integrability of distributions involved in the definition of an almost semi-invariant submanifold from [1].

Let M be an almost semi-invariant submanifold in the Sasakian manifold $\tilde{M}$.

Theorem 3.1, [1]. *The distributions D, $\tilde{D}$, $D \oplus D^{\perp}$, $\tilde{D} \oplus D^{\perp}$, $D \oplus \tilde{D}$ and $D \oplus D^{\perp} \tilde{D}$ are never integrable.*

Theorem 3.2, [1]. (i) *The distribution $D \oplus \{\xi\}$ is integrable if and only if we have*

(3.1) $$h(X, \varphi Y) = h(\varphi X, Y), \text{ for any } X, Y \in \Gamma(D).$$

(ii) *The distribution $D \oplus \{\xi\}$ is integrable and its leaves are totally geodesic immersed in M if and only if we have*

(3.2) $$h(X, Y) \in \Gamma(\nu), \text{ for any } X, Y \in \Gamma(D), \text{ and}$$

(3.3) $$\nabla_X Z \in \Gamma(D^{\perp} \oplus \tilde{D}) \text{ for all } X \in \Gamma(D) \text{ and } Z \in \Gamma(\tilde{D}).$$

Theorem 3.3, [1]. (i) *The distribution $D^{\perp}$ is always integrable.*

(ii) *Any leaf of distribution $D^{\perp}$ is totally geodesic immersed in M if and only if we have*

(3.4) $$h(X, bZ) + \nabla^{\perp}_X cZ \in \Gamma(c\tilde{D} \oplus \nu),$$

for all $X \in \Gamma(D^{\perp})$ and $Z \in \Gamma(D \oplus \tilde{D} \oplus \{\xi\})$.

Theorem 3.4, [1]. (i) *The distribution $D \oplus \tilde{D} \oplus \{\xi\}$ is integrable if and only if we have*

(3.5) $$h(Y, bX) - h(X, bY) + \nabla^{\perp}_Y cX - \nabla^{\perp}_X cY \in \Gamma(c\tilde{D} \oplus \nu),$$

for any X, $Y \in \Gamma(D \oplus \tilde{D} \oplus \{\xi\})$.

(ii) *The distribution $D \oplus \tilde{D} \oplus \{\xi\}$ is integrable and its leaves are totally geodesic immersed in M if and only if we have*

(3.6) $$h(X, bY) + \nabla^{\perp}_X cY \in \Gamma(c\tilde{D} \oplus \nu),$$

for any $X, Y \in \Gamma(D \oplus \tilde{D} \oplus \{\xi\})$.

From Theorems 3.3 and 3.4 we have

Theorem 3.5. *Let M be an almost semi-invariant submanifold of the Sasakian manifold $\tilde{M}$. Then M is locally the Riemannian product $M_1 \times M_2$, where M_1 is a leaf of $D \oplus \tilde{D} \oplus \{\xi\}$ and M_2 is a leaf of $D^{\perp}$ if and only if we have*

(3.7) $$h(X, bY) + \nabla^{\perp}_X cY \in \Gamma(c\tilde{D} \oplus \nu)$$

for all $X \in \Gamma(TM)$ and $Y \in \Gamma(D \oplus \tilde{D} \oplus \{\xi\})$.

4. Almost semi-invariant submanifolds in Sasakian space forms. Let M be an almost semi-invariant submanifold in a Sasakian manifold $\tilde{M}$. We assume that:

(4.1) $$h(X, Z) \in \Gamma(\nu), \text{ for all } X \in \Gamma(D) \text{ and } Z \in \Gamma(\tilde{D} \oplus D^{\perp}).$$

Lemma 4.1. *If the distribution $D \oplus \{\xi\}$ is integrable and the condition (4.1) holds, then we have*

(4.2) $$\tilde{R}(X, \varphi X; Z, cZ) - g(\nabla^{\perp}_X h(\varphi X, Z) - \nabla^{\perp}_{\varphi X} h(X, Z), cZ) = \|A_{cZ} X\|^2 + \|b\nabla_X Z\|^2 - \|\nabla_X bZ\|^2 + 2\|X\|^2 \|cZ\|^2 \text{ for all } X \in \Gamma(D) \text{ and } Z \in \Gamma(\tilde{D} \oplus D^{\perp}).$$

Proof. By using the hypothesis, (2.8) and the equation of Codazzi, for any X, $Y \in \Gamma(D)$ and Z, $W \in \Gamma(D^{\perp} \oplus \tilde{D})$ we obtain

(4.3) $$\tilde{R}(X, Y; Z, cW) = g(\nabla^{\perp}_X h(Y, Z), cW) - g(\nabla^{\perp}_Y h(X, Z), cW) - \eta([X, Y]) g(h(\xi, Z), cW) + g(A_{cW} X, \nabla_Y Z) - g(A_{cW} Y, \nabla_X Z).$$

From (2.15), we have

(4.4) $$\eta([X, Y]) = g(Y, \varphi X) - g(X, \varphi Y) = -2g(X, \varphi Y)$$

for all X, $Y \in \Gamma(D)$.

From (4.1), it follows

(4.5) $$A_{cW} X \in \Gamma(D) \text{ for any } X \in \Gamma(D) \text{ and } W \in \Gamma(\tilde{D} \oplus D^{\perp}).$$

By using (1.2), (2.6), (2.7) and (2.15) we have

(4.6) $$g(A_{cW} Y, \nabla_X Z) = g(A_{cW} Y, \tilde{\nabla}_X Z) = -g(A_{cW} Y, \varphi \tilde{\nabla}_X \varphi Z)$$

for any $Y \in \Gamma(D)$ and Z, $W \in \Gamma(\tilde{D} \oplus D^{\perp})$. Substituting (4.4) and (4.6) in (4.3) and taking account of (2.16) and (2.17), we obtain

(4.7) $$\tilde{R}(X, Y; Z, cW) = g(\nabla^{\perp}_X h(Y, Z) - \nabla^{\perp}_Y h(X, Z), cW) + g(A_{cW} Y, \varphi \tilde{\nabla}_X \varphi Z) - g(A_{cW} X, \varphi \tilde{\nabla}_Y \varphi Z) - 2g(X, \varphi Y) g(cZ, cW),$$

for any X, $Y \in \Gamma(D)$ and Z, $W \in \Gamma(\tilde{D} \oplus D^{\perp})$.

By using (1.5), (2.6), (2.7) and (4.5) in (4.7) we get

(4.8) $$\tilde{R}(X, Y; Z, cW) - g(\nabla^{\perp}_X h(Y, Z) - \nabla^{\perp}_Y h(X, Z), cW) = g(\varphi A_{cW} X, \nabla_Y bZ) - g(\varphi A_{cW} Y, \nabla_X bZ) + g(A_{cW} X, A_{cZ} Y) - g(A_{cW} Y, A_{cZ} X) - 2g(X, \varphi Y) g(cZ, cW),$$

for all X, $Y \in \Gamma(D)$ and Z, $W \in \Gamma(\tilde{D} \oplus D^{\perp})$. Since the distribution $D \oplus \{\xi\}$ is integrable, from (4.5) and (3.1) we obtain

(4.9) $$A_{cW} \varphi X = -\varphi A_{cW} X \text{ for all } X \in \Gamma(D) \text{ and } W \in \Gamma(\tilde{D} \oplus D^{\perp}).$$

We take $Y = \varphi X \in \Gamma(D)$ and $Z = W \in \Gamma(\tilde{D} \oplus D^{\perp})$. Then from (4.8), by using (4.9) we get

(4.10) $$\tilde{R}(X, \varphi X; Z, cZ) - g(\nabla^{\perp}_X h(\varphi X, Z) - \nabla^{\perp}_{\varphi X} h(X, Z), cZ) = g(\varphi A_{cZ} X, \nabla_{\varphi X} bZ) - g(A_{cZ} X, \nabla_X bZ) + 2\|A_{cZ} X\|^2 + 2\|X\|^2 \|cZ\|^2.$$

On the other hand, since $\varphi A_{cZ}X$ and bZ are orthogonal, we have

$$
\begin{aligned}
(4.11)\quad & g(\varphi A_{cZ}X, \nabla_{\varphi X} bZ) = -g(\tilde{\nabla}_{\varphi X}(\varphi A_{cZ}X), bZ) = -g(\varphi\tilde{\nabla}_{\varphi X}(A_{cZ}X), bZ) = \\
& = g(\tilde{\nabla}_{\varphi X}(A_{cZ}X), \varphi bZ) = g([\varphi X, A_{cZ}X] + \tilde{\nabla}_{A_{cZ}X}\varphi X, \varphi bZ) = g(\tilde{\nabla}_{A_{cZ}X}\varphi X, \varphi bZ) = \\
& = g(\varphi\tilde{\nabla}_{A_{cZ}X}X, \varphi bZ) = g(\tilde{\nabla}_{A_{cZ}X}X, bZ) = g([A_{cZ}X, X] + \tilde{\nabla}_X(A_{cZ}X), bZ) = \\
& = g(\tilde{\nabla}_X(A_{cZ}X), bZ) = g(\nabla_X(A_{cZ}X), bZ) = -g(A_{cZ}X, \nabla_X bZ)
\end{aligned}
$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(\tilde{D} + D^{\perp})$. Substituting (4.11) in (4.10) we get

$$
\begin{aligned}
(4.12)\quad & \tilde{R}(X, \varphi X; Z, cZ) - g(\nabla^{\perp}_X h(\varphi X, Z) - \nabla^{\perp}_{\varphi X} h(X, Z), cZ) = 2\|A_{cZ}X\|^2 + \\
& + 2\|X\|^2\|cZ\|^2 - 2g(A_{cZ}X, \nabla_X bZ),
\end{aligned}
$$

for all $X \in \Gamma(D)$ and $Z \in \Gamma(\tilde{D} + D^{\perp})$. Now, from (2.10) — (2.13) and (4.1) we have

$$(4.13)\qquad \nabla_X bZ = A_{cZ}X + b\nabla_X Z \quad \text{for all} \quad X \in \Gamma(D) \text{ and } Z \in \Gamma(\tilde{D} \oplus D^{\perp}).$$

So by (4.12) and (4.13), we get

$$
\begin{aligned}
(4.14)\quad & \tilde{R}(X, \varphi X; Z, cZ) - g(\nabla^{\perp}_X h(\varphi X, Z) - \nabla^{\perp}_{\varphi X} h(X, Z), cZ) = 2\|A_{cZ}X\|^2 + \\
& + 2\|X\|^2\|cZ\|^2 - 2\|\nabla_X bZ\|^2 + 2g(b\nabla_X Z, \nabla_X bZ).
\end{aligned}
$$

From (4.13) we also have

$$(4.15)\qquad \|A_{cZ}X\|^2 = \|\nabla_X bZ\|^2 + \|b\nabla_X Z\|^2 - 2g(\nabla_X bZ, b\nabla_X Z).$$

Then (4.14) becomes (4.2) and the proof of Lemma 4.1 is complete.

Now, suppose that $\tilde{M}$ is a Sasakian space form $\tilde{M}(k)$. Then, by (2.19) we find:

$$(4.16)\qquad \tilde{R}(X, \varphi X; Z, cZ) = -\frac{k-1}{2}\|X^2\|\,\|cZ\|^2,$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(\tilde{D} \oplus D^{\perp})$.

We make the following

Definitions. *An almost semi-invariant submanifold M in a Sasakian manifold $\tilde{M}$ is said to be:*

(i) *$(D, \tilde{D} \oplus D^{\perp})$-geodesic if $h(X, Z) = 0$ for all $X \in \Gamma(D)$ and $Z \in \Gamma(\tilde{D} \oplus D^{\perp})$.*

(ii) *$(D \oplus \tilde{D}, D^{\perp})$-geodesic if $h(X, Y) = 0$ for all $X \in \Gamma(D \oplus \tilde{D})$ and $Y \in \Gamma(D^{\perp})$*

(iii) *a proper almost semi-invariant submanifold if $D \neq \{0\}$, $D^{\perp} \neq \{0\}$ and $\tilde{D} \neq \{0\}$.*

Then from (4.16) and Lemma 4.1 we obtain the following

Lemma 4.2. *Let M be a $(D, \tilde{D} \oplus D^{\perp})$-geodesic almost semi-invariant submanifold in a Sasakian space form $\tilde{M}(k)$. If the distribution $D \oplus \{\xi\}$ is integrable, then we have*

$$(4.17)\qquad \|\nabla_X bZ\|^2 = \|A_{cZ}X\|^2 + \|b\nabla_X Z\|^2 + \frac{k+3}{2}\|X\|^2\|cZ\|^2,$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(\tilde{D} \oplus D^{\perp})$.

Theorem 4.1. *Let M be a $(D, \tilde{D} \oplus D^{\perp})$-geodesic almost semi-invariant submanifold in a Sasakian space form $\tilde{M}(k)$, $k \neq -3$. If the distribution $D \oplus \{\xi\}$ is integrable and its leaves are totally geodesic immersed in M, then M is either an invariant submanifold, or an almost anti-invariant submanifold of $\tilde{M}(k)$.*

Proof. From (4.5) and (3.2) we have

$$(4.18)\qquad A_{cZ}X = 0 \text{ for all } X \in \Gamma(D) \text{ and } Z \in \Gamma(\tilde{D} \oplus D^{\perp}).$$

Then from (4.13) we obtain

$$(4.19)\qquad \nabla_X bZ = b\nabla_X Z \text{ for any } X \in \Gamma(D) \text{ and } Z \in \Gamma(\tilde{D} \oplus D^{\perp}).$$

By using (4.18) and (4.19) in (4.17) we get

$$(4.20)\qquad \frac{k+3}{2}\|X\|^2\|cZ\|^2 = 0$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(\tilde{D} \oplus D^{\perp})$. Thus, the assertion of Theorem 4.1 follows from (4.20).

Definition. *Let M be an almost semi-invariant submanifold in a Sasakian manifold $\tilde{M}$. We say the distribution $\tilde{D}$ is D-parallel if we have*

$$(4.21)\qquad \nabla_X Y \in \Gamma(\tilde{D}) \text{ for all } X \in \Gamma(D) \text{ and } Y \in \Gamma(\tilde{D}).$$

Theorem 4.2. *Let M be a $(D, \tilde{D} \oplus D^{\perp})$-geodesic almost semi-invariant submanifold in a Sasakian space form $\tilde{M}(k)$, $k \neq -3$. If the distribution $D \oplus \{\xi\}$ is integrable and the distribution $\tilde{D}$ is D-parallel, then we have either M is an almost anti-invariant submanifold of $\tilde{M}(k)$, or M is a semi-invariant submanifold of $\tilde{M}(k)$. If M is a proper semi-invariant submanifold that is, $D \neq \{0\}$ and $D^{\perp} \neq \{0\}$, then we have $k < -3$.*

Proof. From (4.5), (4.13) and (4.21) we obtain

$$(4.22)\qquad A_{cY}X = 0 \text{ and } \nabla_X bY = b\nabla_X Y \text{ for all } X \in \Gamma(D) \text{ and } Y \in \Gamma(\tilde{D})$$

and

$$(4.23)\qquad A_{cZ}X = -b\nabla_X Z \text{ and } \nabla_X bZ = 0 \text{ for all } X \in \Gamma(D) \text{ and } Z \in \Gamma(D^{\perp}).$$

Then (4.17) becomes

$$(4.24)\qquad \frac{k+3}{2}\|X\|^2\|cY\|^2 = 0 \text{ for any } X \in \Gamma(D) \text{ and } Y \in \Gamma(\tilde{D})$$

and respectively

$$(4.25)\qquad 2\|A_{cZ}X\|^2 + \frac{k+3}{2}\|X\|^2\,\|cZ\|^2 = 0 \text{ for all } X \in \Gamma(D) \text{ and } Z \in \Gamma(D^{\perp}).$$

Since $k \neq -3$, from (4.24) it follows either $\|X\|=0$ for each $X \in \Gamma(D)$, that is, M is an almost anti-invariant submanifold of $\tilde{M}(k)$, or $\|cY\|=0$ for all $Y \in \Gamma(\tilde{D})$, that is, M is a semi-invariant submanifold of $\tilde{M}(k)$. Also, if M is a proper semi-invariant submanifold of $\tilde{M}(k)$, then (4.25) implies $k < -3$ and the proof is complete.

Theorem 4.3. *Let M be a $(D, \tilde{D}\oplus D^{\perp})$-geodesic almost semi-invariant submanifold in a Sasakian space form $\tilde{M}(k)$, $k > -3$. If the distribution D is nonzero and $D\oplus\{\xi\}$ is integrable then M is an almost invariant submanifold of $\tilde{M}$ (k).*

Proof. Under the hypothesis, the equation (4.17) holds. If the distribution $D^{\perp}\neq\{0\}$, we take a nonzero vector field $Z \in \Gamma(D^{\perp})$. Then $bZ=0$ and $cZ=\varphi Z$ and (4.17) implies

$$(4.26)\qquad \|A_{\varphi Z}X\|^2 + \|b\nabla_X Z\|^2 + \frac{k+3}{2}\|X\|^2\|\varphi Z\|^2 = 0$$

for all $X \in \Gamma(D)$. From (4.26), it follows $k \leqslant -3$. This is a contradiction and the proof is complete.

Theorem 4.4. *Let M be a $(\tilde{D}, D\oplus D^{\perp})$-geodesic proper almost semi-invariant submanifold in $R^{2n+1}(-3)$. If the distribution $D \oplus \{\xi\}$ is integrable and its leaves are totally geodesic immersed in M and the condition*

$$(4.27)\qquad LA_{\varphi Z}X + Lth(X, Z) = 0 \quad \textit{for all} \quad X \in \Gamma(\tilde{D}+D^{\perp}) \quad \textit{and} \quad Z \in \Gamma(D^{\perp})$$

holds, then $M = M_1 \times M_2$ where M_1 is an almost invariant submanifold of $R^{2n+1}(-3)$ and M_2 is an anti-invariant submanifold of $R^{2n+1}(-3)$ such that the structure vector field ξ is normal to M_2.

Proof. First, from (3.2) and (4.5), it follows

$$(4.28)\qquad A_{cZ}X = 0 \text{ for all } X \in \Gamma(D) \text{ and } Z \in \Gamma(\tilde{D}\oplus D^{\perp}).$$

Then (4.13) implies

$$(4.29)\qquad \nabla_X bZ = b\nabla_X Z \text{ for any } X \in \Gamma(D) \text{ and } Z \in \Gamma(\tilde{D}\oplus D^{\perp}).$$

Since $\eta(\nabla_X Z)=0$ for all $X \in \Gamma(D)$ and $Z \in \Gamma(\tilde{D}\oplus D^{\perp})$, from (2.16) and (4.29) follows

$$(4.30)\qquad \nabla_X Z \in \Gamma(D^{\perp}) \text{ for any } X \in \Gamma(D\oplus\{\xi\}) \text{ and } Z \in \Gamma(D^{\perp}).$$

On the other hand, from (2.9), (2.10), (2.12) and (4.27) we obtain

$$(4.31)\qquad b\nabla_X Z = -LA_{\varphi Z}X - Lth(X, Z) = 0$$

for all $X \in \Gamma(\tilde{D}\oplus D^{\perp})$ and $Z \in \Gamma(D^{\perp})$. Then, from (2.15) – (2.17) and (4.31) we get

$$(4.32)\qquad \nabla_X Z \in \Gamma(D^{\perp}), \text{ for all } X \in \Gamma(\tilde{D}\oplus D^{\perp}) \text{ and } Z \in \Gamma(D^{\perp})$$

Thus, from (4.30) and (4.32) we have

$$(4.33)\qquad \nabla_X Z \in \Gamma(D^{\perp}) \text{ for all } X \in \Gamma(TM) \text{ and } Z \in \Gamma(D^{\perp}),$$

that is, the distribution $D^{\perp}$ is parallel. From (4.33) we also have

$$(4.34)\qquad \nabla_X Y \in \Gamma(D\oplus\tilde{D}\oplus\{\xi\}) \text{ for any } X \in \Gamma(TM) \text{ and } Y \in \Gamma(D\oplus\tilde{D}\oplus\{\xi\})$$

that is, the distribution $D\oplus\tilde{D}\oplus\{\xi\}$ is parallel too.

Now, from (4.34) and (2.14) we obtain

$$(4.35)\qquad h(X, bY) + \nabla^{\perp}_X cY = cL\nabla_X Y + fh(X, Y) \in \Gamma(c\tilde{D}\oplus\nu)$$

for all $X \in \Gamma(TM)$ and $Y \in \Gamma(D\oplus\tilde{D}\oplus\{\xi\})$. Then, the assertion of Theorem 4.4 follows from Theorem 3.5.

Remark. By $R^{2n+1}(-3)$ we mean the Sasakian space form with constant φ-sectional curvature $k=-3$ with standard Sasakian structure in the Euclidean space R^{2n+1}(cf. [3], p. 99).

Definition. *Let M be an almost semi-invariant submanifold in a Sasakian manifold $\tilde{M}$. We say that the subbundle $\varphi D^{\perp}$ of $TM^{\perp}$ is $D\oplus\tilde{D}\oplus\{\xi\}$-parallel if we have*

$$(4.36)\qquad \nabla^{\perp}_X N \in \Gamma(\varphi D^{\perp}) \quad \textit{for any} \quad X \in \Gamma(D\oplus\tilde{D}\oplus\{\xi\}) \quad \textit{and} \quad N \in \Gamma(\varphi D^{\perp}).$$

Lemma 4.3. *Let M be a $(D\oplus\tilde{D}, D^{\perp})$-geodesic almost semi-invariant submanifold in a Sasakian manifold $\tilde{M}$. If the distribution $D\oplus\tilde{D}\oplus\{\xi\}$ is integrable and the subbundle $\varphi D^{\perp}$ is $D\oplus\tilde{D}\oplus\{\xi\}$-parallel then we have*

$$(4.37)\qquad A_N bX + bA_N X = 0 \quad \textit{for any} \quad X \in \Gamma(TM) \quad \textit{and} \quad N \in \Gamma(\varphi D^{\perp}).$$

Proof. First, from (4.36), it follows

$$(4.38)\qquad \nabla^{\perp}_X cY \in (c\tilde{D}\oplus\nu) \text{ for all } X, Y \in \Gamma(D\oplus\tilde{D}\oplus\{\xi\}).$$

Then, from (4.38) and (3.5) we obtain

$$(4.39)\qquad g(h(X, bY), N) = g(h(Y, bX), N)$$

for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(\varphi D^{\perp})$. Thus, (4.37) follows from (4.39) by means of (2.8).

Theorem 4.5. *Let M be a $(D\oplus\tilde{D}, D^{\perp})$-geodesic proper almost semi-invariant submanifold in a Sasakian space form $\tilde{M}(k)$. If the distribution $D\oplus\tilde{D}\oplus\{\xi\}$ is integrable and the subbundle $\varphi D^{\perp}$ is $D\oplus\tilde{D}\oplus\{\xi\}$-parallel then we have $k \leqslant -3$.*

Proof. By a straightforward computation, using (2.6) and (2.7), we obtain

$$(4.40)\qquad \nabla_X \varphi N = -bA_N X + \varphi\nabla^{\perp}_X N \text{ for all } X \in \Gamma(D\oplus\tilde{D}\oplus\{\xi\}) \text{ and } N \in \Gamma(\varphi D^{\perp})$$

Then by using (2.21), (2.15)–(2.18) and (4.40) we get

$$(4.41)\qquad (D_X h)(Y, Z) - (D_Y h)(X, Z) = 2g(X, bY)N - h(X, bA_N Y) + h(Y, bA_N X)$$

for all $X, Y \in \Gamma(D\oplus\tilde{D}\oplus\{\xi\})$ and $Z=\varphi N \in \Gamma(D^{\perp})$, $N \in \Gamma(\varphi D^{\perp})$. Substituting (4.41) in (2.20), we obtain

$$(4.42)\qquad -\frac{k-1}{2}g(X, bY)N = 2g(X, bY)N - h(X, bA_N Y) + h(Y, bA_N X)$$

for any $X, Y \in \Gamma(D \oplus \tilde{D} \oplus \{\xi\})$ and $N \in \Gamma(\varphi D^{\perp})$.

In (4.42) we take $X = bY$ and taking account of (4,37) we find

$$2\|A_N bY\|^2 + \frac{k+3}{2}\|bY\|^2\|N\|^2 = 0 \tag{4.43}$$

for all $Y \in \Gamma(D \oplus \tilde{D} \oplus \{\xi\})$ and $N \in \Gamma(\varphi D^{\perp})$. Thus, from (4.43) it results $k \leq -3$ and the proof is complete.

From Theorem 4.5 we have

Corollary 4.1. *Let M be a $(D \oplus \tilde{D}, D^{\perp})$-geodesic almost semi-invariant submanifold in a Sasakian space form $\tilde{M}(k)$, $k > -3$. If the distribution $D \oplus \tilde{D} \oplus \{\xi\}$ is integrable and the subbundle $\varphi D^{\perp}$ is $D \oplus \tilde{D} \oplus \{\xi\}$-parallel, then we have either M is an anti-invariant submanifold of $\tilde{M}(k)$, or M is an almost invariant submanifold of $\tilde{M}(k)$.*

From (3.6) and Theorem 4.5 we also have

Corollary 4.2. *Suppose M satisfies the conditions of Theorem 4.5. If the leaves of distribution $D \oplus \tilde{D} \oplus \{\xi\}$ are totally geodesic immersed in M then we have $k = -3$.*

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SUBMANIFOLDS AND A PINCHING PROBLEM ON THE SECOND FUNDAMENTAL TENSORS

BY

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1. Introduction. M. Okumura in [5] has proved the following: *Let M be an n-dimensional ($n \geq 3$), complete, connected submanifold of an $(n+p)$-dimensional Riemannian manifold of positive constant curvature c, whose mean curvature vector field is parallel with respect to the induced connection of the normal bundle. If M is immersed without minimal point and for a real δ, $0 < \delta < c$, the second fundamental tensors $A_a (a = 1, \ldots, p)$ satisfy*

$$\sum_{a=1}^{p} \operatorname{trace} A_a^2 < 1/n-1 \sum_{a=1}^{p} (\operatorname{trace} A_a)^2 + 2\delta,$$

then M is umbilical with respect to the curvature normal direction.

For codimension 1 ($p = 1$), the present author in [3] has proved a stronger result. The purpose of the present paper is to generalize in some sense the above result of M. Okumura.

2. Preliminaries. Let M be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold $\bar{M}$ of constant curvature c. The Riemannian connections of M and $\bar{M}$ are denoted by ∇ and $\bar{\nabla}$ respectively, whereas the connection of the normal bundle of M in $\bar{M}$ is denoted by D. Let $e_1, \ldots, e_p$ be mutually orthogonal unit normal vectors at a point q of M and extend them to local fields in a neighborhood of q. Thus we have the following equations of Gauss and Weigarten:

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{a=1}^{p} \Sigma g(A_a X, Y) e_a, \tag{2.1}$$

$$\bar{\nabla}_X e_a = -A_a X + D_X e_a, \tag{2.2}$$

where X, Y are tangent vector fields in a neighborhood of M at q, g the Riemannian metric of M induced from that of $\bar{M}$ and A_a the second fundamental tensor with respect to e_a, whereas $D_X e_a$ is the normal part of $\bar{\nabla}_X e_a$ to M. The ambient manifold $\bar{M}$ being of constant curvature c, the curvature tensor K and the scalar curvature R of M are respectively given by