Proof. φ is well defined, since from $s=t$ on $X'\setminus A'$, it follows $\mu(s)=\mu(t)$, $\forall\mu\in S_0^*$ (cf. [1]), hence $s=t$. The other statements are obvious.

Remark. If A' is polar, then $\varphi^*(\mu')$ is an H-measure for any universally bounded μ'.

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ON PRODUCT OF MINIMAL PAIRWISE HAUSDORFF BITOPOLOGICAL SPACES

BY

M. N. MUKHERJEE

1. **Introduction.** M. P. Beri [1] proved that if the non empty product space is minimal Hausdorff, then each component space is also minimal Hausdorff. He left the converse of this theorem as an unsolved one and finally Ikenaga [4] proved it. The concept of minimal pairwise Hausdorff bitopological space was introduced by Raghavan and Reilly [7]. In my paper [6], this concept was further studied in terms of the notion of pairwise almost compactness and pairwise H-closedness.

The purpose of this paper is to study whether the topological product of a family of minimal pairwise Hausdorff spaces is also minimal pairwise Hausdorff and also to study the converse problem.

2. **Preliminaries.** Let $\{(X_\alpha, \tau_\alpha^1, \tau_\alpha^2)/\alpha\in\Lambda\}$, where Λ is an index set, be any family of bitopological spaces. If $X=\prod_{\alpha\in\Lambda}X_\alpha$, then two topologies on X can be constructed in a natural way. Let τ_i, for $i=1, 2$, denote the product topology on X, generated by the family $\{(X_\alpha, \tau_\alpha^i):\alpha\in\Lambda\}$. We shall call (X, τ_1, τ_2) the product bitopological space generated by the family under consideration and shall write $\tau_i=\prod_{\alpha\in\Lambda}\tau_\alpha^i$ for $i=1, 2$. It is obvious that the projection map $p_\alpha:(X, \tau_1, \tau_2)\to(X_\alpha, \tau_\alpha^1, \tau_\alpha^2)$ is pairwise open and pairwise continuous i.e., $p_\alpha:(X, \tau_i)\to(X_\alpha, \tau_\alpha^i)$ is open and continuous, for all $\alpha\in\Lambda$ and $i=1, 2$.

In what follows we shall state a few definitions and theorems which will be used in the sequel.

Definition 2.1. [Kelly [5]] *A bitopological (X, τ_1, τ_2) is said to be pairwise Hausdoff if for any two distinct points x, y in X, there exist a τ_1 open neighbourhood V_x of x and a τ_2 — open neighbourhood V_y of y, so that $V_x\cap V_y=\varnothing$.*

Definition 2.2. [Mukherjee [6]] *In a bitopological space (X, τ_1, τ_2) a filter F is called a τ_i open filter (for $i=1, 2$) if F has a base consisting of τ_i open sets.*

Definition 2.3. [Mukherjee [6]] *A bitopological space (X, τ_1, τ_2) is said to be τ_i almost compact with respect to τ_j, for $i, j=1, 2$ and $i\neq j$, if every τ_i open filter has a τ_i cluster point. The space (X, τ_1, τ_2) will be called pairwise almost compact if it is τ_1 almost compact with respect to τ_2 and τ_2 almost compact with respect to τ_1.*

In my paper [6], the following theorem was established.

Theorem 2.4. *In a bitopological space (X, τ_1, τ_2) the following statements are equivalent :*

(a) (X, τ_1, τ_2) is pairwise almost compact.

(b) if $\mathcal{C}$ is a system of τ_i open sets, having finite intersection property then

$$\cap\{\bar{c}^{\tau_j} : c \in \mathcal{C}\} \neq \varnothing, \quad \text{for} \quad i, j = 1, 2 ; \quad i \neq j.$$

(c) If $X = \bigcup_{\alpha \in \Lambda} G_\alpha$, where each $G_\alpha (\alpha \in \Lambda)$ is a τ_i open set, there is a finite sub collection $\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$ of $\{G_\alpha : \alpha \in \Lambda\}$ such that $X = \bigcup_{k=1}^{n} \overline{G_{\alpha_k}}^{\tau_j}$.

(d) Every τ_i ultra open filter is convergent in (X, τ_j), where $i, j = 1, 2$ and $i \neq j$.

The following definition and characterization theorem for minimal pairwise Hausdorff spaces were given by Roghavan and Reilly [7].

Definition 2.5. *A bitopological space (X, τ_1, τ_2) is said to be minimal pairwise Hausdorff if it is pairwise Hausdorff and if (X, ψ_1, ψ_2) is pairwise Hausdorff with $\psi_1 \subset \tau_1$ and $\psi_2 \subset \tau_2$, then $\psi_1 = \tau_1$ and $\psi_2 = \tau_2$.*

Theorem 2.6. *A pairwise Hausdorff space (X, τ_1, τ_2) is minimal pairwise Hausdorff if and only if every τ_i open filterbase on X with a unique τ_j adherent point p is τ_i convergent to p, where $i, j = 1, 2$; $i \neq j$.*

In my paper [6], the following theorem was established

Theorem 2.7. *If (X, τ_1, τ_2) is minimal pairwise Hausdorff, then X is either τ_1 almost compact with respect to τ_2 or τ_2 almost compact with respect to τ_1.*

3. **Product of Minimal Pairwise Hausdorff Spaces.** In what follows, unless otherwise stated, $\{(X_\alpha, \tau_\alpha^1, \tau_\alpha^2) : \alpha \in \Lambda\}$ shall denote an arbitrary family of bitopological spaces and by (X, τ_1, τ_2) we shall mean the product bitopological space.

Proposition 3.1. *(X, τ_1, τ_2) is pairwise Hausdorff if and only if $(X_\alpha, \tau_\alpha^1, \tau_\alpha^2)$ is pairwise Hausdorff, for each $\alpha \in \Lambda$.* For proof see [2], [8].

Theorem 3.2. If the product space (X, τ_1, τ_2) is minimal pairwise Hausdorff, then each factor space-$(X_\alpha, \tau_\alpha^1, \tau_\alpha^2)$ is also so, where $\alpha \in \Lambda$.

Proof. If possible, let $(X_{\alpha_0}, \tau_{\alpha_0}^1, \tau_{\alpha_0}^2)$ be not minimal pairwise Hausdorff for some $\alpha_0 \in \Lambda$.

Then there exists a pair of topologies $\psi_{\alpha_0}^1, \psi_{\alpha_0}^2$ on X_{α_0} such that $\psi_{\alpha_0}^1 \subset \tau_{\alpha_0}^1$, $\psi_{\alpha_0}^2 \subset \tau_{\alpha_0}^2$, $(X_{\alpha_0}, \psi_{\alpha_0}^1, \psi_{\alpha_0}^2)$ is pairwise Hausdorff and either $\psi_{\alpha_0}^1 \neq \tau_{\alpha_0}^1$, or $\psi_{\alpha_0}^2 \neq \tau_{\alpha_0}^2$. For definiteness, let $\psi_{\alpha_0}^1 \subsetneq \tau_{\alpha_0}^1$. Then there exists $V_{\alpha_0}^1 \in \tau_{\alpha_0}^1$ such that $V_{\alpha_0}^1 \notin \psi_{\alpha_0}^1$, consider the product space of the family $\{(X_\alpha, \tau_\alpha^1, \tau_\alpha^2) : \alpha \in \Lambda, \alpha \neq \alpha_0\} \cup \{(X_{\alpha_0}, \psi_{\alpha_0}^1, \psi_{\alpha_0}^2)\}$ and call it (X, ψ_1, ψ_2). Since (X, τ_1, τ_2) is pairwise Hausdorff, by proposition 3.1, $(X_\alpha, \tau_\alpha^1, \tau_\alpha^2)$ is so, for each $\alpha \in \Lambda$ and also $(X_{\alpha_0}, \psi_{\alpha_0}^1, \psi_{\alpha_0}^2)$ is pairwise Hausdorff. Hence again by proposition 3.1, it follows that (X, ψ_1, ψ_2) is pairwise Hausdorff. It is obvious that $\psi_1 \subset \tau_1$ and $\psi_2 \subset \tau_2$. Now, $V = p_{\alpha_0}^{-1}(V_{\alpha_0}^1) \in \tau_1$ but $V \notin \psi_1$. For, if $V \in \psi_1$, then $p_{\alpha_0}(V) = p_{\alpha_0}(p_{\alpha_0}^{-1}(V_{\alpha_0}^1)) = V_{\alpha_0}^1 \in \psi_{\alpha_0}^1$ which is not the case. $\psi_1 \neq \tau_1$. Thus (X, ψ_1, ψ_2) is a pairwise Hausdorff

bitopological space such that $\psi_1 \subsetneq \tau_1$, $\psi_2 \subset \tau_2$. This contradicts the minimality of the pairwise Hausdorff space (X, τ_1, τ_2) and the theorem is proved.

We now give an example which will decisively establish that the converse of the above theorem is, in general, not true.

Example 3.3. Let X_1 be the set of all real numbers and Y_1 denote the set of all non-negative reals. Consider the bitopological spaces (X_1, τ_1, τ_2) and (Y_1, ψ_1, ψ_2), where τ_1, τ_2 denote respectively the discrete and cofinite topologies on X_1; ψ_2 is the usual topology on Y_1 and ψ_1 on Y_1 is given by

$$\psi_1 = \{\Phi\} \cup \{V \cup (x, \infty) / x \in Y_1, V \in \psi_2\}.$$

Raghavan and Reilly [7] proved that (X_1, τ_1, τ_2) is minimal pairwise Hausdorff and Fletcher, Hoyle, Patty [3] showed that (Y_1, ψ_1, ψ_2) is pairwise Hausdorff and pairwise compact. Then by a result of Raghavan and Reilly (See Theorem 2 of [7]), (Y_1, ψ_1, ψ_2) is minimal pairwise Hausdorff.

Let $X = X_1 \times Y_1$ and σ_1, σ_2 respectively be the topologies on X given by $\sigma_1 = \tau_1 \times \psi_1$, $\sigma_2 = \tau_2 \times \psi_2$. We shall show that (X, σ_1, σ_2) is neither σ_1 almost compact with respect to σ_2 nor σ_2 almost compact with respect to σ_1. Then by theorem 2.7 it will follow that (X, σ_1, σ_2) is not minimal pairwise Hausdorff.

Let $\mathcal{G}_1 = \{\{x\} : x \in X_1\}$ be a τ_1 open cover of X_1 and let $\mathcal{G}_2$ be any ψ_1 open cover of Y_1. Then $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$ is a σ_1 open cover of X. If possible, let $\mathcal{G}$ have a finite subfamily $\mathcal{G}'$, the σ_1 closures of whose members cover X. Then there exists a finite subcollection.

$$\mathcal{G}_1' = \{\{x_1\}, \{x_2\}, \ldots, \{x_n\}\} \text{ (say) of } \mathcal{G},$$

and a finite subcollection $\mathcal{G}_2'$ of $\mathcal{G}_2$ such that $\mathcal{G}' = \mathcal{G}_1' \times \mathcal{G}_2'$.

Let $x^0 \in X$ be such that $x^0 < \min(x_1, x_2, \ldots, x_n)$.

Then $x^0 \in X_1 - \{x_1, x_2, \ldots, x_n\} \in \tau_2$.

Then for any $y \in Y_1$, and any ψ_2 open set U_y containing y, $(x^0, y) \in [X_1 - \{x_1, x_2, \ldots, x_n\}] \times U_y = V(\text{say}) \in \sigma_2$ and clearly V does not intersect any member of $\mathcal{G}'$. Hence $\mathcal{G}$ cannot have a finite subcollection, the σ_2 closures of whose members cover X. Hence, by theorem 2.4, X is not σ_1 almost compact with respect to σ_2.

Again, consider any τ_2 open cover $\mathcal{F}_1$ of X_1, and $\mathcal{F}_2$ consisting of all sets of the form $\left(y - \frac{1}{2}, y + \frac{1}{2}\right) \cap Y_1$, for $y \in Y_1$. Then $\mathcal{F}_2$ is a ψ_2 open cover of Y_1 and consequently $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ is a σ_2 open cover of X. If possible, let it have a finite subcollection $\mathcal{F}'$ such that the σ_1 closures of members of $\mathcal{F}'$ cover X. Then there exist a finite subcollection $\mathcal{F}_1'$ (say) of $\mathcal{F}_1$ and a finite subcollection

$$\mathcal{F}_2' = \left\{\left(y_i - \frac{1}{2}, y_i + \frac{1}{2}\right) \cap Y_1 : i = 1, 2, \ldots, m ; y_i \in Y_1\right\} \text{ (say) of } \mathcal{F}_2 \text{ such that}$$

$\mathcal{F}' = \mathcal{F}_1' \times \mathcal{F}_2'$.

Let $y_0 > 2 + \max(y_1, y_2, \ldots, y_m)$.

Then $y_0 \in \left[\left(y_0 - \frac{1}{2},\ y_0 + \frac{1}{2}\right) \cap Y_1\right] \cup (y_0, \infty) = W_{y_0}$ (say) $\in \psi_1$.

Now, let $p \in Y_1$ and let G_p be any τ_1 open neighbourhood of p, the $(p, y_0) \in G_p \times W_{y_0} \in \sigma_1$ but $G_p \times W_{y_0}$ does not intersect any member of F'. Hence F has no finite subcollection, the σ_1 closures of whose members cover X and consequently X is not σ_2 almost compact with respect to σ_1.

We have just now shown that product of minimal pairwise Hausdorff spaces may not be minimal pairwise Hausdorff. In the nextstheorem our task will be to investigate as to under what additional condition the converse of theorem 3.2 may be true. For this purpose we require the following

Lemma 3.4. *If $(X_\alpha, \tau_\alpha^1, \tau_\alpha^2)$ is pairwise almost compact for each $\alpha \in \Lambda$, then (X, τ_1, τ_2) is also pairwise almost compact.*

Proof. Let $\mathcal{F}^i = \{F_\nu^i : \nu \in I\}$, where I is some index set, be a τ_i open filterbase in X (for $i = 1, 2$). For each $\alpha \in \Lambda$, $\mathcal{F}_\alpha^i = \{p_\alpha(F_\nu^i) : \nu \in I\}$ is clearly a τ_α^i open filterbase in $(X_\alpha, \tau_\alpha^1, \tau_\alpha^2)$, since $p_\alpha : (X, \tau_i) \to (X_\alpha, \tau_\alpha^i)$ is open for $i = 1, 2$.

$(X_\alpha, \tau_\alpha^1, \tau_\alpha^2)$ being pairwise almost compact, $\mathcal{F}_\alpha^i$ has a τ_α^j adherent point x_α (say) $\in X_\alpha$, for $i, j = 1, 2$, $i \neq j$. Consider the point $x = (x_\alpha) \in X$.

We show that x is a τ_j cluster point of $\mathcal{F}^i$.
Let V be a τ_j open neighbourhood of x in (X, τ_1, τ_2) and let $V_\alpha = X_\alpha$ except for the indices $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$. Then $V_{\alpha_k} = p_{\alpha_k}(V)$ is a τ_α^j neighbourhood of x_{α_k} in $(X_{\alpha_k}, \tau_{\alpha_k}^1, \tau_{\alpha_k}^2)$ for $k = 1, 2, \ldots, n$; $i \neq j$, $i, j = 1, 2$. If $F^i \in \mathcal{F}^i$ be arbitrary, then $p_{\alpha_k}(F^i) \cap V_{\alpha_k} \neq \Phi$, for $k = 1, 2, \ldots, n$, since x_{α_k} is a $\tau_{\alpha_k}^j$ adherent point of $\mathcal{F}_{\alpha_k}^i$ and $p_{\alpha_k}(F^i) = F_{\alpha_k}^i$ (say) $\in \mathcal{F}_{\alpha_k}^i$. Then clearly $V \cap F^i \neq \varnothing$. This shows that each τ_j open neighbourhood of x has non-null intersection with each member F^i of $\mathcal{F}^i$. Consequently $\mathcal{F}^i$ has x as a τ_j cluster point, proving that (X, τ_1, τ_2) is pairwise almost compact.

Theorem 3.5. *If each factor space $(X_\alpha, \tau_\alpha^1, \tau_\alpha^2)$ is pairwise almost compact and minimal pairwise Hausdorff, where $\alpha \in \Lambda$, then the product space (X, τ_1, τ_2) is minimal pairwise Hausdorff (and also pairwise almost compact by Lemma 3.4).*

Proof. Let $\mathcal{B}_1$ be a τ_1 open filter base having a unique τ_2-adherent point $x^0 = (x_\alpha^0)\ (\in X)$ and $\mathcal{B}_2$ be a τ_2 open filterbase having the same unique τ_1 adherent point x^0. We shall show that $\mathcal{B}_i$ is τ_i convergent to x^0 for $i = 1, 2$. Then by theorem 2.6 it will follow that (X, τ_1, τ_2) is minimal pairwise Hausdorff.

For each $\alpha \in \Lambda$, let $\mathcal{B}_\alpha^i = \{p_\alpha(F^i) / F^i \in \mathcal{B}_i\}$, for $i = 1, 2$. Then clearly $\mathcal{B}_\alpha^i$ is a τ_α open filter base in $(X_\alpha, \tau_\alpha^1, \tau_\alpha^2)$. Since $\mathcal{B}_i$ has τ_j adherent x^0, it is obvious that x_α^0 is a τ_α^j-adherent point of $\mathcal{B}_\alpha^i$, for $i, j = 1, 2$, $i \neq j$.

Let $\beta \in \Lambda$ be arbitrarily fixed. We shall show that x_β^0 is the unique τ_β^j adherent point of $\mathcal{B}_\beta^i$ ($i \neq j$, $i, j = 1, 2$). In order to show this, suppose $x_\beta \in X_\beta$ such that $x_\beta \neq x_\beta^0$. Let now, $X' = X_\beta \times \prod_{\alpha \neq \beta} X_\alpha$, then X' is pairwise homomorphic to X. Put $\prod_{\alpha \neq \beta} X_\alpha = Y$ and $\prod_{\alpha \neq \beta} \tau_\alpha^i = \psi_i$ for $i = 1, 2$. Then $X = X_\beta \times Y$. For an arbitrary $y \in Y$, consider the point $x = (x_\beta, I)$ of X. As

$x \neq x^0$ (since $x_\beta \neq x_\beta^0$), x is not a τ_j adherent point of $\mathcal{B}_i$, where $i \neq j$; $i, j = 1, 2$. Thus there exists some τ_j open neighbourhood $W_y(x) = V_Y(x_\beta) \times U(y)$ of x where $V_Y(x_\beta)$ is a τ_β^j-neighbourhood of x_β and $U(y)$ is a ψ_j open neighbourhood of y, and some $F_y \in \mathcal{B}_i$ such that

$$[V_y(x_\beta) \times U(y)] \cap F_Y = \varnothing.$$

As y runs over Y, we have $Y = \bigcup_{y \in Y} U(y)$.

Now, Y being a product of pairwise almost compact spaces, is also pairwise almost compact, by Lemma 3.4 and hence by theorem 2.4 there exists a finite number of points $y_1, y_2, \ldots, y_n$ in Y such that $Y = \bigcup_{k=1}^{n} \{\overline{U(y_k)}^{\psi_i}\}$.

Since $[V_{Y_k}(x_\beta) \times U(Y_k)] \cap F_{Y_k} = \varnothing$, for $k = 1, 2, \ldots, n$ it follows that $[V_{y_k}(x_\beta) \times \overline{U(y_k)}^{\psi_i}] \cap F_{Y_k} = \varnothing$ for $k = 1, 2, \ldots, n$. Now $V(x_\beta) = \bigcap_{k=1}^{n} V_{Y_k}(x_\beta) \in \tau_\beta^j$ and let $F^i \in \mathcal{B}_i$ be such that $F^i \subset \bigcap_{k=1}^{n} F_{Y_k}$.

$$\text{Then } \Phi = \bigcup_{k=1}^{n} \{V_{y_k}(x_\beta) \times \overline{U(y_k)}^{\psi_i} \cap F^i\} \supset \bigcup_{k=1}^{n} \{V(x_\beta) \times \overline{U(y_k)}^{\psi_i}\} \cap F^i = (V(x_\beta) \times Y) \cup F^i \Rightarrow V(x_\beta) \cap p_\beta(F^i) = \varnothing.$$

But $V(x_\beta)$ is a τ_β^j neighbourhood of x_β. Hence x_β is not τ_β^j adherent point of $\mathcal{B}_\beta^i$ for $i, j = 1, 2, i \neq j$. In other words, $\mathcal{B}_\beta^i$ has a unique τ_β^j cluster point x_β^0, for $i, j = 1, 2$ and $i \neq j$. $(X_\beta, \tau_\beta^1, \tau_\beta^2)$ being minimal pairwise Hausdorff it follows by theorem 2.6 that $\mathcal{B}_\beta^i$ is τ_β^i convergent to x_β^0, for $i = 1, 2$.

We next show that $\mathcal{B}_i$ is τ_i convergent to x^0, for $i = 1, 2$. Let $W(x^0) = \prod_{\alpha \in \Lambda} V(x_\alpha^0)$ be a τ_i open neighbourhood of x^0, where $V(x_\alpha^0)$ is a τ_α^i open neighbourhood of x_α^0 for $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_m$ and $V(x_\alpha^0) = X_\alpha$ for $\alpha \neq \alpha_1, \alpha_2, \ldots \alpha_m$. For each $\alpha_t (t = 1, 2, \ldots, m)$ there exists $F_{\alpha_t} \in \mathcal{B}_{\alpha_t}^i$ such that $F_{\alpha_t} \subset V(x_{\alpha_t}^0)$, since $\mathcal{B}_{\alpha_t}^i$ is $\tau_{\alpha_t}^i$ convergent to $x_{\alpha_t}^0$, for $i = 1, 2$. Hence $p_{\alpha_t}^{-1} V(x_{\alpha_t}^0) \supset p_{\alpha_t}^{-1}(F_{\alpha_t}) \supset F^t \in \mathcal{B}_i$, where $p_{\alpha_t}(F^t) = F_{\alpha_t}$, for $t = 1, 2, \ldots, m$. Let $F \in \mathcal{B}_i$ be such that $F \subset \bigcap_{k=1}^{m} F^t$.

Since $W(x^0) = \prod_{\alpha} V(x_\alpha^0) = \bigcap_{t=1}^{m} p_{\alpha_t}^{-1}(V(x_{\alpha_t}^0)) \supset \bigcap_{t=1}^{m} F^t \supset F \in \mathcal{B}_i$,

it follows that $\mathcal{B}_i$ is τ_i convergent to x^0, for $i = 1, 2$ and then it follows that (X, τ_1, τ_2) is minimal pairwise Hausdorff.

This completes the proof.

Remark 3.6. By virtue of theorem 2.7 we notice that every minimal pairwise Hausdorff space (X, τ_1, τ_2) is either τ_1 almost compact with respect to τ_2 or τ_2 almost compact with respect to τ_1. Thus the condition assumed in the above theorem along with the minimal pairwise Hausdorffness of the component spaces, is not very strong. Also, it is shown in [6] that minimal pairwise Hausdorff property may not imply that the space is pairwise almost compact. Hence the condition is not altogether superfluous.

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BOL LOOPS OF ORDER 12

BY

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1. **Introduction.** Recently Burn [4] has shown that there exist exactly six non-associative Bol loops of order 8. The Object of this paper is to prove that (up to isomorphisms) there are only two Bol loops of order 12 which are not groups and Moufang loops.

For the definition of a loop, the reader should consult Bruck [3]. Albert [1] defines a right multiplication $R(a)$ $a\in L$, as a permutation of a loop $(L,.)$

$$R(a): x\to x\,.\,\alpha. \quad \forall x\in L. \tag{1}$$

We call the set $\Pi=\{R(a) [a\in L\}$ the right regular representation of $(L,.)$ or briefly, the representation of L.

A Bol loop L is a loop in which we have

$$(xy\,.\,z)y=x(yz\,.\,y) \text{ for all } x, y, z\in L. \tag{2}$$

Strictly speaking, (2) defines a right Bol loop. (For Bol loop, see [2], [6], [7], [8, 9]). The following theorems due to Burn [4] will be used in the investigations.

Theorem 1. *If Π is the representation of a loop L, then L is a Bol loop if and only if α, $\beta\in\Pi$ implies $\alpha\beta\alpha\in\Pi$.*

Theorem 2. *If Π is a representation of a finite Bol loop and $\alpha\in\Pi$, then α is a product of disjoint cycles of equal length.*

Theorem 3. *The order of an element of a finite Bol loop divides the order of the loop.*

Theorem 4. *If $R(a)$ and $R(b)$ are right multiplications of a Bol loop and $R(a)^2=R(b)^2=1$, then $R(a)R(b)=R(b)R(a)$ if and only if $aR(b)=bR(a)$.*

2. In this section we show that if Π is a representation of a finite Bol loop of order 12, then Π can not contain an element of order 4.

Theorem 5. *If Π is a representation of a finite Bol loop of order 12, then Π can not contain two elements of order 4 and two element of order 3.*

Proof. We assume that α, $\beta\in\Pi$, such that $\alpha^4=1$ and $\beta^3=1$. Without loss of generality, we take

$$\alpha = R(2)=(1\ 2\ 3\ 4)\ (5\ 6\ 7\ 8)\ (9\ 10\ 11\ 12),$$

$$\alpha^2 = R(3)=(1\ 3)\ (2\ 4)\ (5\ 7)\ (6\ 8)\ (9\ 11)\ (10\ 12),$$

$$\alpha^3 = R(4)=(1\ 4\ 3\ 2)\ (5\ 8\ 7\ 6)\ (9\ 12\ 11\ 10), \text{ and}$$