10. Mişicu, M. — *Mecanica mediilor deformabile. Fundamentele elasticităţii structurale*, Edit. Acad. R. S. România, Bucureşti, 1967
11. Eringen, A. C. — *Theory of micropolar elasticity. Fracture*, 2, Academic Press, New-York, 1968
12. Nowaki, W. — *Theory of micropolar elasticity*. Springer, Vien, 1970
13. Teodorescu P. P. — *Dinamica corpurilor liniar elastice*. Edit. Acad. R. S. România, Bucureşti, 1972
14. Soos, E. — *Modele discrete şi continue ale solidelor*. Edit. ştiinţifică, 1974
15. Ieşan, D. — *Unele probleme fundamentale ale teoriei liniare a mediilor elastice de tip Cosserat. Probleme actuale în mecanica solidelor*, vol. I, Edit. Acad. R. S. România Bucureşti, 1975
16. Şandru, N. — *Unele probleme la limită în elasticitatea micropolară liniară. Probleme actuale în mecanica solidelor*, Vol. I, Edit. Acad. R. S. România, Bucureşti, 1975
17. Kröner, I. A. — *Elasticity theory with long range cohesive forces*. Int. J. Solids, Struct. 3, 731, 1967
18. Edelen, B. G. B., Green, E. A., Laws N. — *Nonlocal continuum mechanics*, Arch. Rat. Mech. Anal, 43, 1971, 36—44
19. Eringen, C. A., and Edelen D. G. B. — *On nonlocal elasticity*. Int. J. Eng. Sci. 10, 233—248, 1972
20. Eringen C. A. — *Nonlocal polar elastic continua*. Int. J. Eng. Sci. 10, 1—16, 1972
21. Rogula, D. — *On nonlocal continuum theories of elasticity*. Arch. Mech. Warszawa, 25, 233—251, 1973
22. Eringen C. A. — *Theory of nonlocal thermoelasticity*. Int. J. Eng. Sci. 12, 1063—1077, 1974
23. Rymarz, C. — *Boundary problems of the nonlocal theory*. Proc. Vibrations Problems, 15, 355—372, 1974
24. Edelen, D. G. B. — *Nonlocal field theories*. Continuum Physics, vol. IV, (Edited by Eringe). Academic Press, New York, 1976
25. Eringen C. A. — *Nonlocal polar field theories*. Continuum Physics, vol. IV (Edited by Eringen), Academic Press, New York, 1976
26. Green E. A. and Naghdi M. P. — *On non-local continuum mecanics*. Math. Proc. Camb. Phil. Soc. 83, 1978, 307—319
27. Borş, C. I. — *On nonlocal elastic continua*. An. şt. Univ. Iaşi, Supliment la tomul XXV, S. I a, 1979. 49—52
28. Borş, C. I. — *A nonlocal continuum with asymmetric stress tensor*. Bul. Inst. polit. Iaşi, tomul XXVI(XXX), S.I, f. 3—4, 1980, 99—102
29. Borş, C. I. — *Sur la Thérmodynamique des milieux continus non locaux*. C.R. Acad. Sci. Paris, t. 290, 1980, Serie, B, 427—430
30. Borş, C. I. — *Deformable solid with microstructure having a symmetric stress tensor*. An. şt. Univ. Iaşi, tomul XXVII, S. Ia, f. 1, 1981, 177—184

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BOUNDARY VALUE PROBLEMS IN MICROPOLAR THERMOELASTICITY

BY

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1. Introduction. There are many papers and books which study the stationary vibrations in elastic solids. This kind of problems have been initiated by V. D. Kupradze, the results of his researches and of others, especially from his school, or from elsewhere, up to 1976, being published in [1].

In our previous papers [5]—[7] some problems concerning the stationary vibrations of an isotropic, homogeneous and centrosymmetric solid in the linear theory of micropolar thermoelasticity [2], have been studied. The same theory has been discussed by Agniasvili [8].

In this paper two interior and two exterior boundary value problems of the above mentioned theory, which have been introduced in [6], are considered. For the study of these problems we have deduced in Sect. 3 some Green identities from which, by using the same method as in [1], [3]—[5], [7], [8], the integral representation formulae for the regular solution of the differential equations of the governing theory have been given. Starting from the results of Sect. 3, we introduce in Sect. 4 two micropolar thermoelastic surface potentials for which some properties are given. In Sect. 5, we reformulate the two interior and the two exterior boundary value problems by means of some boundary operators introduced in Sect. 3.

In the last section these problems are reduced to singular integral equations of Fredholm type. The study of these singular integral equations allow to prove existence theorems for the considered problems.

2. Basic Equations. We consider an isotropic, homogeneous and centrosymmetric micropolar thermoelastic solid occupying either a three-dimensional domain D^+, bounded by a closed Lyapunov surface $S \in L_1(\alpha)$, $0 < \alpha \leq 1$, or the infinite domain D^-, where $D^- = E_3 / D^+ \cup S$, E_3 being the three-dimensional Euclidian space.

If we consider that this solid is acting on his surface S by stresses, couple-stresses and heat flux varying harmonicaly on time, then, in the absence of the body force, couple-body force and heat source, the basic differential equations which give the amplitudes $\mathbf{u}$, φ, θ of the displacement vector, the rotation vector and the increament of temperature in the point $x(x_1, x_2, x_3)$, respectively, are [4]

$$
(2.1)\qquad \begin{cases} [(\mu+\varkappa)\Delta+\rho\omega^2]\mathbf{u}+(\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div}\mathbf{u}+2\alpha\operatorname{rot}\varphi-\nu\operatorname{grad}\theta=0, \\ 2\alpha\operatorname{rot}\mathbf{u}+[(\gamma+\varepsilon)\Delta+J\omega^2-4\alpha]\varphi+(\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\varphi=0, \\ i\omega\eta\operatorname{div}\mathbf{u}+(\Delta+i\omega/\varkappa)\theta=0, \end{cases}
$$

where λ, μ, α, β, γ, ε are mechanical constants ; ν, $\varkappa$, η are constants depending on the thermal properties of the body ; ρ is the density of the material, J is the rotational inertia, ω is the frequency of the vibrations, $\Delta=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2+\partial^2/\partial x_3^2$ is Laplace's operator, x_1, x_2, x_3 are the coordinates of the point x in the rectangular coordinate system $Ox_1x_2x_3$ and $i=\sqrt{-1}$.

The equations (2.1) may be written in the matricial form

$$
(2.2)\qquad B(\partial/\partial x,\omega)V(x)=0,
$$

where B is the differential operator of the micropolar thermoelasticity of an isotropic, homogeneous and centrosymmetric solid in the case of stationary vibrations, whose matrix of the 7×7 type can be easely found from (2.1) and

$$
(2.3)\qquad V(x)=\begin{pmatrix}\mathbf{u}(x)\\ \varphi(x)\\ \theta(x)\end{pmatrix}=\begin{pmatrix}U(x)\\ \theta(x)\end{pmatrix},\quad x\in D^+ \text{ or } x\in D^-.
$$

If we denote by $\mathbf{t}(x)$, $\mathbf{m}(x)$ and $h(x)$ the stress vector, the couple-stress vector and the heat flux, respectively, all acting on a surface element containing the point $x\in E_3$, with the unit normal vector $\mathbf{n}_x$, oriented from D^+ to D^-, then [2], [3], [5]

$$
(2.4)\qquad \begin{cases} \mathbf{t}(x)=\lambda\mathbf{n}_x\operatorname{div}\mathbf{u}(x)+2\mu\partial\mathbf{u}(x)/\partial\mathbf{n}_x+(\mu-\alpha)\mathbf{n}_x\times\operatorname{rot}\mathbf{u}(x)+2\alpha\mathbf{n}_x\times\varphi(x)-\nu\theta(x)\mathbf{n}_x, \\ \mathbf{m}(x)=\beta\mathbf{n}_x\operatorname{div}\varphi(x)+2\gamma\partial\varphi(x)/\partial\mathbf{n}_x+(\gamma-\varepsilon)\mathbf{n}_x\times\operatorname{rot}\varphi(x), \\ h(x)/k=\partial\theta(x)/\partial n_x,\quad k=\text{const.} \end{cases}
$$

The equations (2.3) can be also written in the form

$$
(2.5)\qquad \begin{pmatrix}\mathbf{t}(x)\\ \mathbf{m}(x)\\ h(x)/k\end{pmatrix}=R(\partial/\partial x,n_x)V(x),
$$

where R is a matricial differential operator of 7×7 type which in the case $x\in S$ may be called a boundary value operator of the B operator.

Let us notice that

$$
(2.6)\qquad R(\partial/\partial x,n_x)V(x)=\begin{pmatrix}T(\partial/\partial x,n_x)U(x)-\nu\theta(x)\mathbf{n}_x\\ M(\partial/\partial x,n_x)\varphi(x)\\ \partial\theta(x)/\partial n_x\end{pmatrix}
$$

where T is a differential matrix of 3×6 type, while M is of 3×3 type (see e.g.[5]). Obviously,

$$
(2.7)\qquad T(\partial/\partial x,n_x)U(x)=\mathbf{t}(x)+\nu\theta(x)\mathbf{n}_x,\quad M(\partial/\partial x,n_x)\varphi(x)=\mathbf{m}(x).
$$

With the T and M operators introduced, R may be written in the form

$$
(2.8)\qquad R(\partial/\partial x,n_x)=\left\|\begin{array}{c|ccc|c} \multicolumn{4}{c|}{\multirow{3}{*}{$T(\partial/\partial x,n_x)$}} & -\nu n_1\\ \multicolumn{4}{c|}{} & -\nu n_2\\ \multicolumn{4}{c|}{} & -\nu n_3\\ \hline \multirow{4}{*}{$O_{4\times 3}$} & \multicolumn{3}{c|}{} & 0\\ & \multicolumn{3}{c|}{M(\partial/\partial x,n_x)} & 0\\ & \multicolumn{3}{c|}{} & 0\\ \cline{2-5} & 0 & 0 & 0 & \partial/\partial n_x \end{array}\right\|,
$$

where $O_{4\times 3}$ is the nul matrix and n_1, n_2, n_3 are the components of $\mathbf{n}_x$.

In our previous paper [4] we denoted by $\tilde{B}(\partial/\partial x,\omega)$ the adjoint operator in Lagrange's sense of the operator $B(\partial/\partial x,\omega)$ and we have proved the Green identity for the bounded domain D^+.

$$
(2.9)\qquad \int_{D^+}[V(x)\,.\,\tilde{B}(\partial/\partial x,\omega)\tilde{V}(x)-\tilde{V}(x)\,.\,B(\partial/\partial x,\omega)V(x)]dx = \int_S[V(x)\,.\,\tilde{R}(\partial/\partial x,nx)\tilde{V}(x)-\tilde{V}(x)\,.\,R(\partial/\partial x,n_x)V(x)]dS_x,
$$

where $\tilde{R}(\partial/\partial x,n_x)$ is a boundary value operator associated to $\tilde{B}(\partial/\partial x,\omega)$, obtained from $R(\partial/\partial x,n_x)$ by changing ν with $i\omega\eta$ in the seventh column ; $\tilde{V}(x)$ is a vector function with seven components, similar to $V(x)$ from (2.3) ; the point means the scalar product in a space with 7 dimensions. It is supposed here that V and $\tilde{V}$ are of the $C^2(D^+)\cap C^1(\overline{D^+})$ class, where $\overline{D^+}=D^+\cup S$.

If we suppose that the body occupies the infinite domain D^-, V and $\tilde{V}$ are of class $C^2(D^-)\cap C^1(\overline{D^-})$ and satisfy the radiation conditions defined in [6] and [7], then we have the following Green identity for the infinite domain D^-(see [7])

$$
(2.10)\qquad \int_{D^-}[V(x)\,.\,\tilde{B}(\partial/\partial x,\omega)\tilde{V}(x)-\tilde{V}(x)\,.\,B(\partial/\partial x,\omega)V(x)]dx = -\int_S[V(x)\,.\,\tilde{R}(\partial/\partial x,n_x)\tilde{V}(x)-\tilde{V}(x)\,.\,R(\partial/\partial x,n_x)V(x)]dS_x.
$$

3. New Green Identities and Integral Representation. Starting from (2.9) and (2.10), we can easily see that we also have the following two Green identities

$$
(3.1)\qquad \int_{D^+}[V(x)\,.\,\tilde{B}(\partial/\partial x,\omega)\tilde{V}(x)-\tilde{V}(x)\,.\,B(\partial/\partial x,\omega)V(x)]dx= \int_S[\mathscr{K}(\partial/\partial x,n_x)V(x)\mathscr{L}(\partial/\partial x,n_x)\tilde{V}(x)-\tilde{\mathscr{K}}(\partial/\partial x,n_x)\tilde{V}(x)\,.\,\mathscr{L}(\partial/\partial x,n_x)V(x)]dS_x,
$$

$$\int_{D^-}[V(x)\,.\,\tilde{B}(\partial/\partial x,\omega)\tilde{V}(x)-\tilde{V}(x)\,.\,B(\partial/\partial x,\omega)V(x)]dx=$$

(3.2)
$$=-\int_{D^-}[\mathcal{T}(\partial/\partial x,n_x)V(x).\mathcal{L}(\partial/\partial x,n_x)\tilde{V}(x)-\tilde{\mathcal{T}}(\partial/\partial x,n_x)\tilde{V}(x)\,.\,\mathcal{L}(\partial/\partial x,n_x)V(x)]dS_x,$$

where the new boundary value operators $\mathcal{T}$ and $\mathcal{L}$ are differential matrices of 7×7 type, whose experessions in terms of T and M are

(3.3)
$$\mathcal{T}=\left\Vert\begin{array}{c|c|c} \multicolumn{2}{c|}{-T} & \begin{matrix}-\nu n_1\\-\nu n_2\\-\nu n_3\end{matrix}\\ \hline O_{4\times 3} & \begin{matrix}E_{3\times 3}\\ 0\ 0\ 0\end{matrix} & \begin{matrix}0\\0\\0\\1\end{matrix}\end{array}\right\Vert,\quad \mathcal{L}=\left\Vert\begin{array}{c|c} E_{3\times 3} & O_{3\times 4}\\ \hline O_{4\times 3} & \begin{matrix}M & \begin{matrix}0\\0\\0\end{matrix}\\ 0\ 0\ 0 & \partial/\partial n_x\end{matrix}\end{array}\right\Vert,$$

$E_{3\times 3}$ being here the unit matrix of the third order.

The operator $\tilde{\mathcal{T}}$ is obtained from $\mathcal{T}$ by changing ν with $i\omega\eta$ in the seventh column.

Let us now apply Green's identity (3.1) both to the regular vector $V(y)$ satisfying (2.2), and to the vector $\tilde{V}(y)=\tilde{\Gamma}^{(k)}(y,x;\omega,\nu)$, $1\leqslant k\leqslant 7$, where $\tilde{\Gamma}^{(k)}$ is the the k order column of the matrix $\tilde{\Gamma}$ of fundamental solutions of the operator $\tilde{B}(\partial/\partial x,\omega)$ (see [5]).

It is known that [5]

(3.4)
$$\tilde{B}(\partial/\partial y,\omega)\tilde{\Gamma}^{(k)}(y,x;\omega,\nu)=-\delta^{(k)}\delta(y-x),$$

where $\delta(y-x)$ is the Dirac distribution, and $\delta^{(k)}=(\delta_1^{(k)},\delta_2^{(k)},\ldots\delta_7^{(k)})$, $\delta_j^{(k)}$ being the Kronecker symbol.

Taking into account (3.4) and

(3.5)
$$\int_{D^+}V(y)\,.\,\delta^{(k)}\delta(y-x)dy=l(x)v_k(x),\ x\in E_3,$$

where v_k is the k order component of $V(y)$ and

$$l(x)=\begin{cases}1 & \text{if } x\in D^+,\\ 1/2 & \text{if } x\in S,\\ 0 & \text{if } x\in D^-,\end{cases}$$

then from (3.1) we get

(3.6)
$$l(x)v_k(x)=\int_S\tilde{\mathcal{T}}(\partial/\partial y,n_y)\tilde{\Gamma}^{(k)}(y,x;\omega,\nu)\,.\,\mathcal{L}(\partial/\partial y,n_y)V(y)dS_y-$$
$$-\int_S\mathcal{L}(\partial/\partial y,n_y)\tilde{\Gamma}^{(k)}(y,x;\omega,\nu)\,.\,\mathcal{T}(\partial/\partial y,n_y)V(y)dS_y.$$

Considering that

(3.7)
$$\Gamma^T(x,y;\omega,\nu)=\Gamma(y,x;\omega,\nu),$$

where $\Gamma(x,y;\omega,\nu)$ is the matrix of fundamental solutions of the operator $B(\partial/\partial x,\omega)$, then the identity (3.7) can be written in the following form

(3.8)
$$l(x)V(x)=\int_S(\tilde{\mathcal{T}}(\partial/\partial y,n_y)\Gamma^T(x,y;\omega,\nu))^T\mathcal{L}(\partial/\partial y,n_y)V(y)dS_y-$$
$$-\int_S(\mathcal{L}(\partial/\partial y,n_y)\Gamma^T(x,y;\omega,\nu))^T\mathcal{T}(\partial/\partial y,n_y)V(y)dS_y,$$

which for $x\in D^+$ is an integral representation formula for the regular solution of (2.2) in D^+. The subscript T means the transpose operation.

To obtain an integral representation formula in D^- for the regular solution of (2.2), we take in (3.2) $\tilde{V}(y)=\tilde{\Gamma}^{(k)}(y,x;\omega,\nu)$, $1\leqslant k\leqslant 7$, and for $V(y)$ a regular solution of (2.2) in D^- satisfying the radiation conditions. This formula can be written in the vector form as

(3.9)
$$\mu(x)V(x)=\int_S(\mathcal{L}((\partial/\partial y,n_y)\Gamma^T(x,y;\omega,\nu))^T\mathcal{T}(\partial/\partial y,n_y)V(y)dS_y-$$
$$\int_S(\tilde{\mathcal{T}}(\partial/\partial y,n_y)\Gamma^T(x,y;\omega,\nu))^T\mathcal{L}(\partial/\partial y,n_y)V(y)dS_y,$$

where

$$\mu(x)=\begin{cases}1 & \text{if } x\in D^-,\\ 1/2 & \text{if } x\in S,\\ 0 & \text{if } x\in D^+.\end{cases}$$

When $x\in D^-$, (3.9) gives an integral representation for the regular solution in D^- of (2.2).

4. Surface Potentials and their Properties. As in the classical thermoelasticity [4], (3.8) and (3.9) lead us to introduce two surface micropolar thermoelastic potentials, namely,

(4.1)
$$L(x;\varphi)=\int_S(\mathcal{L}(\partial/\partial y,n_y)\Gamma^T(x,y;\omega,\nu))^T\varphi(y)dS_y,$$

(4.2)
$$H(x;\psi)=\int_S(\tilde{\mathcal{T}}(\partial/\partial y,n_y)\Gamma^T(x,y;\omega,\nu))^T\psi(y)dS_y,$$

where φ and ψ are unknown vector functions with seven components. We shall suppose that φ and ψ are of $C^{0,\beta}(S)$ class, where $0<\beta<\alpha$.

In the following we shall give some theorems on the potentials (4.1) (4.2).

Theorem 4.1. *The first derivatives with respect to x_i of $L(x;\varphi)$ belong to the class $C^{0,\beta}(\overline{D^+})$ or $C^{0,\beta}(\overline{D^-})$ and the $\mathscr{H}$-operator of $L(x;\varphi)$ tends to finite limits as the points x tends to the boundary point $z \in S$ from inside or from outside, these limits being respectively equal to*

$$(4.3)\qquad \begin{aligned}[\mathscr{H}(\partial/\partial z, n_z)L(z;\varphi)]^{\pm} &= \mp 1/2\varphi(z) + \\ &+ \int_S \mathscr{H}(\partial/\partial z, n_z)(\mathscr{L}(\partial/\partial y, n_y)\Gamma^T(z, y;\omega,\nu))^T\varphi(y)]\,dS_y\end{aligned}$$

where the symbol "+" ("−") corresponds to the limit from $D^+(D^-)$ and the integral is to understand in the sense of Cauchy's principal value.

The proof of this theorem is based on the fact that some components of the vector $\mathscr{H}(\partial/\partial x, n_x)L(x;\varphi)$, $x \in D^+$ or $x \in D^-$ are single layer potentials, the others being double layer ones, and for these types of potentials discontinuity relations have been given in [3]–[5].

Similarly, we may also formulate

Theorem 4.2. *The first derivatives with respect to x_i of $H(x;\psi)$ belong to the class $C^{0,\beta}(\overline{D^+})$ or $C^{0,\beta}(\overline{D^-})$* and the *$\mathscr{L}$-operator of $H(x;\psi)$ tends to finite limits as the points x tends to the boundary point $z \in S$ from inside or from outside, these limits being respectively equal to*

$$(4.4)\qquad \begin{aligned}[\mathscr{L}(\partial/\partial z, n_z)H(z;\psi)]^{\pm} &= \pm \tfrac{1}{2}\psi(z) + \\ &+ \int_S \mathscr{L}(\partial/\partial z, n_z)(\bar{\mathscr{H}}(\partial/\partial y, n_y)\Gamma^T(z, y;\omega,\nu))^T\psi(y)\,dS_y.\end{aligned}$$

Theorem 4.3. *The potentials $L(x;\varphi)$ and $H(x;\psi)$ satisfy (2.2) for x nonlocated on S.*

Theorem 4.4. *The potentials $L(x;\varphi)$ and $H(x;\psi)$ satisfy the radiation conditions.*

The proof of this theorems result from the asymptotic estimates of the elements of the matrix of fundamental solutions $\tilde{\Gamma}(x, y;\omega,\nu)$ given in [7].

5. **Boundary Value Problems.** In a previous paper [6] we have defined thirty two boundary value problems for D^+, called interior problems and denoted by $(p.q.r)^+$, where $p, q \in \{I, II, III, IV\}$, $r \in \{I, II\}$, and also thirty two exterior problems for $D^{\pm}$, denoted by $(p.q.r)^-$. Some of these problems, namely, $(I.II.II)^{\pm}$ and $(II.I.I)^{\pm}$, may be now reformulated in terms of the operators $\mathscr{L}$ and $\mathscr{H}$, respectively, as follows.

Interior Problems. *To find the vector function $V(x) \in C^2(D^+) \cap C^1(\overline{D^+})$, satisfying both* (2.2) *and one of the following conditions*

$$(5.1)\qquad [\mathscr{L}(\partial/\partial z, n_z)V(z)]^+ = F_i(z),\quad z \in S,$$

$$(5.2)\qquad [\mathscr{H}(\partial/\partial z, n_z)V(z)]^+ = G_i(z),\quad z \in S,$$

where F_i, G_i are prescribed vector functions with seven components belonging to the $C^{0\beta}(S)$ class.

These mean the determination of $\mathbf{u}$, $\boldsymbol{\varphi}$ and θ, satisfying (2.1) in D^+, when on the boundary, there are given either the displacements, the couple-stress vector $\mathbf{m}$ and the heat flux h, in the case of the problem $(I.II.II)^+$, or the stress vector t, the rotation φ and the temperature θ, when the problem $(II.I.I)^{\pm}$ is concerned.

Exterior Problems. *To find the vector function $V(x) \in C^2(D^-) \cap C^1(\overline{D^-})$ satisfying* (2.2), *the radiation conditions and one of the conditions*

$$(5.3)\qquad [\mathscr{L}(\partial/\partial z, n_z)V(z)]^- = F_e(z),\quad z \in S,$$

$$(5.4)\qquad [\mathscr{H}(\partial/\partial z, n_z)V(z)]^- = G_e(z),\quad z \in S,$$

where F_e, G_e are known vector functions of the $C^{0,\beta}(S)$ class.

Obviously, (5.3) is the boundary condition of the problem $(I.II.II)^-$, while (5.4) is the boundary condition of the problem $(II.II.I)^-$.

6. **Reduction of the Problems to Singular Integral Equations.** Let us look for the solutions of our problems as follows:

— for the problems $(I.II.II)$,$^{\pm}$ $V(x) = H(x;\psi)$,

— for the problems $(II.I.I)^{\pm}$, $V(x) = L(x;\varphi)$,

where the potential densities φ and ψ are unknown vector functions belonging to the $C^{0,\beta}(S)$ class.

By virtue of the theorems 4.3 and 4.4, $V(x) = H(x;\psi)$ and $V(x) = L(x;\varphi)$ verify (2.2) and also satisfy, when $x \in D^-$, the radiation conditions (5.1).

Let us now impose these functions to satisfy the boundary conditions (5.1)–(5.4).

By using the discontinuity theorems 4.1 and 4.2, from (4.3), (4.4) and (5.1)–(5.4), we obtain

$$(6.1)\qquad \pm\tfrac{1}{2}\psi(z) + \int_S \mathscr{L}(\partial/\partial z, n_z)(\bar{\mathscr{H}}(\partial/\partial y, n_y)\Gamma^T(z, y;\omega,\nu))^T\psi(y)\,dS_y = F_{i,e}(z),$$

$$(6.2)\qquad \mp\tfrac{1}{2}\varphi(z) + \int_S \mathscr{H}(\partial/\partial z, n_z)(\mathscr{L}(\partial/\partial y, n_y)\Gamma^T(z, y;\omega,\nu))^T\varphi(y)\,dS_y = G_{i,e}(z),$$

where the subscript i from the right-hand side of (6.1) and (6.2) corresponds to the interior problems, while e corresponds to the exterior problems

By $L^{\pm}(\psi)$ we denote the operators generated by the left-hand side of the equations (6.1). The notations $H^{\pm}(\varphi)$ will have an analogous sense. Therefore,

$$(6.3)\qquad L^{\pm}(\psi)(z) = \pm\tfrac{1}{2}\psi(z) + \int_S \mathscr{L}(\partial/\partial z, n_z)(\bar{\mathscr{H}}(\partial/\partial y, n_y)\Gamma^T(z, y;\omega,\nu))^T\psi(y)\,dS_y,$$

$$(6.4)\qquad H^{\pm}(\varphi)(z) = \mp\tfrac{1}{2}\varphi(z) + \int_S \mathscr{H}(\partial/\partial z, n_z)(\mathscr{L}(\partial/\partial y, n_y)\Gamma^T(z, y;\omega,\nu))^T\varphi(y)\,dS_y.$$

The kernel of the first operators $L^{\pm}(\psi)(z)$ is $K_L(x, y) = \mathscr{L}(\partial/\partial z, n_z)(\bar{\mathscr{H}}(\partial/\partial y, n_y)\Gamma^T(z, y;\omega,\nu))^T$, while $K_H(x, y) = \mathscr{H}(\partial/\partial z, n_z)(\mathscr{L}(\partial/\partial y, n_y)\Gamma^T(z, y;

$\omega, \nu))^T$ is the kernel of the operators $H^{\pm}(\varphi)(z)$. Using both the expressions of the matrix of fundamental solutions $\Gamma(z, y; \omega, \nu)$ [5] and (3.3), we can show that K_I and K_{II} are singular kernels of class $G^*(2, \alpha, \alpha)$ (see [1], ch. IV, §3, p. 150), and hence (6.1), (6.2) are singular integral equations.

In the same way as in the general case presented in [1], we can prove that $L^{\pm}(\psi)$ and $H^{\pm}(\varphi)$ are operators of normal type, and therefore they assume a regularization in $L_p^{(8)}(S)$ — the linear space of functions having the form $V=(v_1, v_2, \ldots, v_8)$, summed over S in the p-th power, the norm being

$$\|V\|=\left\{\int_S \left(\sum_1^8 |v_i|^2\right)^{p/2} dS_y\right\}^{1/p}, \quad p > 1. \tag{6.5}$$

It follows that the Fredholm theorems are valid for the singular integral equations (6.1) and (6.2) in the space $L_p^{(8)}(S)$. On the other hand, taking into account that $S \in L_1(\alpha)$, and that $L^{\pm}(\psi)$, $H^{\pm}(\varphi)$ are operators of normal type, if $F_{i,s}, G_{i,s} \in C^{0,\beta}(S)$, then any solution of (6.1) and (6.2), of class $L_p^{(8)}(S)$ belongs also to $C^{0,\beta}(S)$ class.

Therefore, the existence of the solutions of the problems (II. I. I)$^{\pm}$ and (I.II.II)$^{\pm}$ is reduced to the study of the singular integral equations (6.1) and (6.2). This study can be made in the same manner as in the case of the classical thermoelasticity [4].

REFERENCES

1. Kupradze, V. D., Gegelia, T. G., Baseleisvili, M. O., Burculadz T. V. — *Trehmernye zadaci matematiceskoi teorii uprugostii termouprugosti*, Izd. „Nauka", Moskva (1976)
2. Nowacki, W. — *Couple Stresses in the Theory of Thermoelasticity*. III, Bull. Acad. Polou. Sci., Sér. Sci. Techn., 14, *8* (1966)
3. Ieşan, D. — *Unele probleme fundamentale ale teoriei liniare a mediilor elastice de tip Cosserat, Probleme actuale în mecanica solidelor*, vol. I. Sub redacţia lui P. P. Teodorescu. Edit. Acad. R.S.R., Bucureşti (1975)
4. Kupradze, V. D., Burculadze, T. V. — *Granicinye zadaci termouprugosti*, Diff. Uravn., 5, *1* (1969)
5. Crăciun, I. — *Potential Method in Micropolar Thermoelasticity*, Rev. Roum. Sci. Techn. Sér. Mec. Appl., 22, *1* (1977)
6. Crăciun, I. — *Radiation Conditions and Uniqueness Theorems in Micropolar Thermoelasticity*, Rev. Roum. Sci. Techn., Sée. Méc. Appl., 23, *2* (1978)
7. Crăciun, I. — *Green Formula for an Infinite Domain with a Cavity and Integral Representation of a Regular Solution in Micropolar Thermoelasticity*, Bul. Inst. Polit. Iaşi, XXIV (XXVIII), s. I a, fasc. *1—2*, fasc. *1—2* (1978)
8. Agniasvili, M. R. — *Nekotorye kraevye zadaci termomomentnoi teorii uprugosti*, Raboty po zadaciam mehaniki splosnyh sred, Inct. Prikl. Mat. Tbilisi, G.U.T. (1976).

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ON THE DEFORMATION OF ALMOST CYLINDRICAL BEAMS

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1. **Introduction.** The problem of deformation of almost cylindrical beams in the theory of elasticity was initiated by Panov [1], Rukhadze [2] and Borş [3] and it was developed in various papers (see, e.g. [4]—[8]).

In the present paper we give a method, more simple than those known, for solving the problem of deformation of an almost cylindrical beam subjected to body forces and lateral boundary tractions which are polynomials in the axial coordinate. The ends of the beam are subjected to appropriate stress resultants. We consider the case of an almost cylindrical beam made of an isotropic and inhomogeneous elastic material. The Lamé coefficients are assumed to be independent of the axial coordinate.

Our method presents a new treatment of the problem even for inhomogeneous elastic bodies. Moreover, the present paper points out the contribution due to the form of the beam in the solution. In fact, the method presents the solution in terms of the solutions of two problems, $\mathfrak{L}_0$ and $\mathfrak{L}_1$. The solution of the problem $\mathfrak{L}_0$ represents the solution of the initial formulated problem as though the beam would be a cylindrical beam. The solution of the problem $\mathfrak{L}_1$ is the solution of an Almansi type problem for a cylindrical beam. This last solution is due to the form of the beam. Both solutions can be determined by using the method described in [9], [10].

The proposed method is then used in order to study the Saint-Venant's problem for a frustrum of cone made of an isotropic and homogeneous elastic material.

2. **Statement of the problem.** Throughout this paper a rectangular Cartesian coordinate system Ox_k $(k=1, 2, 3)$ is used. We shall employ the usual summation and differentiation conventions: Greek subscripts are understood to range over the integers (1,2), whereas Latin subscripts — unless otherwise specifed — to the range (1, 2, 3); summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding coordinate.

We consider *an almost cylindrical beam* limited by two planes $x_3=0$ and $x_3=l$, and a lateral surface F, given by

$$f(x_1(1-\varepsilon x_3), \quad x_2(1-\varepsilon x_3))=0, \tag{1}$$

where ε is a small parameter, the square and higher powers of which can be neglected. Let B be the interior of the beam and Σ the generic cross-section of the beam. We assume that Σ is a C^1-smooth domain [11]. We denote by Σ_{x_3} the cross-section located in the plane $x_3=\text{const}$. The x_3-axis of our coordinate system will be directed along the line passing through the centroids of the cross-sections of the beam.

In what follows we consider the linear theory of elasticity for inhomogeneous and isotropic materials. Let u_i denote the components of the displacement vector field. The components of the strain tensor are given by